Portugal. Math. (N.S.)
Vol. 75, Fasc. 1, 2018, 11–65
DOI 10.4171/PM/2010

Portugaliae Mathematica

Ricci flow on cone surfaces

Daniel Ramos

Abstract. We study the evolution under the Ricci flow of surfaces with singularities of cone type. Firstly we provide a complete classification of gradient Ricci solitons on surfaces, which is of independent interest. Secondly, we prove that closed cone surfaces with cone angles less or equal to π converge, up to rescaling, to closed soliton metrics under the Ricci flow.

Mathematics Subject Classification: 53C44

Keywords: Ricci flow, Ricci solitons, conical singularities, uniformization theorem

1. Introduction

The classical uniformization theorem for smooth closed surfaces states that every smooth closed surface admits a Riemannian metric with constant curvature $+1$, 0 or -1, according to its Euler characteristic χ being positive, zero or negative, respectively. If we consider non-smooth surfaces, for instance orbifolds or more generally surfaces with cone singularities, it is well known that some closed surfaces do not admit Riemannian metrics (in the smooth part) with constant curvature, such as the so called "teardrop" and "football" orbifolds. It is a natural question to find a canonical or natural metric for these objects.

The uniformization theorem has many proofs and traces back to 19th century, but one modern proof can be made using Ricci flow. A *Ricci flow* is a PDE evolution equation for a time-dependent Riemannian metric $g(t)$ on a smooth manifold $\mathscr{M}$, according to

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} \tag{1}$$

with a given initial condition $g(0) = g_0$. It was introduced in 1982 by R. Hamilton [Ham82] in the context of 3-manifolds, and later used by him [Ham88] and by B. Chow [Cho91b] on surfaces. The general idea is that the evolution under the

Ricci flow spreads out the curvature, so the metric becomes more regular across the manifold. Eventually, the metric can converge to a constant curvature metric, or more generally, to the so-called *solitons* of the Ricci flow, which are the fixed points of the flow as a dynamical system (up to diffeomorphism and homothety). This is the behavior on surfaces (up to scaling normalizations) but it is much more complex in higher dimension. In 2002 and 2003 G. Perelman made major advances to the understanding of the singularities of Ricci flow in dimension three [Per02], [Per03b], [Per03a], which eventually led him to prove the Poincaré conjecture and Thurston's geometrization. The fact that the uniformization theorem can be consistently proved by Ricci flow was established in 2006 by X. Chen, P. Lu and G. Tian [CLT06].

The purpose of this article is to apply Ricci flow to uniformize surfaces with cone-like singularities. First reference to this subject in the literature are the works of L.-F. Wu and B. Chow in early 90s, [Wu91], [CW91], [Cho91a], that deal with some "orbifold solitons" on the teardrop and the football that Hamilton found in [Ham88], when he noted that cone singularities arise naturally in the study of solitons. These works assume a group-equivariant definition of the flow for orbifolds and don't study the existence of the flow in a general cone surface. More recently, the series of H. Yin [Yin10], [Yin13], [Yin16], and the work of R. Mazzeo, Y. Rubinstein and N. Sesum [MRS15], study the problem of the short-time existence on a cone surface, following different approaches. We appeal to these results for this article. Yet another approach was followed by D. H. Phong, J. Song, J. Sturm and X. Wang [PSSW14] for the existence. While all these works contain some descriptions of the long-time behaviour of the flow and its convergence, none of them use Perelman's analysis of singularities by rescaling blow-up. We propose this method to analyze the behaviour of the flow on cone surfaces with Euler characteristic $\chi > 0$, which we find successful under the additional assumption that the cone angles are $\leq\pi$. This restriction has been later removed in the works of [MRS15] and [PSSW15].

The structure of the article is the following. In the first part (Section 2) we make an exhaustive enumeration of all complete gradient Ricci solitons on smooth and cone surfaces, both compact and noncompact, with an arbitrary lower bound on the curvature; and we give explicit constructions for them:

Theorem 1.1. *All gradient Ricci solitons on a surface, smooth everywhere except possibly on a discrete set of cone-like singularities, complete, and with curvature bounded below fall into one of the following families:*

(1) *Steady solitons:*
   (a) *Flat surfaces.*
   (b) *The smooth cigar soliton.*
   (c) *The cone-cigar solitons of angle $\alpha \in (0, +\infty)$.*

(2) *Shrinking solitons:*
   (a) *Spherical surfaces.*
   (b) *Teardrop and football solitons, on a sphere with one or two cone points.*
   (c) *The shrinking flat Gaussian soliton on the plane.*
   (d) *The shrinking flat Gaussian cones.*

(3) *Expanding solitons:*
   (a) *Hyperbolic surfaces.*
   (b) *The $\alpha\beta$-cone solitons, with a cone point of angle $\beta > 0$ and an end asymptotic to a cone of angle $\alpha > 0$.*
   (c) *The smooth blunt α-cones.*
   (d) *The smooth cusped α-cones in the cylinder, asymptotic to a hyperbolic cusp in one end and asymptotic to a cone of angle $\alpha > 0$ in the other end.*
   (e) *The flat-hyperbolic solitons on the plane, that are universal coverings of the cusped cones.*
   (f) *The expanding flat Gaussian soliton on the plane.*
   (g) *The expanding flat Gaussian cones.*

Theorem 1.1 unifies and completes the list of particular cases that were scattered through the literature, and it is of independent interest beyond the study of cone surfaces. Our method finds all solitons smooth or with cone points, but we listed above only the ones that are complete and with bounded curvature. Non-complete solitons and unbonded-below curvature also appear in our study, but we discard them since their role as fixed points of the flow (modulo isometries and homotheties) is somehow diluted (non-uniquenes or non-existence of solutions). A similar classification for smooth solitons, using a different analysis, was found in [BM15]; that classification lists complete and incomplete solitons, but not solitons with cone singularities. Pictures in our Appendix A provide the first visualizations for many of these solitons.

In the second part of the article (Sections 3 and 4) we study the evolution of a surface with cone points along the Ricci flow. Our uniformization result is the following:

Theorem 1.2. *Let $\left(\mathscr{M}, (p_1, \ldots, p_n), g_0\right)$ be a closed cone surface, and assume that the cone angles are less than or equal to π. Then there is an angle-preserving Ricci flow that converges, up to rescaling, to either*

- *a constant nonpositive curvature metric, if $\hat{\chi}(\mathscr{M}) \leq 0$, or*
- *a spherical (constant positive curvature) metric, a teardrop soliton or a football soliton, if $\hat{\chi}(\mathscr{M}) > 0$,*

where $\hat{\chi}(\mathscr{M})$ is the conic Euler characteristic of the cone surface.

We start by giving in Section 3.1 an outline of a proof of the uniformization of smooth surfaces via Ricci flow, using the results of Hamiton and Perelman. Next we discuss the three main obstacles we need to address to make this proof valid in the setting of cone manifolds: First, the existence of the flow (Section 3.2), for which we invoke the short-time existence result in [MRS15]. Second, adapting the maximum principles and Harnack inequalities (Section 3.3). Third, establishing compactness theorems for classes of cone manifolds, to perform Perelman's rescaling blow-ups (postposed to Section 4). With these elements, we present the proof of Theorem 1.2 in Section 3.4.

In contrast to Section 3, the compactness theorems exposed in Section 4 use techniques of metric geometry, and most of them are independent of the flow, hence the different section. We find that the compactness theorems that guarantee the existence of limits of sequences of rescaled surfaces require the additional assumption of small angles, namely less than or equal to π, in order to guarantee that the cone structure at the limit is the same as in the sequence. If this condition is not satisfied, other phenomena might happen, like cone points collapsing together. For a compactness theorem in the case of collapsing points, the reader can confer with the later work of C. Débin [Déb16].

Our use of compactness of classes of cone surfaces is one of the main differences of our work with respect to [Yin16] and [MRS15]. We obtain a more precise control of the cone points in the case of angles $\leq \pi$, when points don't collapse together, but we could not adapt this technique to describe the collapse of cone points. The results in [MRS15], [PSSW14] and [PSSW15] point out that collapsing together of cone points actually happens in some conditions with many cone points with angles bigger than π, so the limit is a soliton with at most two cone points (as in the classification in Theorem 1.1). A description of how this phenomenon happens is not yet as precise as the case of small angles.

A note about chronology: This paper was first completed and made public as a PhD thesis in December 2013 [Ram14]. The complete classification of two-dimensional gradient Ricci solitons was found and published independently and almost simultaneously in [BM15] and [Ram13] with the differences noted above. The works [Yin10], [Yin13] were prior to this work, although we based ourselves in the existence theorem of [MRS15], which was announced in the survey [IMS11] and was communicated personally in more detail to us in 2012. The convergence theorem in [MRS15] was obtained independently to ours. The works [Yin16], [PSSW14] and [PSSW15] are posterior to this work. The work [Déb16] is also posterior, but it was also communicated to us in 2014.

Readers interested in a more extended exposition of this article can consult [Ram14].

2. Gradient Ricci solitons on smooth and cone surfaces

A *Ricci soliton* is a special type of self-similar solution of the Ricci flow (1), in the form

$$g(t) = c(t)\phi_t^*(g_0) \tag{2}$$

where for each t, $c(t)$ is a constant and ϕ_t is a diffeomorphism. The family ϕ_t is the flow associated to a (maybe time-dependent) vector field $X(t)$. If this vector field is the gradient field of a function, $X = \operatorname{grad} f$, the soliton is said to be a *gradient* soliton. In this case, differenciating (2) and evaluating at $t = 0$ gives

$$\mathrm{Ric} + \operatorname{Hess} f + \frac{\varepsilon}{2} g = 0. \tag{3}$$

for $g = g_0$, where $\varepsilon = \dot{c}(0)$. The soliton is said to be *shrinking*, *steady* or *expanding* if the constant ε is negative, zero or positive respectively. This constant can be normalized to be -1, 0 or $+1$ respectively, being this equivalent to reparameterize the time t. Therefore, a gradient Ricci flow is a triple $(\mathscr{M}, g, f)$ satisfying (3).

In our case, $\mathscr{M}$ is a surface. Thus, $\mathrm{Ric} = \frac{R}{2} g$, and hence the soliton equation (3) becomes

$$\operatorname{Hess} f + \frac{1}{2}(R + \varepsilon) g = 0. \tag{4}$$

In dimension two we have the remarkable property that all nonconstant curvature gradient solitons are rotationally symmetric. Let $J : T\mathscr{M} \to T\mathscr{M}$ be an almost-complex structure on the surface $\mathscr{M}$, that is, a 90° rotation on the positive orientation sense. Then the vector field $J(\operatorname{grad} f)$ is a Killing vector field (see [CCCY03], pp. 241–242, [CCG+07], p. 11, [Cao96] and [CLT06]). This vector field may be null if the $\operatorname{grad} f$ field itself is null, but otherwise the flow of $J(\operatorname{grad} f)$ constitutes a continuous group acting by isometries that makes the surface symmetrical. Let us assume that the surface may have some marked points (cone singularities) which are not locally diffeomorphic to regular points.

Lemma 2.1. *Let $(\mathscr{M}, g, f)$ be a gradient Ricci soliton on a surface (possibly with cone singularities). Then, at least one of the following holds:*

(1) *$\mathscr{M}$ has constant curvature.*

(2) *$\mathscr{M}$ is rotationally symmetric (i.e. admits a $\mathbb{S}^1$-action by isometries).*

(3) *$\mathscr{M}$ admits a quotient that is rotationally symmetric.*

Besides, if the soliton has not constant curvature, no more than two cone points may exist.

Proof (cf. [CLT06]). We will discuss in terms of $\operatorname{grad} f$. If $\operatorname{grad} f \equiv 0$, then by the soliton equation (4) we have $R = -\varepsilon$ and the curvature is constant. Let us assume then that f is not constant everywhere. Therefore $J(\operatorname{grad} f)$ is a nontrivial Killing vector field and its line flow, ϕ_t, is a one-parameter group acting over $\mathscr{M}$ by isometries.

Suppose that $\operatorname{grad} f$ has at least one zero in a point $O \in \mathscr{M}$. The point O is a zero of the vector field $J(\operatorname{grad} f)$, so it is a fixed point of ϕ_t for every t. Then, ϕ_t induces ϕ_t^* acting on $T_O\mathscr{M}$ by isometries of the tangent plane, so we conclude that the group $\{\phi_t\}$ is $\mathbb{S}^1$ acting by rotations on the tangent plane. Via the exponential map on O, the action is global on $\mathscr{M}$ and therefore the surface is rotationally symmetric. This is the case of closed smooth surfaces.

Suppose now that $\operatorname{grad} f$ has no zeroes but the surface contains a cone point P. Then the flowlines of ϕ_t cannot pass through P, because there is no local diffeomorphism between a cone point and a smooth one. So this point P is fixed by ϕ_t for every t and, via the exponential map, ϕ_t induces ϕ_t^* acting on $C_P\mathscr{M}$ the tangent cone (space of directions) on P. Again, a continuous one-parameter subgroup of the metric cone $C_P\mathscr{M}$ must be the $\mathbb{S}^1$ group acting by rotations. Besides, if other cone points were to exist, these should also be fixed by the already given $\mathbb{S}^1$ action. This implies that no more than two cone points can exist on $\mathscr{M}$, for otherwise the minimal geodesics joining P with two or more conical points would be both fixed and exchanged by some $\mathbb{S}^1$ group element. Note that in the case of two cone points, these need not to have equal cone angles.

Finally suppose that $\operatorname{grad} f$ has no zeroes and the surface has no cone points. Then the surface is smooth and the flowlines of $\operatorname{grad} f$ are all of them isomorphic to $\mathbb{R}$ (no closed orbits can appear for the gradient of a function) and foliate the surface. The action of ϕ_t exchanges the fibres of this foliation. The parameter of ϕ_t is $t \in \mathbb{S}^1$ or $t \in \mathbb{R}$. In the first case, $\mathbb{S}^1$ is acting on $\mathscr{M}$ and it is rotationally symmetric. In the second case, $\mathscr{M} \cong \mathbb{R}^2$, and the flowline ϕ_t of the Killing vector field induces a $\mathbb{Z}$-action by isometries by $x \mapsto \phi_1(x)$ that acts freely on $\mathscr{M}$ since no point is fixed by ϕ_t for any $t \neq 0$ (if $\phi_t(p) = p$, then all fibres are fixed and every point in each fibre also is, so $\phi_t = id$). Then the quotient by this action is topologically $\mathscr{M}/_\sim \cong \mathbb{R} \times \mathbb{S}^1$ and is rotationally symmetric. We will find nontrivial examples of these solitons as cusped expanding solitons and their universal coverings. □

Being rotationally symmetric allows us to endow $\mathscr{M}$ with polar coordinates (r, θ) such that the metric is given by

$$g = dr^2 + h^2(r)\, d\theta^2$$

where $r \in I \subseteq \mathbb{R}$ is the radial coordinate, and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the periodic angular coordinate. The function $h(r)$ does not depend on θ because of the rotational symmetry; and similarly, the potential function f only depends on the r coordinate, since $\operatorname{grad} f$ is a radial vector field. Surfaces not rotationally symmetric but with a rotationally symmetric quotient also admit these coordinates, changing only $\theta \in \mathbb{R}$.

In these coordinates, the Gaussian curvature is given by

$$K = \frac{R}{2} = \frac{-h''}{h},$$

and the Hessian of a radial function $f(r)$ is given by

$$\operatorname{Hess} f = f''\, dr^2 + hh'f'\, d\theta^2.$$

On that rotationally symmetric setting, the soliton equation (4) becomes

$$\operatorname{Hess} f + \frac{1}{2}(R+\varepsilon)g = \left(f'' - \frac{h''}{h} + \frac{\varepsilon}{2}\right) dr^2 + \left(hh'f' + \left(-\frac{h''}{h} + \frac{\varepsilon}{2}\right)h^2\right) d\theta^2 = 0,$$

which is equivalent to the second order ODEs system

$$\begin{cases} f'' - \frac{h''}{h} + \frac{\varepsilon}{2} = 0 \\ \frac{h'}{h} f' - \frac{h''}{h} + \frac{\varepsilon}{2} = 0. \end{cases} \tag{5}$$

We combine both equations to obtain

$$\frac{f''}{f'} = \frac{h'}{h},$$

and integrating this equation,

$$\ln f' = \ln h + C$$

so

$$f' = ah$$

for some $a > 0$. Hence, substituting on the system we obtain a single ODE,

$$h'' - ahh' - \frac{\varepsilon}{2} h = 0.$$

We summarize the computations in the following lemma,

Lemma 2.2. *Let $(\mathscr{M}, g, f)$ be a gradient Ricci soliton on a surface with non-constant curvature. Then $\mathscr{M}$ admits coordinates (r, θ), with $r \in I \subseteq \mathbb{R}$ and $\theta \in \mathbb{S}^1$ or $\theta \in \mathbb{R}$, such that the metric takes the form $g = dr^2 + h^2(r)\, d\theta^2$ for some function $h = h(r)$ satisfying*

$$h'' - ahh' - \frac{\varepsilon}{2} h = 0, \tag{6}$$

for some $a > 0$ and $\varepsilon = -1, 0, 1$. The potential function $f = f(r)$ satisfies $f' = ah$.

Setting $h' = u$, the second order ODE (6) is equivalent to a vectorial first order ODE

$$\begin{cases} h' = u \\ u' = \left(au + \frac{\varepsilon}{2}\right) h. \end{cases} \tag{7}$$

The solutions to system (7) are functions $h(r)$ that define rotationally symmetric metrics on the cylinder $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$. This cylinder may be pinched in one or both ends, thus changing the topology of the surface. The pinching appears as zeros of h. Compactness condition of the surface is equivalent to the boundary conditions

$$h(0) = 0 \qquad \text{and} \qquad h(A) = 0$$

for some $A > 0$. In this case, one or two cone angles may appear,

$$h'(0) = \frac{\alpha_1}{2\pi} \qquad \text{and} \qquad h'(A) = -\frac{\alpha_2}{2\pi}$$

where α_1 and α_2 are the cone angles. Smoothness conditions would be $h'(0) = 1$ and $h'(A) = -1$, plus the condition

$$h^{(2k)} = 0$$

at $r = 0$ and $r = A$ for all $k \geq 0$. This condition ensures $\mathscr{C}^\infty$ regularity ([CCG+07], Lemma A.2). This condition holds on our solitons, since derivating $2k$ times the equation (6) we obtain

$$h^{(2+2k)} - a\Big(\sum_{i+j=2k} h^{(i)} h^{(j+1)}\Big) - \frac{\varepsilon}{2} h^{(2k)} = 0.$$

Since $i + j$ is an even number, both i and j must be even or odd. In both cases, there is an even index in $h^{(i)} h^{(j+1)}$. Thus inductively, if all even-order derivatives vanish at $r = 0$ up to order $2k$, then also vanishes the $2k+2$ derivative at $r = 0$ (idem at $r = A$).

We shall study the system (7) for steady, shrinking and expanding solitons to obtain a complete enumeration of gradient Ricci solitons on surfaces of non-constant curvature.

2.1. Closed solitons of constant curvature. Trivial examples of solitons are surfaces of constant curvature. Further, these are the only closed smooth solitons in dimension 2 (cf. [CLT06]).

Lemma 2.3. *The only solitons over a compact smooth surface are those of constant curvature.*

Proof. We multiply equation (6) by h' to get

$$h'h'' - ah(h')^2 + \varepsilon \frac{hh'}{2} = 0$$

and integrate on $[0, A]$ to obtain

$$\frac{(h')^2}{2}\bigg|_0^A - a\int_0^A h(h')^2\,dr + \varepsilon \frac{h^2}{4}\bigg|_0^A = 0.$$

If we look for rotationally symmetric closed smooth solitons, $h(0) = h(A) = 0$ and $h'(0) = -h'(A) = 1$, therefore

$$0 = -a\int_0^A h(h')^2\,dr$$

which implies $a = 0$. Thus $f' = 0$, there is no gradient vector field, no Killing vector field, constant curvature and the soliton is a homothetic fixed metric. □

Note that if there is no vector field, there is no need to be rotationally symmetric. More generally, rotationally symmetric closed solitons with two equal angles satisfy $h'(0) = -h'(A) = \frac{\alpha}{2\pi}$ and the same argument applies.

Lemma 2.4. *The only solitons over a compact surface with two equal cone points are shrinking spherical surfaces.*

Proof. In this case, equation (6) with $a = 0$ turns into

$$h'' - \frac{\varepsilon}{2}h = 0$$

that can be explicitly solved. For $\varepsilon = 1$ the solution is

$$h(r) = c_1 e^{r/\sqrt{2}} + c_2 e^{-r/\sqrt{2}}$$

but the closedness condition $h(0) = h(A) = 0$ implies $c_1 = c_2 = 0$. Thus there are no expanding solitons with two equal cone points besides the constant curvature ones.

For $\varepsilon = 0$, the solution is $h(r) = c_1 r + c_2$, that can't have two zeroes unless $h \equiv 0$. Finally, for $\varepsilon = -1$ the solution is $h(r) = c_1 \sin(r/\sqrt{2}) + c_2 \cos(r/\sqrt{2})$, and by the closedness $c_2 = 0$. This metric is locally the round sphere. $\square$

We have examined all possible cases with $a = 0$, which account for the families (1-a), (2-a) and (3-a) in Theorem 1.1. We will assume henceforth that $a \neq 0$ and f is not constant.

2.2. Steady solitons. In this subsection we study the steady case ($\varepsilon = 0$) of rotationally symmetric solitons. The equation (6) reduces to

$$h'' - ahh' = 0 \tag{8}$$

and the system (7) to

$$\begin{cases} h' = u \\ u' = auh \end{cases} \tag{9}$$

The phase portrait of (9) is shown in Figure 1.

This phase portrait has a line of fixed points at $\{u = 0\}$, that account for the trivial steady solitons consisting on a flat cylinder of any fixed diameter (or their universal covering, the flat plane). No other critical points are present. Only the right half-plane $\{h > 0\}$ is needed, since we can take $h > 0$ in the metric definition.

Every integral curve of the system lies on a parabola. This follows from manipulating system (9)

$$u' = ahh' = a\left(\frac{h^2}{2}\right)'$$

and hence

$$u = a\frac{h^2}{2} + C.$$

In another terminology, the function

$$H(h,u) = a\frac{h^2}{2} - u$$

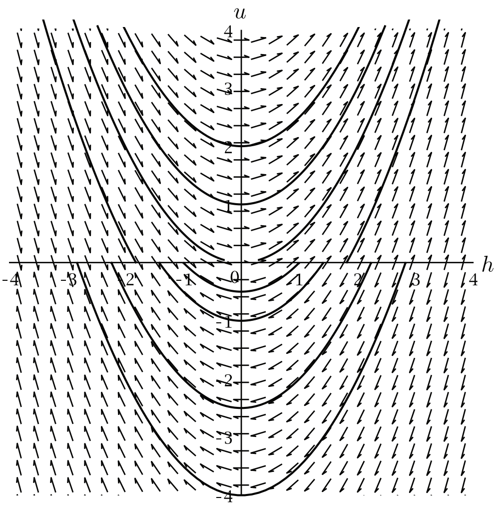

Figure 1. Phase portrait of the system (9) with $a = 1$.

is a first integral of the system (9). Furthermore, we can finish the integration of the equation

$$h' = a\frac{h^2}{2} + C$$

by writting

$$\frac{h'}{C + \left(\sqrt{\frac{a}{2}}h\right)^2} = 1.$$

The solution to this ODE is

$$h(r) = \sqrt{\frac{2C}{a}}\tan\left(\sqrt{\frac{2}{aC}}r + D\right) \tag{10}$$

if $C > 0$;

$$h(r) = \sqrt{\frac{-2C}{a}}\tanh\left(\sqrt{\frac{2}{-aC}}r + D\right) \tag{11}$$

if $C < 0$; and

$$h(r) = \frac{1}{D - \frac{a}{2}r} \tag{12}$$

if $C = 0$.

Now, let us examine each type of solution. If $C > 0$, the parabola lies completely on the upper half-plane $\{u > 0\}$. The equation (10) implies that $h \to \infty$ for some finite value of r, and hence the metric is not complete. Furthermore, the Gaussian curvature K of the metric satisfies

$$u' = -Kh$$

and since u is increasing on these solutions, the curvature is not bounded below. The case $C = 1$ is sometimes called the *exploding soliton* in the literature ([CCG+07]).

If we look at $C = 0$, the parabola touches the origin of coordinates, and its right hand branch defines a metric on the cylinder. The value of $D = \frac{1}{h(0)}$ can be set so that $D = h(0) = 1$ just reparameterizing r. With this parameterization, $r \in \left(-\infty, \frac{2}{a}\right)$. For $r \leq 0$, the function h is well defined and determines a negatively curved metric that approaches a cusp as $r \to -\infty$. However, for $r \in \left[0, \frac{2}{a}\right)$ the metric is not complete and its curvature tends to $-\infty$ as $t \to \frac{2}{a}$.

We look now at the case $C < 0$, first for the solutions lying in the lower half-plane $\{u < 0\}$. We can assume $h(0) = 0$, $C = u(0)$, $D = 0$ and $r < 0$ (this means that $-r$ is the arc-parameter of the meridians). All these arcs of parabolas join a point on the $\{h = 0\}$ axis with a point on the $\{h' = u = 0\}$ axis. This means that the cylinder is pinched in one end, and approaches a constant diameter cylinder on the other end. The metrics are complete on the cylindrical end, because from equation (11) $h \to cst$ as $r \to -\infty$. The curvature on these metrics is bounded and positive, since u and $u' < 0$ are bounded.

Some of these metrics are smooth, the particular cases of $C = u(0) = h'(0) = -1$. Note that derivating the equation $h'' = ahh'$ and evaluating at $r = 0$ one sees that all even-order derivatives vanish and the surface is truly $\mathscr{C}^\infty$ at this point. These are the so called *cigar solitons*. There are actually infinitely many of them, adjusting the value of a and changing the diameter of the asymptotic cylinder, although all of them are homothetic and hence it is said to exist *the* cigar soliton. All the other metrics have a cone point at $r = 0$, whose angle is $-2\pi h'(0)$.

The only remaining case to inspect is the solutions with $C < 0$ lying on the upper half plane $\{u > 0\}$. These unbounded arcs of parabolas rise from the axis $\{u = 0\}$. We can assume (changing D and reparameterizing r) that $r \in [0, +\infty)$. The metric is complete in $r \to +\infty$ because of equation (11), however, these metrics fail to be complete on $r = 0$, having a metric completion with boundary $\mathbb{S}^1$. The curvature on these metrics is negative and not bounded below.

This completes the classification of the steady solitons in Theorem 1.1, Part 1. Pictures of a cigar soliton and a cone-cigar soliton are shown in Figures 6 and 7 in Appendix A.

2.3. Shrinking solitons. In this subsection we study the shrinking solitons ($\varepsilon = -1$), besides the round sphere and the spherical footballs with two equal cone angles found in Subsection 2.1. When $\varepsilon = -1$, the metric of $\mathscr{M}$ is determined by a real-valued function $h(r)$ satisfying the second order ODE

$$h'' - ahh' + \frac{h}{2} = 0, \tag{13}$$

or equivalently the system

$$\begin{cases} h' = u \\ u' = \left(au - \frac{1}{2}\right)h. \end{cases} \tag{14}$$

The phase portrait of this ODE system with $a = 1$ is shown in Figure 2. This phase portrait has a critical point at $(h, u) = (0, 0)$ of type center, and a horizontal isocline (points such that $u' = 0$) at the line $u = \frac{1}{2a}$. Each curve on this hu-plane corresponds to a solution h, and the intersection with the vertical axis $\{h = 0\}$ are at $u(0)$ and $u(A)$, which stand for the cone angles. Indeed, only half of each curve is enough to define the soliton, the one lying in the $\{h > 0\}$ half-plane, since we can choose the sign of h because only h^2 is used to define the metric.

All curves in the phase portrait represent rotationally symmetric soliton metrics over, a priori, a topological cylinder. Closed curves (that intersect twice

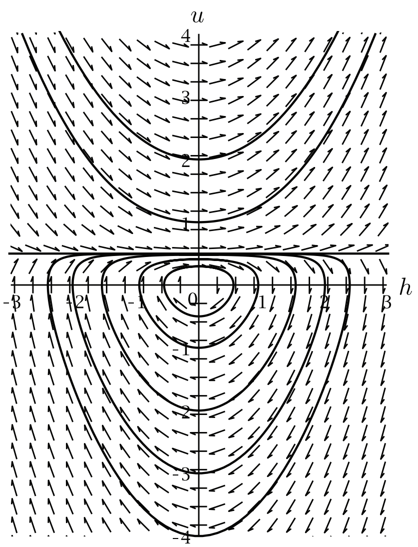

Figure 2. Phase portrait of the system (14) with $a = 1$.

the axis $\{h=0\}$) are actually metrics over a doubly pinched cylinder, thus a topological sphere with two cone points, giving the so called *football solitons*. Open curves only intersect once the $\{h=0\}$ axis, and hence are metrics over a topological plane. If the intersection of any curve with the $\{h=0\}$ axis occurs at $u=\pm 1$, then the metric extends smoothly to this point (truly $\mathscr{C}^{\infty}$ since derivating (13) all even-order derivatives vanish at this point). For instance, in Figure 2 there is only one curve associated to a *teardrop soliton*, namely the one intersecting the vertical axis at some value $u(0)\in\left(0,\frac{1}{2}\right)$ and at $u(A)=-1$. There is also a smooth soliton metric on $\mathbb{R}^2$, namely the one associated with the curve passing through $(h,u)=(0,1)$, and all other curves represent solitons over cone surfaces. The separatrix line, $u=\frac{1}{2a}$, represents the solution $h(r)=\frac{r}{2a}+c_0$, which stands for the metric $dr^2+\frac{1}{4a^2}r^2\,d\theta^2$. This is a flat metric on the cone of angle $\frac{\pi}{a}$, a cone version of the shrinking Gaussian soliton, and we call it a *shrinking Gaussian cone soliton* (the smooth *shrinking Gaussian soliton* is the case $a=\frac{1}{2}$).

Let us focus on the compact shrinking solitons.

Lemma 2.5. *For every pair of values* $0<\alpha_1<\alpha_2<\infty$, *there exist a unique value* $a>0$ *such that the equation* (6) *has one solution satisfying the boundary conditions* $h'(0)=\frac{\alpha_1}{2\pi}$ *and* $h'(A)=-\frac{\alpha_2}{2\pi}$.

Equivalently, the lemma asserts that there exists a value a such that the phase portrait of the system (7) has one solution curve that intersect the vertical axis $\{h=0\}$ at $u(0)=\frac{\alpha_1}{2\pi}$ and $u(A)=-\frac{\alpha_2}{2\pi}$.

Proof. We can normalize the system by

$$\begin{cases} v = ah \\ w = au \end{cases}$$

so that on this coordinates the system becomes

$$\begin{cases} v' = w \\ w' = \left(w-\frac{1}{2}\right)v \end{cases} \tag{15}$$

This would be the same system as (14) with $a=1$, shown in Figure 2.

The system (15) has the following first integral,

$$H(v,w)=v^2-2w-\ln|2w-1|$$

that is, the solution curves of the system are the level sets of H. Indeed, derivating $H\big(v(r),w(r)\big)$ with respect to r,

$$\begin{aligned}\frac{\partial}{\partial r}H(v,w) &= 2vv' - 2w' - \frac{2w'}{2w-1}\\ &= 2vw - 2\left(w-\frac{1}{2}\right)v\left(1+\frac{1}{2w-1}\right) = 0.\end{aligned}$$

The cone angle conditions are $\alpha_1 = \frac{2\pi w(0)}{a}$, $\alpha_2 = -\frac{2\pi w(A)}{a}$, while $v(0) = v(A) = 0$. Thus, the function w evaluated at 0 and A satisfies

$$H(0,w) = 2w + \ln|2w-1| = C$$

for some $C \in \mathbb{R}$. This is equivalent, via $2w - 1 = -y$ and $e^C = k$, to the equation

$$|y| = ke^{y-1} \tag{16}$$

(cf. [Ham88]). Although not expressable in terms of elementary functions, this equation has three solutions for y, one for negative y and two for positive y (see Figure 3). The two positive solutions of (16) are the intersection of the exponential function e^{y-1} with the line $\frac{1}{k}y$ with slope $\frac{1}{k}$. These two positive solutions are associated to a compact connected component of $H(v,w) = C$, whereas the negative solution is associated to a noncompact component of H that represent noncompact soliton surfaces. The two positive solutions of (16) exist only when $k \in (0,1)$ and actually these two solutions are equal when $k = 1$ and the line is tangent to the exponential function at $y = 1$. These two solutions y_1, y_2 of (16) are therefore located on $(0,1)$ and $(1,+\infty)$ respectively, and can be expressed as

$$y_1 = 1 - p, \qquad y_2 = 1 + q$$

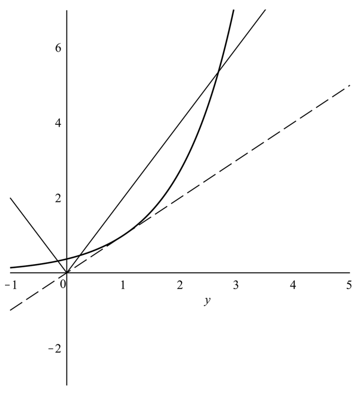

Figure 3. The graphs of the exponential e^{y-1} and $\frac{1}{k}|y|$.

with $p, q \geq 0$. The two cone angles, having assumed $\alpha_1 < \alpha_2$, are then expressed as

$$\alpha_1 = 2\pi h'(0) = 2\pi u(0) = \frac{2\pi w(0)}{a} = \frac{2\pi}{a}\frac{1-y_1}{2} = \frac{\pi}{a}p$$

$$\alpha_2 = -2\pi h'(A) = -2\pi u(A) = -\frac{2\pi w(A)}{a} = -\frac{2\pi}{a}\frac{1-y_2}{2} = \frac{\pi}{a}q$$

and their quotient is

$$\frac{\alpha_1}{\alpha_2} = \frac{p}{q}.$$

Let $\Psi : (0,1) \to \mathbb{R}$ be the mapping

$$k \mapsto \Psi(k) = \frac{p}{q}.$$

The function Ψ is injective and the quotient $\Psi(k)$ ranges from 0 to 1 when varying $k \in (0,1)$. This is proven in [Ham88], Lem 10.7, we can visualize its graph in Figure 4. Therefore, for any pair of chosen angles $\alpha_1 < \alpha_2$ there exists $k = \Psi^{-1}\left(\frac{\alpha_1}{\alpha_2}\right)$, such that the equation (16) has two positive solutions y_1, y_2. This yields two values $p = 1 - y_1$, $q = y_2 - 1$, and finally we recover

$$a = \frac{\alpha_1}{\pi p} = \frac{\alpha_2}{\pi q}.$$

This value makes the system (7) and the equation (6) to have the required solutions. $\square$

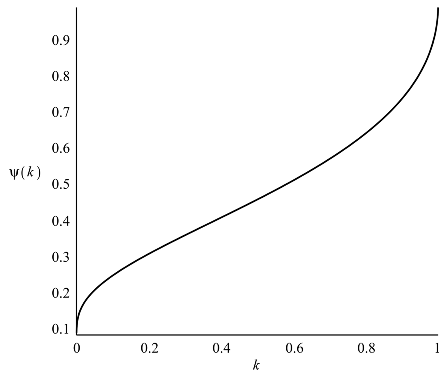

Figure 4. The function $\Psi(k)$.

This completes the classification of the shrinking solitons in Theorem 1.1, Part 2. Pictures of a football soliton and a teardrop soliton are shown in Figures 8 and 9 in Appendix A.

2.4. Expanding solitons. We end our classification with the expanding solitons ($\varepsilon = 1$). The equation (6) and the system (7) are in this case

$$h'' - ahh' + \frac{h}{2} = 0 \tag{17}$$

and

$$\begin{cases} h' = u \\ u' = \left(au + \frac{1}{2}\right)h. \end{cases} \tag{18}$$

The phase portrait of (18) is shown in Figure 5. We can rescale the system (18) with the change

$$\begin{cases} v = ah \\ w = au \end{cases}$$

so that on this coordinates the system becomes

$$\begin{cases} v' = w \\ w' = \left(w + \frac{1}{2}\right)v \end{cases} \tag{19}$$

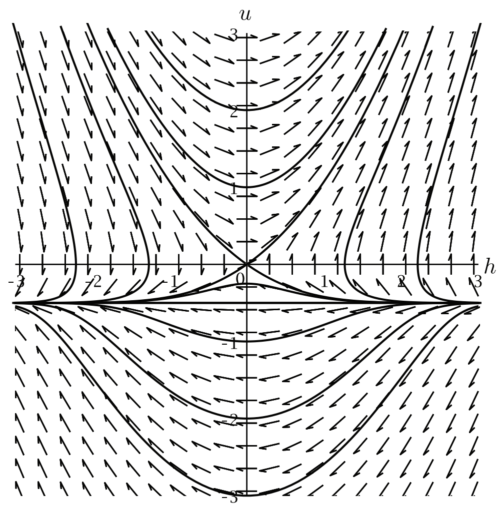

Figure 5. Phase portrait of the system (18) with $a = 1$.

which is exactly the system (18) with $a = 1$, shown in Figure 5. We will study the trajectories of the normalized system and next we will discuss the geometrical interpretation of each trajectory.

The system (19) has a critical point ($v' = w' = 0$) at $(0, 0)$. It has an horizontal isocline ($w' = 0$) at the line $L = \{w = -\frac{1}{2}\}$ (in the unnormalized system it is at $L = \{u = -\frac{1}{2a}\}$), that is also an orbit solution, and hence no other trajectory can cross it. The vertical axis $\{v = 0\}$ is also an horizontal isocline. The horizontal axis $\{w = 0\}$ is, on the other hand, a vertical isocline ($v' = 0$).

The linearization of the system (19) at the critical point $(0, 0)$ is

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ w + \frac{1}{2} & v \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

The matrix of the linearized system at the critical point is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$, that has determinant $-\frac{1}{2} < 0$ and hence the critical point is a saddle point. The eigenvalues of this matrix are $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$, with eigenvectors respectively $(\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$. These eigenvectors determine the two principal directions of the saddle point, from which four separatrix curves are emanating.

The system (19) has the following first integral,

$$H(v, w) = v^2 - 2w + \ln|2w + 1|$$

that is, the solution curves of the system are the level sets of H. Indeed, derivating $H\big(v(r), w(r)\big)$ with respect to r,

$$\begin{aligned} \frac{\partial}{\partial r} H(v, w) &= 2vv' - 2w' + \frac{2w'}{2w + 1} \\ &= 2vw - 2\left(w + \frac{1}{2}\right) v \left(1 - \frac{1}{2w + 1}\right) = 0. \end{aligned}$$

From the system, and more apparently from the first integral, it is clear that the phase portrait is symmetric with respect to the axis $\{v = 0\}$. We will only study then the trajectories on the right-hand half-plane $\{v > 0\}$. Actually this restriction agrees with the geometric assumption of $h > 0$.

We first inspect the separatrix S emanating (actually sinking) from the critical point at the direction $(\sqrt{2}, -1)$. The associated eigenvalue is $-1/\sqrt{2}$ and hence the trajectory is approaching the saddle point (hence the sinking). The curve S lies in the $w' > 0$ region, and cannot cross the horizontal isocline L. Therefore, the separatrix when seen backwards in r must be decreasing and bounded, and hence must approach a horizontal asymptote. This asymptote must be L, since if the trajectory were lying in the region $w' > \delta > 0$ for infinite time, it would come from $w = -\infty$, which is absurd since it cannot cross the isocline L. Therefore, over this separatrix S, $v \to +\infty$ and $w \to -\frac{1}{2}$ as $r \to -\infty$.

We discuss three cases of trajectories: those above S, below S, and the trajectory S itself.

Trajectories above S. Any such trajectory eventually enters the upper right quadrant, $\{v > 0, w > 0\}$. Then $v' > 0$ and $w' > 0$ and hence the curve moves upwards and rightwards. More carefully, it is easy to see that $\lim_{r \to +\infty} \frac{v(r)^2}{2w(r)} = 1$, so the orbit approaches a parabola (as in the steady and shrinking cases). All these solutions have unbounded positive w. This means that the Gaussian curvature of the associated metric $K = -\left(au + \frac{1}{2}\right) = -\left(w + \frac{1}{2}\right)$ is not bounded below, and we will discard them.

Trajectories below S. All these curves intersect the axis $\{v = 0\}$, and we can consider the origin of the r coordinate as such that the intersection point with the axis occurs at $r = 0$. Then, the region of the curves parameterized by $r < 0$ lies in the $v > 0$ half-plane. Since the curves are below S, they lie in the lower right quadrant and hence $v' < 0$ and $v \to +\infty$ as $r \to -\infty$. If the curve lies over L, then $w' > 0$, and if it lies below L, then $w' < 0$. This means that the isocline is repulsive forward in r and attractive backwards in r. Therefore any curve lying below S will have an asymptote as $r \to -\infty$ and as before this must be the isocline L, that is, $v \to +\infty$ and $w \to -\frac{1}{2}$ as $r \to -\infty$.

Let us remark that the trajectories below S are parameterized for $r \in (-\infty, 0]$, although a priori it could be $r \in (-M, 0]$ for some maximal M (and hence $v \to +\infty$ as $r \to -M$, and these would represent noncomplete metrics). This is not the case, since the trajectories are approaching $v' = -\frac{1}{2}$, and hence v' is bounded ($|v'| < 1$ for r less than some $r_0 < 0$), so v cannot grow to $+\infty$ for finite r-time.

These trajectories have bounded w and therefore bounded curvature on the associated metric. More specifically, the curves above the isocline L will give metrics with negative curvature, and curves below L will give metrics with positive curvature. These curves will intersect the $\{h = 0\}$ axis at $b < 0$, and the associated metric will have a cone point of angle

$$\beta = -2\pi b$$

at the point of coordinate $r = 0$. On the other end, the function $h(r)$ is asymptotic to $-\frac{1}{2a} r$ (recall that the parameter is $r \in (-\infty, 0]$) and the metric will be asymptotic to the wide part of a flat cone of angle

$$\alpha = \frac{\pi}{a}.$$

We call these solitons the *$\alpha\beta$-cone solitons*. These solitons have positive curvature if $\alpha < \beta$ $\left(b < \frac{-1}{2a}\right)$ and negative if $\alpha > \beta$ $\left(b > \frac{-1}{2a}\right)$. In the case $\alpha = \beta$ $\left(b = \frac{-1}{2a}\right)$ we are in the case of the isocline L. This line $h' = u = -\frac{1}{2a}$, has as solution the parameterization

$$h(r) = -\frac{1}{2a}r + C$$

which represents a *flat expanding Gaussian cone soliton*, with cone angle $\frac{\pi}{a}$. The special case $a = \frac{1}{2}$ yields a smooth metric at $r = 0$, thus we have a flat metric on the plane known as the *flat expanding Gaussian soliton*.

Other remarkable cases are those with $\beta = 2\pi$ $(b = -1)$, because the cone point at the apex is now blunted and the surface is smooth (we can check from equation (17) that all even-order derivatives vanish at $r = 0$), we call them the *blunt α-cone solitons*. The angle α may be less or greater than 2π and the curvature is positive or negative respectively. However, only the first case can be embedded symmetrically in $\mathbb{R}^3$. The existence of this family was described by a different method by H.-D. Cao in [Cao97] in the context of Kähler–Ricci solitons.

Trajectory of the separatrix S. This curve is parameterized by $r \in \mathbb{R}$, and $(v, w) \to (0, 0)$ as $r \to +\infty$ and $(v, w) \to \left(+\infty, -\frac{1}{2}\right)$ as $r \to -\infty$ (this follows from the Grobman–Hartman theorem in the end near the saddle point, and from the asymptotic L on the other end). We can give a more detailed description of the asymptotics. As $r \to -\infty$, we know that $w \to -\frac{1}{2}$, this is $\lim_{r\to-\infty} \frac{v'}{-\frac{1}{2}} = 1$. Then, applying the l'Hôpital rule, $\lim_{r\to-\infty} \frac{v}{-\frac{1}{2}r} = 1$, or $v(r) \sim -\frac{1}{2}r$ as $r \to -\infty$. This is valid for all the trajectories asymptotic to the horizontal isocline. Similarly, as $r \to +\infty$, we know that $v, w \to 0$, but furthermore we know that their quotient tends to the slope of the eigenvector determining the separatrix, i.e. $\lim_{r\to+\infty} \frac{v}{w} = \lim_{r\to+\infty} \frac{v}{v'} = \frac{-1}{\sqrt{2}}$, which is to say $\lim_{r\to+\infty} (\ln v)' = \lim_{r\to+\infty} \frac{v'}{v} = -\sqrt{2}$. Then, by l'Hôpital rule, $\lim_{r\to+\infty} \frac{(\ln v)'}{-\sqrt{2}} = \lim_{r\to+\infty} \frac{\ln v}{-\sqrt{2}r} = 1$. This is, $v(r) \sim e^{-\sqrt{2}r}$ as $r \to +\infty$.

Geometrically, the separatrix corresponds to the limiting case of $\alpha\beta$-cone solitons when the angle β tends to zero. In this case the parameter r is not on $(-\infty, 0]$ but on the whole $\mathbb{R}$ and thus $h(r)$ defines a smooth complete metric on the cylinder. As $r \to +\infty$, the function $h(r)$ is asymptotic to $\frac{1}{a}e^{-\sqrt{2}r}$, that defines a hyperbolic metric of constant curvature -2. This hyperbolic metric on a cylinder is called a *hyperbolic cusp*. The separatrix S represents a soliton metric that approaches the thin part of a hyperbolic cusp in one end, and the wide part of a flat cone on the other. There is still freedom to set the angle α, and we call these the *cusped α-cone solitons*.

Finally, there is still one more family of two-dimensional gradient solitons, namely the universal cover of the cusped α-cones. These solitons are metrics on

$\mathbb{R}^2$ locally isometric to the cusped cones. These solitons are not rotationally symmetric, but translationally symmetric, i.e. there is not a $\mathbb{S}^1$ group but a $\mathbb{R}$ group acting by isometries. The plane $\mathbb{R}^2$ with any of this metrics has a fixed direction (given by $\operatorname{grad} f$) such that a straight line following this direction (that is also a geodesic of the soliton metric) transits gradually from a region of hyperbolic curvature on one end to a region of flat curvature on the other. Any translation on the direction perpendicular to $\operatorname{grad} f$ (this is, in the direction of $J(\operatorname{grad}(f))$) is an isometry on these metrics. We call these *flat-hyperbolic soliton planes*.

This completes the classification of the expanding solitons in Theorem 1.1, Part 3; which finishes the proof or the whole Theorem. Pictures of some expanding solitons are shown in Figures 10 to 13 in Appendix A.

2.5. Solitons embedded into $\mathbb{R}^3$. We end the section with a visualization remark. Some of the solitons we described above can be embedded into $\mathbb{R}^3$ and then visualized numerically as surfaces with the inherited metric from the ambient Euclidean space. If we want to keep the rotational symmetry apparent, however, we cannot embed into $\mathbb{R}^3$ a cone point of angle greater than 2π, and we can't embed a rotational surface whose parallels have length $L = 2\pi h(R)$ if $R < \frac{L}{2\pi}$.

In order to do this, we use the metric in polar coordinates $(r, \theta) \in [0, A] \times [0, 2\pi]$,

$$dr^2 + h(r)^2\, d\theta^2.$$

We recall that r is the arc parameter of the $\{\theta = cst\}$ curves (meridians), and that the $\{r = cst\}$ curves (parallels) are circles of radius $h(r)$ parameterized by $\theta \in [0, 2\pi]$. Therefore, we can use the h, θ as polar coordinates on the plane, and find an appropriate third coordinate z (height). When we put the stacked parallels of radius $h(r)$ at height $z(r)$, we obtain a rotational surface whose meridians have length parameter r. Thus,

$$dr^2 = dh^2 + dz^2$$

or equivalently

$$\frac{dz^2}{dr^2} = 1 - \frac{dh^2}{dr^2}$$

which defines $z = z(r)$ as satisfying

$$(z')^2 = 1 - u^2$$

with the convention that $h' = u$. Hence, to obtain an embedded surface satisfying the soliton system (7) it is sufficient to integrate the first order vector ODE

$$\begin{cases} h' = u \\ u' = \left(au + \frac{\varepsilon}{2}\right)h \\ z' = \sqrt{1-u^2} \end{cases}$$

with initial conditions $h(0) = 0$, $u(0) = b$, $z(0) = 0$. Once obtained a numerical solution for $h(r)$, $u(r)$ and $z(r)$, we can fix a value $A > 0$ and then plot the set of points

$$\left\{ \left(h(r)\cos\theta, h(r)\sin\theta, z(r)\right) \in \mathbb{R}^3 \mid r \in [0, A],\ \theta \in [0, 2\pi] \right\}.$$

In Appendix A we display some embedded solitons.

3. Cone surfaces evolving along Ricci flow

3.1. Uniformization of smooth surfaces. In this section we present a proof of the uniformization theorem using Ricci flow, by using the techniques of Hamilton and Perelman. On this proof, the surface is endowed with an arbitrary metric, and we let it evolve according to Ricci flow (for the question of existence of the flow, see [Ham88]). We analyse the behavior of this evolving metric to conclude that it converges (maybe up to suitable rescalings) to a constant curvature metric. This argumental line will be adapted to the framework of cone surfaces in Section 3.4.

3.1.1. Surfaces with $\chi(\mathscr{M}) \leq 0$. In the case $\chi \leq 0$ we use the argument in [Ham88]. We use the normalized version of the Ricci flow,

$$\frac{\partial}{\partial t} g = (r - R)g$$

with $g(0) = g_0$, where r is the average scalar curvature, a constant defined by $r = \frac{\int R\, d\mu}{\int d\mu} = \frac{4\pi\chi(\mathscr{M})}{\mathrm{Area}(\mathscr{M})}$, that depends on the topology by Gauss–Bonnet theorem. This normalization is a time-dependent rescaling so that the area of the compact surface is kept invariant.

The evolution of the scalar curvature for the normalized Ricci flow is

$$\frac{\partial}{\partial t} R = \Delta R + R^2 - rR$$

Therefore, by the maximum principle, if $-C < R < -\varepsilon < 0$ at $t = 0$, then

$$re^{-\varepsilon t} \leq r - R \leq Ce^{rt}.$$

Thus, if $R < 0$, then the metric is defined for all time and converges exponentially fast to a metric of constant negative curvature.

Further, we introduce a measure of how much a flow differs from a soliton. The soliton equation for the normalized Ricci flow is

$$\operatorname{Hess} f - \frac{1}{2}(R - r)g = 0 \tag{20}$$

for some function f. In normalized Ricci flow there are no distinctions between shrinking, steady or expanding solitons, since the area is fixed and therefore no homothetic factor applies.

Although that equation has no solution when the Ricci flow is not a soliton, it is always possible to solve the traced equation (which is a Poisson PDE). The *potential* f of a Ricci flow is the solution of

$$\Delta f = R - r$$

normalized to have mean value zero, $\int f = 0$. The soliton quantity is

$$M := \operatorname{Hess} f - \frac{1}{2}(R - r)g = \operatorname{Hess} f - \frac{1}{2}\Delta f g$$

and vanishes iff the flow is a soliton. We also define

$$h = \Delta f + |\nabla f|^2$$

that helps controlling $R = h + |\nabla f|^2 + r \leq h + r$.

The evolution of these quantities under the normalized Ricci flow is

$$\frac{\partial}{\partial t} h = \Delta h - 2|M|^2 + rh$$

$$\frac{\partial}{\partial t}|M|^2 = \Delta|M|^2 - 2|\nabla M|^2 - 2R|M|^2$$

and using the maximum principle we deduce that for any initial metric, there is a constant C such that

$$-C \leq R \leq Ce^{rt} + r.$$

In particular, for any initial metric the normalized Ricci flow has solution for all time $t \in [0, +\infty)$ (since the curvature remains bounded).

This implies that if $r < 0$, then the metric converges exponentially fast to a metric of constant negative curvature, which proves the uniformization for $\chi < 0$.

If $R \geq c > 0$, then $|M|^2 \leq Ce^{-ct}$ for all time. This implies that the metric converges exponentially fast to a soliton metric. Together with the classification of solitons, this proves the uniformization for surfaces of positive curvature. However, proving that a surface with $\chi > 0$ (the sphere) eventually develops positive curvature is much harder and required Hamilton and Chow to develop several Harnack inequalities and monotone entropies. We will take a different path on the case $\chi > 0$ using Perelman's techniques.

The case $r = \chi = 0$ can be solved by similar techniques applying the maximum principle to the evolution of several other quantities, such as $|\nabla f|^2$ and ∇R (this argument is in the style of Bernstein–Bando–Shi estimates, and is different from Hamilton's original, see for instance [CK04], Sec 5.6).

3.1.2. Surfaces with $\chi(\mathscr{M}) > 0$. From now on, we use the unnormalized Ricci flow and assume that $\chi(M) > 0$. On this situation, the flow develops a singularity on finite time. For, the evolution of the area of the surface is

$$\frac{d}{dt}\operatorname{Area}(\mathscr{M}) = \int_{\mathscr{M}} \frac{d}{dt}\,d\mu = \int_{\mathscr{M}} -R\,d\mu = -4\pi\chi(\mathscr{M})$$

and since $\chi(\mathscr{M}) > 0$, the area or $\mathscr{M}$ is a decreasing linear function of t and it collapses to zero in finite time.

Singularity formation is a phenomenon that may occur in the long time behaviour of the n-dimensional Ricci flow. By [Ham95b], Thm. 8.1 (see also [Top06], Sec 5.3), if $g(t)$ is a Ricci flow defined on a maximal time interval $[0, T)$ and $T < \infty$, then $\sup_{\mathscr{M}}|\mathrm{Rm}|(\cdot, t) \to \infty$ as $t \to T$. That means that the only obstruction to continue the evolution of the Ricci flow further in time is the appearance of points with infinite curvature in finite time.

To study the singularities of the flow, and the phenomena of exploding curvature or vanishing volume, the technique is to use parabolic rescalings. A *parabolic rescaling* is a transformation of the evolving metric into the form

$$g(t) \mapsto \lambda^2 g\left(\frac{t}{\lambda^2}\right) = \tilde{g}(t).$$

It is useful because if $g(t)$ is a Ricci flow, then so is $\tilde{g}(t)$. Besides, distances get multiplied by λ, time gets multiplied by λ^2 and scalar curvature gets divided by λ^2. The idea is to pick a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$, and construct a sequence of rescaled pointed Ricci flows keeping the curvature bounded. The hope is to find a convergent subsequence of these pointed flows, using techniques of classes (spaces) of manifolds, and compactness theorems for classes of manifolds. If this process of iterated rescalings is successful and we obtain a limit, we call this a *blow-up* of the singularity.

The compactness theorems for classes of manifolds usually need as a hypothesis a control of the injectivity radius of the manifold. This allows one to consider the sequence of manifolds as a sequence of metrics on a fixed chart, with its coordinate functions defined over a fixed open subset of $\mathbb{R}^n$. However, controling the injectivity radius is difficult, because topology and curvature affect it. Fortunately, under bounded curvature, controlling volume is equivalent to controlling the injectivity radius. More precisely ([BBI01], Thm 10.6.8, [CGT82], Thm 4.3), if $(\mathscr{M}^n, g)$ is a smooth Riemannian n-manifold, $p \in \mathscr{M}$, and $|\mathrm{Rm}| \leq r^{-2}$ in $B(p,r)$, then $\frac{\mathrm{inj}(p)}{r}$ is bounded below if and only if $\frac{\mathrm{Vol}(B(p,r))}{r^n}$ is bounded below (note that all quantities involved, $\frac{\mathrm{Rm}}{r^{-2}}$, $\frac{\mathrm{inj}}{r}$, $\frac{\mathrm{Vol}}{r^n}$, are scale-invariant). Thus, one can focus on controlling the volume, instead of the more elusive injectivity radius.

This fact motivated Perelman to define the notion of κ-noncollapsed manifolds (and flows). A Riemannian manifold $(\mathscr{M}^n, g)$ is *κ-noncollapsed at scale ρ* in $p \in \mathscr{M}$ if $\forall r < \rho$ it is satisfied that

$$|\mathrm{Rm}|(x) \leq \frac{1}{r^2} \ \forall x \in B(p,r) \quad \Rightarrow \quad \frac{\mathrm{Vol}\big(B(p,r)\big)}{r^n} \geq \kappa.$$

This means that every ball with radius $r < \rho$ and bounded curvature has a volume of at least κr^n. Observe that for any given smooth compact manifold, one can find κ, ρ small enough such that the manifold is κ-noncollapsed at scale ρ. The breaktrhough of Perelman is that κ and ρ do not degrade under the Ricci flow, so κ-noncollapse is perserved under the flow ([Per02], see also [KL08], Thm 26.2, [CZ06], Thm 3.4.2).

Proposition 3.1 (Noncollapsing theorem (Perelman)). *Given numbers $n \in \mathbb{N}$, $T < \infty$, $\rho, K, c > 0$, there exists $\kappa > 0$ such that the following holds: Let $\big(M^n, g(t)\big)$ be a Ricci flow defined on $[0, T)$ such that*

- *$|Rm|$ is bounded on every compact subinterval $[0, T'] \subset [0, T)$.*
- *$\big(M, g(0)\big)$ is complete with $|Rm| < K$ and $\mathrm{inj}\big(M, g(0)\big) \geq c > 0$.*

Then the Ricci flow is κ-noncollapsed at scale ρ (that is, every time t slice of the flow is a κ-noncollapsed manifold at scale ρ, with uniform κ). Furthermore, κ is (non-strictly) decreasing in T, while all other constants fixed.

Perelman developed several proofs of the noncollapsing theorem. One of them, the "comparison geometry approach" based in the so-called *$\mathscr{L}$-geodesics* uses entirely integral quantities, which are not affected by perturbations in sets of measure zero, such as isolated cone points.

The κ-noncollapsing property allows one to find limits of sequences of rescaled Ricci flows. Since we use sequences of dilations around points of high positive curvature, the limit flow will have positive curvature, and since we are dilating the time before the singular moment, the limit flow will be ancient. Perelman coined the word κ-solution for those Ricci flows that enjoy these good properties. A *κ-solution* is a Ricci flow ancient $\big(t \in (-\infty, T]\big)$, nonflat $(Rm \neq 0)$, with curvature operator Rm positive definite and bounded in each time-slice $(|Rm| < C)$, and κ-noncollapsed at all scales.

The technique of parabolic rescalings, together with the noncollapsing theorem, allows to find a model for the singular times of the flow.

Proposition 3.2. *Let $\big(\mathscr{M}^2, g(t)\big)$ be a Ricci flow on a surface, defined on $[0, T)$, which becomes singular at time T. Let $\kappa, \rho > 0$ and assume $\big(\mathscr{M}, g(0)\big)$ is κ-noncollapsed at scale ρ. There is a sequence of times $t_i \to T$ such that, if $Q_i = \max R(\cdot, t_i)$ and p_i is the point that achieves the maximum of R at time t_i, then the sequence of pointed Ricci flows $\big(\mathscr{M}, g_i(t), p_i\big)$ with*

$$g_i(t) = Q_i g\left(\frac{t}{Q_i} + t_i\right)$$

has a subsequence that converges to a κ-solution.

This theorem in dimension three is the so-called Canonical Neighbourhood theorem of Perelman, [Per02], Thm 12.1, and it requires a subsequent classification of κ-solutions to understand the local model of the singularities of the flow. In dimension two, the picture is much simpler, since as we see below, all κ-solutions turn out to be solitons.

The enhanced properties of κ-solutions allow an in-depth analysis that can't be done for a general Ricci flow. An important feature is that one can find a soliton "buried" inside every κ-solution (Perelman [Per02], see also [KL08] Prop 39.1). More specifically, the limit backwards in time to $t = -\infty$ is, after rescaling, a gradient shrinking soliton, called the *asymptotic soliton*.

The proof of the asymptotic soliton theorem uses the same $\mathscr{L}$-geodesics theory as in the noncollapsing theory, and additionally it uses a maximum principle, or more accurately, a Harnack inequality from [Ham93]. Harnack inequalities are elaborated uses of the maximum principle where one computes not only the evolution of a function f in terms of its second derivatives (Δf), but the also the evolution of the convexity $(Q = \Delta f)$ in terms of fourth order derivatives (ΔQ) . See [Ram14], Ch. 3.2–3.5 for a quick survey of these results.

To prove that all κ-solutions are solitons, one can use in the smooth case [Per02], Cor. 11.3 (clarified by [Ye04]), but we propose the following proof:

Lemma 3.1. *Let $(\mathscr{M}^2, g(t))$ be a κ-solution of the Ricci flow on a smooth surface. Then it is a soliton.*

Proof. Given a κ-solution we construct its asymptotic soliton by taking a sequence of times $\bar{\tau}_k \to +\infty$; then picking appropriate $q_k \in \mathscr{M}$; and then constructing the sequence of rescalings

$$g_k(t) = \frac{1}{\bar{\tau}_k} g(\bar{\tau}_k t)$$

that subconverges to a shrinking soliton.

From the classification of the solitons on surfaces on Section 2, the only smooth, complete, nonflat, gradient shrinking soliton on a surface is the shrinking round sphere, which is compact. Therefore, the limit of rescalings is not only locally diffeopmorphic to the original surface, but also globally diffeomorphic and hence $\mathscr{M}$ is compact.

The (dynamic) soliton equation for unnormalized Ricci flow is

$$\mathrm{Ric} + \mathrm{Hess}\, f + \frac{1}{2t} g = 0.$$

Although not always exists a function f solving that, one can solve the traced equation

$$\Delta f = -\left(R + \frac{1}{t}\right)$$

to obtain a potential function f for the flow. This is a Poisson equation ($\Delta f = h$) over a compact manifold, which by operator theory has a unique solution f with mean value zero, $\int_{\mathscr{M}} f\, d\mu = 0$, if and only if $\int_{\mathscr{M}} h\, d\mu = 0$. In our case it is satisfied since

$$\int_{\mathscr{M}} R + \frac{1}{t}\, d\mu = \int_{\mathscr{M}} R\, d\mu + \frac{1}{t} \operatorname{Area} \mathscr{M} = 4\pi\chi(\mathscr{M}) + \frac{1}{t} \operatorname{Area} \mathscr{M} = 0.$$

We define then the soliton quantity

$$M = \mathrm{Hess}\, f + \frac{1}{2}\left(R + \frac{1}{t}\right) g$$

which vanishes iff the Ricci flow is a gradient shrinking soliton. We would like to consider the evolution of $|M|^2$, but this is not a scale invariant quantity. Instead, we use the quantity $t^2|M|^2$ and obtain

$$\frac{\partial}{\partial t} t^2|M|^2 = \Delta(t^2|M|^2) - 2t^2|\nabla M|^2 \leq \Delta(t^2|M|^2),$$

and by the maximum principle, $\max_{\mathscr{M}} t^2|M|^2$ is decreasing on t.

On the sequence of rescalings, by monotonicity, taking $t_k = -\bar{\tau}_k$, $\forall x_0 \in \mathscr{M}$ and $\forall t_0 \in (-\infty, T)$,

$$t_0^2|M|^2_{g(t_0)}(x_0, t_0) \leq \max_{\mathscr{M}} t_k^2|M|^2_{g(t_k)}(\cdot, t_k) \qquad \forall t_k < t_0.$$

By the rescaling invariance,

$$t_k^2|M|^2_{g(t_k)}(\cdot, t_k) = |M|^2_{g_k(-1)}(\cdot, -1).$$

Using that the limit of rescalings is a soliton, for all $\varepsilon > 0$ there exists $k > 0$ big enough (t_k negative big enough) such that $|M|^2_{g_k(-1)}(\cdot, -1) < \varepsilon$. Putting all together, for all (x_0, t_0) and for all $\varepsilon > 0$ we conclude that $|M|^2_{g(t_0)}(x_0, t_0) < \varepsilon$, so $M \equiv 0$, which proves that the κ-solution is actually a soliton. $\square$

Thus, the surface evolving under Ricci flow converges, up to rescaling, to a round sphere. This finishes the proof of the uniformization theorem for smooth surfaces with $\chi(\mathscr{M}) > 0$.

3.2. Existence of Ricci flow on cone surfaces. A cone surface $\big(\mathscr{M}, (p_1, \dots, p_n), g\big)$ is a topological surface $\mathscr{M}$ and marked points $p_1, \dots, p_n \in \mathscr{M}$ equipped with a smooth Riemannian metric g on $\mathscr{M} \backslash \{p_1, \dots, p_n\}$, such that every point p_i admits a local chart where the metric takes some model form.

The marked points are part of the boundary of the domain where we try to run a PDE, and hence, some boundary conditions apply. We have some freedom to choose these boundary conditions, but they carry implications for the flow. For instance, in [Ram15] we construct a Ricci flow that instantaneously removes cone points; or in [GT11] it is constructed an instantaneously complete cusp-generating flow, from a surface with some points removed. Here we look for an angle-preserving flow that allows the uniformization of cone surfaces, that is, the model structure of the metric around a marked point is invariant under the flow.

The metric model we used in Section 2, call it *polar geodesic coordinates*, takes a disc with coordinates (ρ, θ) where the metric is written

$$g = d\rho^2 + h^2\, d\theta^2$$

with $h = h(\rho, \theta)$ a smooth function $h : D \to \mathbb{R}$, satisfying

$$h(0) = 0, \qquad \frac{\partial h}{\partial \rho}(0) = \frac{\alpha_i}{2\pi}, \qquad \frac{\partial^{2k} h}{\partial \rho^{2k}}(0) = 0 \tag{21}$$

for some $\alpha_i \in (0, 2\pi]$ (the cone angles). Here, $\rho > 0$ is the arclength parameter measuring the distance to the singular point, and $\theta \in [0, 2\pi]/\sim$ is proportional to the angle.

A second metric model, call it *conformal to a flat cone*, considers the metric of a standard Euclidean cone with angle α $g_0 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2\, d\theta^2$ on a disc (r, θ), and writes a general cone metric as a conformal deformation of the Euclidean one,

$$g = e^{2\phi(r,\theta)}\left(dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2\, d\theta^2\right)$$

for a function $\phi(r, \theta)$ with certain regularity. The Ricci flow equation on surfaces is $\frac{\partial}{\partial t} g(t) = -Rg$. If $g(t) = e^{2\phi(t,x)} g_0$, then the flow becomes

$$\frac{\partial}{\partial t}\phi(t) = e^{-2\phi}\Delta_{g_0}\phi + R_{g_0}. \tag{22}$$

We appeal to the following existence result by Mazzeo, Rubinstein and Sesum:

Proposition 3.3 ([MRS15], Prop 3.12). *Let $\left(\mathscr{M}, (p_1, \dots, p_n), g_0\right)$ be a cone surface. Let $\bar{g}_0$ be a background reference metric such that on a neighbourhood of a cone point p_i is flat and takes the conic form $\bar{g}_0 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2\, d\theta^2$. Let $g_0 = e^{2\phi_0}\bar{g}_0$ be a cone metric, where $\phi_0 \in \mathscr{D}_b^{k,\delta}(\tilde{\mathscr{M}})$, and let $g(t) = e^{2\phi(t)} g_0$. Then, there is a unique solution $\phi \in \mathscr{D}^{k+\delta,(k+\delta)/2}([0, T] \times \tilde{\mathscr{M}})$ to (22) with $\phi|_{t=0} = 0$, provided T is sufficiently small.*

This theorem uses the b-calculus introduced by R. B. Melrose [Mel93], that gives an explicit description of the asymptotic behaviour of the functions near the cone point. The idea is to "desingularize" the point by a blow-up, substituting the cone point by the S^1 boundary $\{r = 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ one obtains a manifold with boundary $\tilde{\mathscr{M}}$. In polar coordinates, one changes the space of derivative operators to $\mathscr{V}_b = \left\{r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$ instead of the usual $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. This is equivalent to restrict to vector fields on $\tilde{\mathscr{M}}$ tangent to the boundary. Then it is constructed $\mathscr{C}_b^{k,\delta}(\tilde{\mathscr{M}})$ as the space of functions with k derivatives (taken in $\mathscr{V}_b$), and after taking all the derivatives, the result is on a Hölder space $\mathscr{C}^{0,\delta}(\tilde{\mathscr{M}})$. Similarly, the space $\mathscr{C}_b^{k+\delta,(k+\delta)/2}([0, T] \times \tilde{\mathscr{M}})$ is the space of functions of space and time, with i space-derivatives (taken in $\mathscr{V}_b$), j time-derivatives (in the usual sense), $i + 2j \le k$, and after taking all the derivatives, the result is on a Hölder space $\mathscr{C}^{\delta,\delta/2}$. Finally, the Hölder-Friedrichs domain for these functions is

$$\mathscr{D}_b^{k+\delta,(k+\delta)/2}([0,T]\times\tilde{\mathscr{M}}) = \{u \in C_b^{k+\delta,(k+\delta)/2} \mid \Delta_g u \in C_b^{k+\delta,(k+\delta)/2}\}.$$

Lemma 3.2. *A metric in polar geodesic coordinates with $h \in \mathscr{C}^\infty$ can be written as conformal to a flat cone with $\phi \in \mathscr{D}_b^{k,\delta}$ for all $k > 0$. Conversely, a metric conformal to a flat cone with $\phi \in \mathscr{D}_b^{k',\delta}$ can be written as polar geodesic coordinates for h satisfying* (21) *up to certain k.*

Proof. We express a cone metric in the two coordinate charts: conformal coordinates with respect to a cone, and polar geodesic coordinates.

$$g = e^{2u}\left(dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2\, d\theta^2\right) = d\rho^2 + h^2\, d\xi^2.$$

For simplicity, we will assume that the functions u, h are radial, i.e. $u = u(r)$, $h = h(\rho)$. The general case $u = u(r,\theta)$, $h = h(\rho,\xi)$ follows the same structure, only involving more terms on $\frac{\partial u}{\partial \theta}$ and $\frac{\partial h}{\partial \xi}$.

The Gaussian curvature can be expressed as

$$K = -\Delta_g = -e^{-2u}\Delta u = -\frac{1}{h}\frac{\partial^2 h}{\partial \rho^2}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. The change of coordinates can be achieved by the transformation

$$d\rho = e^u\, dr \qquad \xi = \theta \qquad h = \left(\frac{\alpha}{2\pi}\right) r e^u$$

Assume we have a metric in polar geodesic coordinates, given by $h(\rho) \in \mathscr{C}^\infty\big([0,A)\big)$ and satisfying (21). We need to check that

$$u(r) = \ln h - \ln r - \ln\left(\frac{\alpha}{2\pi}\right)$$

and $\Delta_g u$ belong to $\mathscr{C}_b^k$ for any k, i.e. when applying k times the operator $r\frac{\partial}{\partial r}$ the result is in $\mathscr{C}^0$. Indeed,

$$r\frac{\partial}{\partial r}u = r\frac{\partial}{\partial r}\left(\ln h - \ln r - \ln\frac{\alpha}{2\pi}\right) = r\frac{1}{h}\frac{\partial h}{\partial \rho}\frac{\partial \rho}{\partial r} - 1 = re^u\frac{1}{h}\frac{\partial h}{\partial \rho} - 1 = \frac{2\pi}{\alpha}\frac{\partial h}{\partial \rho} - 1$$

which is $\mathscr{C}^0$ and tends to 0 as $r \to 0$. Since the derivative operator is

$$r\frac{\partial}{\partial r} = r\frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = re^u\frac{\partial}{\partial \rho} = \left(\frac{2\pi}{\alpha}\right)h\frac{\partial}{\partial \rho},$$

when applied further, results on functions with the same regularity as h and the derivatives of h. For the Laplacian,

$$r\frac{\partial}{\partial r}\Delta_g u = -r\frac{\partial}{\partial r}K = \frac{2\pi}{\alpha}h\frac{\partial}{\partial \rho}\left(\frac{1}{h}\frac{\partial^2 h}{\partial \rho^2}\right) = \frac{2\pi}{\alpha}\left(\frac{\partial^3 h}{\partial \rho^3} + K\frac{\partial h}{\partial \rho}\right).$$

Hence, all further derivations respect to $h\frac{\partial}{\partial \rho}$ will yield terms in K and derivatives of h. Since $h = \frac{\alpha}{2\pi}\rho + O(\rho^3)$, the curvature has a limit as $\rho \to 0$,

$$\lim_{\rho \to 0} K = \lim_{\rho \to 0}\frac{-h_{\rho\rho}}{h} = \lim_{\rho \to 0}\frac{O(\rho)}{\frac{\alpha}{2\pi}\rho + O(\rho^3)} = C,$$

and hence K is a $\mathscr{C}^0$ function. Therefore, $\Delta_g u$ is in $\mathscr{C}_b^k$. □

3.3. Barrier maximum principles for cone manifolds. We now develop some maximum principles for cone manifolds. First we develop a generic maximum principle for functions on cone surfaces. Next, we show an ad hoc maximum principle for the Harnack inequality that proves the asymptotic soliton theorem.

One standard formulation of the maximum principle states that the maximum of a function $u = u(x,t)$ evolving according to

$$\frac{\partial u}{\partial t} \le \Delta u \qquad t \in [0, T)$$

occurs at $t = 0$ or at the boundary of the domain. Heuristically, an interior maximum would have gradient zero and Hessian negative defined. So it would have negative Laplacian and the function on that point would be decreasing when fixed on that point, giving greater values backwards in time. Therefore the space maximum cannot increase in time, and hence the maximum is at $t = 0$, or at the boundary.

This argument fails if u is not at least $\mathscr{C}^2$ on the interior of the domain, in particular if the domain itself contains a cone point, because a nonsmooth maximum point no longer needs to have zero gradient or negative Laplacian. We can workaround this problem if we are able to find a way to guarantee that the maximum cannot occur on the cone point. The way for achieving this is constructing a new function $\bar{u}_\varepsilon$ depending on a parameter $\varepsilon > 0$ such that $\bar{u}_\varepsilon$ tends uniformly to u when $\varepsilon \to 0$; and on a small neighbourhood, $\bar{u}_\varepsilon$ is strictly increasing over radial lines leaving the cone point (thus not having a maximum at the cone point). We call this new $\bar{u}_\varepsilon$ a *barrier function* for u.

Through all this section, we will assume that $(\mathscr{M}, \{p_1, \dots, p_n\}, g)$ is a cone surface such that the metric on a neighbourhood of a cone point is written as $g = dr^2 + h(r,\theta)^2\, d\theta^2$ for some analytic $h : [0, A) \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^+$, such that

- $h(0,\theta) = 0$, $\forall\theta$ (the neighbourhood is a disc),
- $h_r(0,\theta) = \frac{\alpha}{2\pi}$ (where α is the cone angle), and
- $|h_{rr}| \le Ch$ (bounded curvature).

Recall (cf. with Section 2) that on that metric, the Gaussian curvature and the Hessian and Laplacian of a radial function are given by

$$K = -\frac{h_{rr}}{h}, \qquad \operatorname{Hess} f = f_{rr}\, dr^2 + hh_r f_r\, d\theta^2, \qquad \Delta f = f_{rr} + \frac{h_r}{h} f_r.$$

Further, using the control on h_r and h_{rr} on the Taylor expansion of h, we obtain that for a fixed θ,

$$h(r,\theta) = \frac{\alpha}{2\pi} r + O(r^3) \qquad \text{and} \qquad r\frac{h_r}{h} - 1 = O(r^2).$$

In particular, $\frac{h_r}{h} \sim \frac{1}{r}$ as $r \to 0$.

Now we define a helpful function (cf. [Jef05]) that we will use later to build the barriers.

Lemma 3.3. *Let U be a topological disk, with given polar coordinates $(r,\theta) \in (0, r_0) \times [0, 2\pi)$, a cone angle at the origin, and a smooth Riemannian metric outside the cone point with bounded curvature. Let $0 < \delta < 1$. Then the function given by $(r,\theta) \mapsto r^\delta$ satisfies*

- $\operatorname{grad} r^\delta$ *is pointing away from the cone point, and with norm tending to $+\infty$ as we approach the vertex.*
- $\Delta r^\delta > 0$ *if r small enough.*

Proof. The gradient vector is

$$\operatorname{grad} r^\delta = \delta r^{\delta-1} \frac{\partial}{\partial r}$$

so it is clear that it points away from the origin and its norm tends to ∞ as $r \to 0$. The Laplacian of r^{δ} is

$$\Delta r^{\delta} = \delta(\delta - 1)r^{\delta-2} + \frac{1}{h}\frac{\partial h}{\partial r}\delta r^{\delta-1} = \delta r^{\delta-2}\left(\delta - 1 + \frac{1}{h}\frac{\partial h}{\partial r}r\right).$$

Since $r\frac{1}{h}\frac{\partial h}{\partial r} \to 1$ as $r \to 0$, then $\Delta r^{\delta} > 0$ for r small enough. $\square$

This lemma allows us to construct a barrier function that proves the maximum principle on closed cone surfaces:

Theorem 3.1. *Let $(\mathscr{M}, (p_1, \dots, p_n), g_0)$ be a closed cone surface, and let $u \in C^{2,1}(\mathscr{M} \times (0, T], g_0)$ such that*

$$\frac{\partial u}{\partial t} \le \Delta u$$

Let $(x_0, t_0) \in \mathscr{M} \times [0, T]$ such that realizes the maximum of u over space and time,

$$u(x_0, t_0) = \max_{\mathscr{M}\times[0,T]} u$$

then $t_0 = 0$.

The notation $C^{2,1}(\mathscr{M} \times (0, T], g_0)$ means functions $\mathscr{C}^2$ in space and $\mathscr{C}^1$ in time with bounded $\mathscr{C}^2$-norm, this norm taken with respect to the metric g_0.

Proof. Applying the maximum principle over the open set $\mathscr{M}\backslash\Sigma$, where we denote $\Sigma = \{p_1, \dots, p_n\}$, the maximum of u is achieved on $t = 0$ or, maybe, on $t > 0$ and $p \in \Sigma$. We will rule out the latter case. Assume by contradiction that (p, t_0), $p \in \Sigma$, is the maximum of u over $\mathscr{M} \times [0, T]$.

Let U be a small neighbourhood of p such that we can dispose polar coordinates (r, θ), and $\Delta r^{\delta} > 0$ for some $0 < \delta < 1$, by Lemma 3.3. Let $\varepsilon > 0$, and define over U the function

$$\bar{u} = u + \varepsilon r^{\delta}.$$

It satisfies

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial u}{\partial t} \le \Delta u \le \Delta(u + \varepsilon r^{\delta}) = \Delta\bar{u}.$$

Applying the maximum principle to the open set $U\backslash\{p\}$, $\max_{\bar{U}\times[0,T]} \bar{u}$ lies on $t = 0$ or on $x \in \partial U \cup \{p\}$. We claim that the latter cannot happen. Indeed, $\bar{u}$

cannot have a maximum on $x = p$ (i.e. $r = 0$) because grad $\bar{u}$ is pointing away from p with infinite norm when $r = 0$, and grad $u|$ is bounded, so $\bar{u}$ is strictly increasing on radial directions leaving p. On the other hand, the original u has no maxima on $\partial U \times (0, T]$ because they would be interior points in $\mathscr{M} \backslash \Sigma \times [0, T]$. Since $\bar{u} \to u$ uniformly as $\varepsilon \to 0$, $\bar{u}$ cannot either have maxima on $\partial U \times (0, T]$; specifically, for any $\varepsilon < \varepsilon_0 = \frac{1}{2}(\max_{\bar{U} \times (0,T]} u - \max_{\partial U \times (0,T]} u)$, the function $\bar{u}$ cannot have maxima on ∂U because this value would be at most $\max_{\partial U} u + \varepsilon$ that is less than $u(p, t_0)$.

Therefore, $\max \bar{u}$ is on $t = 0$ and again since $\bar{u} \to u$ uniformly, $\max u$ is on $t = 0$. $\square$

Now we look for a cone version of Harnack inequality for Ricci flows. Recall from Section 3.1 that the only point we need a Harnack inequality is in the proof of the Asymptotic soliton of a κ-solution. This Harnack inequality is given in [Ham93] and states that if $(\mathscr{M}, g(t))$ is a Ricci flow with nonnegative curvature operator, then certain quantity ($Z = M_{ij} W^i W^j + 2P_{kij} U^{ki} W^j + R_{ijkl} U^{ij} U^{kl}$) is nonnegative. This is proven using a maximum (minimum) principle that involves creating a barrier function for the spatial infinity, given by the following lemma.

Proposition 3.4 ([Ham93], Lem 5.2). *For any $C, \eta > 0$ and any compact set K in space-time, we can find functions $\psi = \psi(t)$ and $\varphi = \varphi(x, t)$ such that*

(1) *$\delta \leq \psi \leq \eta$ for some $\delta > 0$, for all t;*

(2) *$\varepsilon \leq \varphi \leq \eta$ on the compact set K for some $\varepsilon > 0$, for all t. Furthermore, $\varphi(x, t) \to \infty$ if $x \to \infty$, i.e. the sets $\{x \,|\, \varphi(x, t) < M\}$ are compact for all t and all M;*

(3) *$\frac{\partial \varphi}{\partial t} > \Delta \varphi + C\varphi$;*

(4) *$\frac{\partial \psi}{\partial t} > C\psi$;*

(5) *$\varphi \geq C\psi$.*

The functions in this smooth case are:

$$\varphi = \varepsilon e^{At} f(x), \qquad \psi = \delta e^{Bt}$$

with ε, δ small and A B sufficiently large. The function $f(x)$ depends only on the position, $f(x) \to +\infty$ as x goes to ∞ (the distance to a fixed basepoint tends to infinity), but the derivatives of f are bounded. Then, φ is a space barrier for the infinity and a time barrier for $t = 0$. The function ψ is only a time barrier.

The only point we need is to change φ to be a barrier also at the cone points.

Lemma 3.4. *Let $(\mathcal{M}, (p_1, \dots, p_n), g)$ be a cone surface. There is some $C > 0$ and a function $\mu = \mu(x)$ satisfying*

(1) $\mu \geq 1/C$

(2) $\mu \to +\infty$ *as x tends to a cone point.*

(3) $\Delta\mu \leq C$

Proof. On a local chart around p_i, we can assume that $g = dr^2 + h^2\,d\theta^2$ for $r \in [0, r_0)$, for some r_0 uniform on the surface. Without loss of generality, we can assume $r_0 = 1$.

It suffices to use a smooth interpolation between $\mu = -\ln r$ for $r < \frac{1}{2}$ and $\mu = \ln 2$ for $r > \frac{1}{2}$ (assume that the interpolation only affects a very small neighbourhood of $r = \frac{1}{2}$. This function μ obviously satisfies (1) and (2). To see (3), we only need to check it for the case $\mu = -\ln r$ for r small. This gives us,

$$\Delta\mu = \mu_{rr} + \frac{h_r}{h}\mu_r = \frac{1}{r^2}\left(1 - r\frac{h_r}{h}\right) = \frac{1}{r^2}O(r^2) = O(1)$$

and hence $\Delta\mu$ is bounded on $\left[0, \frac{1}{2}\right)$. Finally, glue all the functions defined on neighbourhoods of the cone points, and define $\mu = \ln 2$ outside these neighbourhoods. □

Now we construct the new barriers.

Lemma 3.5. *For any $C, \eta > 0$ and any compact set K in space-time not containing cone points, the functions $\psi = \psi(t)$ and $\tilde{\varphi} = \tilde{\varphi}(x, t)$ defined as*

$$\tilde{\varphi} = \varepsilon e^{At}\big(f(x) + \mu(x)\big), \qquad \psi = \delta e^{Bt}$$

satisfy

(1) $\delta \leq \psi \leq \eta$ *for some $\delta > 0$, for all t;*

(2) $\varepsilon \leq \tilde{\varphi} \leq \eta$ *on the compact K for some $\varepsilon > 0$, for all t. Furthermore, $\tilde{\varphi}(x, t) \to \infty$ if $x \to \infty$ or $x \to \Sigma = \{p_1, \dots, p_n\}$, i.e. the sets $\{x \mid \tilde{\varphi}(x, t) < M\}$ are compact for all t and all M;*

(3) $\frac{\partial\tilde{\varphi}}{\partial t} > \Delta\tilde{\varphi} + C\tilde{\varphi}$;

(4) $\frac{\partial\psi}{\partial t} > C\psi$;

(5) $\tilde{\varphi} \geq C\psi$.

Proof. We have defined

$$\tilde{\varphi} = \varphi + \varepsilon e^{At}\mu$$

where φ is the function on the smooth case on Lemma 3.4. Thus, items 1 and 4 have not changed. Item 2 follows from the fact that $\mu \to +\infty$ as x tends to a cone point. Item 5 is immediate, $C\psi \leq \varphi \leq \tilde{\varphi}$.

We check item 3:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{\varphi} = \left(\frac{\partial}{\partial t} - \Delta\right)\varphi + \varepsilon e^{At}(A\mu - \Delta\mu).$$

Since $\Delta\mu \leq C' \leq (C')^2\mu$, we have $A\mu - \Delta\mu \geq \left(A - (C')^2\right)\mu \geq C''\mu$ if A is big enough. Hence,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{\varphi} > C\varphi + C''\varepsilon e^{At}\mu > C'''\tilde{\varphi},$$

for possibly different constants C's. □

3.4. Uniformization of cone surfaces. We finally assemble the properties obtained on the previous sections to reconstruct a proof for the uniformization of certain cone surfaces, as done with the smooth case.

Recall that for cone surfaces there is a suitable modified definition of Euler characteristic,

$$\hat{\chi}(\mathcal{M}) = \chi(\mathcal{M}) + \sum_{i=1}^{n} \beta_i$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the underlying topological surface, and $\beta_i = \frac{\alpha_i}{2\pi} - 1$ are the angle parameters of the cone points. In the case of orbifolds, this definition makes the conic Euler characteristic multiplicative with respect to branched coverings (i.e. if $\tilde{\mathcal{M}} \to \mathcal{M}$ is an n-to-one branched covering, then $\hat{\chi}(\tilde{\mathcal{M}}) = n\hat{\chi}(\mathcal{M})$). Furthermore, with this definition the Gauss–Bonnet formula holds,

$$\int_{\mathcal{M}} K\, d\mu = 2\pi\hat{\chi}(\mathcal{M}),$$

and the evolution of the area of the surface under the Ricci flow is still

$$\frac{\partial}{\partial t}\operatorname{Area}(\mathcal{M}) = \int_{\mathcal{M}} \frac{\partial}{\partial t}\, d\mu = \int_{\mathcal{M}} R\, d\mu = -4\pi\hat{\chi}(\mathcal{M}).$$

In the case $\hat{\chi}(\mathcal{M}) \leq 0$, the area does not tend to zero, and there are also no isolated infinite-curvature singularities; since the rescaling blow-up of such singularities would bring as a limit a noncompact κ-solution, which is impossible. Therefore, the flow is defined for all $t > 0$. In this case the most easy way to study the flow is to use the normalized Ricci flow, as in the smooth case. Substituting the maximum principle with the cone maximum principle from Section 3.3, the result is the same, and the surface converges to a constant curvature cone metric.

In the case $\hat{\chi}(\mathscr{M}) > 0$, the area tends to zero in finite time and there must be an infinite-curvature singularity, which, as before, cannot be isolated and must happen at the same time as the area collapses to zero.

In the smooth case, the only obstruction to the continuation of the flow is the explosion of the curvature at some point, as we saw in Section 3.1. To prove this, one picks a flow defined on $t \in [0, T)$ and assumes uniformly bounded curvature. One selects a sequence $t_i \to T$ and then the metrics $g(t_i)$ are equivalent to $g(0)$. By a compactness theorem, we get a limit as $t_i \to T$ and get a metric $g(T)$ that can serve as initial data for a continuation of the flow.

In the case of cone surfaces, the same result applies. However, the compactness theorem must be examined. A priori, other phenomena associated with the cone points might prevent a continuation of the flow, such as two cone points collapsing close together, or a limit of cone points with certain angle that converge to a different cone angle. Fortunately, in Section 4 we prove that, provided uniform bounds on the injectivity radius and the curvature, the cone structure is preserved under Gromov–Hausdorff limits. The uniform bound for the injectivity radius on $[0, t_i)$ comes automatically from the finite distortion of the metric for finite time (and hence finite distortion of the distances). If we assume bounded curvature on $[0, t_i)$, then we can apply Gromov's compactness theorem to get a G–H limit, this limit is smooth except on the limit of the cone points, and the cone angle of the limit is the same as in $g(0)$. Therefore, the continuous extension is not altered by the cone points, and the smooth extension applies (locally) to any neighbourhood of any smooth point. Let us remark that no maximum principle is required for the continuous extension of the metric. We have proven the following result.

Theorem 3.2. *Let $\mathscr{M}$ be a smooth closed cone surface, $\Sigma \subset \mathscr{M}$ a discrete set of cone points, and $g(t)$ a Ricci flow on a maximal time interval $[0, T)$ and $T < \infty$, then $\sup_{\mathscr{M} \setminus \Sigma} |R(\cdot, t)| \to \infty$ as $t \to T$.*

If the curvature explodes to infinity for finite time, we can perform a sequence of pointed parabolic rescalings. In the smooth case, the κ-noncollapsing property allows us to get a pointed limit flow, that is a κ-solution. In the cone setting, the same result applies, but we need to impose a restriction on the magnitude of the cone angles. In order to have κ-noncollapsing, the cone angles must be less than 2π. This is natural since a cone angle bigger than 2π has metric curvature $-\infty$ (in the sense of Alexandrov), so even Gromov–Hausdorff convergence could fail. More interestingly, in order to keep the same cone structure in the limit, we need to restrict ourselves to surfaces with cone angles less than or equal to π (this in particular covers the case of all orbifolds, with cone angles $\frac{2\pi}{n}$ for $n \in \mathbb{N}$). If this restriction is dropped, collision of cone points might happen in the limit.

The theory of $\mathscr{L}$-geodesics is still valid on cone surfaces. This is due to the following fact.

Lemma 3.6. *Let $(\mathscr{M}, g(t))$ be a smooth closed cone surface with cone angles less than 2π. Then, an $\mathscr{L}$-geodesics that minimizes the $\mathscr{L}$-length between two smooth points does not pass across any cone point.*

Sketch of the proof. It is analogous to classical geodesics (as a length-minimizing path). On a smooth surface, a minimizing geodesic does not have sharp angles. Let $\gamma(t)$ be an arc-parameterized path with a sharp angle in $p = \gamma(t*)$, it has an angle bisector $\eta = \dot{\gamma}(p)^+ - \dot{\gamma}(p)^-$ (the bisector of the inner angle, which is less than π).

Then a smooth variation of γ with normal direction Y and $Y(p) = \eta$ of would shorten the length of the path. This is proven by using the first variation of the length-energy functional $L = \int_{t_1}^{t_2} |\dot{\gamma}(t)|^2 \, dt$, which is

$$\delta_Y L = 2 \int_{t_1}^{t_2} \langle \nabla_X Y, X \rangle \, dt$$

or when there is a sharp angle,

$$\delta_Y L = 2 \int_{t_1}^{t_2} \langle \nabla_X Y, X \rangle \, dt - \langle Y(p), \eta \rangle,$$

where X is the tangent vector to the curves in the variation and Y is the normal vector to the variation. In the same way, the tangent vector to a geodesic on a cone surface that passes through a cone angle, will form one (or two) angles less than π at the cone point. Taking a variation in the direction of the bisector of that angle will decrease the length.

The same occurs with $\mathscr{L}$-geodesics. A spacetime path $\gamma(\tau)$, parameterized backwards in time ($t = -\tau$) is an $\mathscr{L}$-geodesic if it minimizes the $\mathscr{L}$-functional

$$\mathscr{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau} \big(R(\gamma(\tau)) + |\gamma(\tau)|^2 \big) \, d\tau$$

between its endpoints. The first variation [KL08], Sec 17 is

$$\delta_Y \mathscr{L} = \int_{\tau_1}^{\tau_2} \sqrt{\tau} (\langle Y, \nabla R \rangle + 2 \langle \nabla_X Y, X \rangle) \, d\tau.$$

Using that ∇R is bounded in a neighbourhood of the cone point (and that it keeps so under the Ricci flow), the same shortening variation trick applies to this functional. $\square$

Hence, integral quantities (such as Perelman's reduced volume) on a cone surface are defined the same way as in the smooth case, up to a set of singular cone points that have zero measure. Since the noncollapsing theorem is independent of Harnack inequalities and maximum principles, we have directly the same noncollapsing theorem for cone surfaces.

Now, noncollapsing together with Lemma 4.2 has the following important consequence for surfaces with cone angles less or equal than π.

Theorem 3.3. *Let $\big(\mathcal{M}, (p_1, \dots, p_n), g(t)\big)$ be an angle-preserving Ricci flow on a cone surface defined on $[0, T]$, such that $|K| < C$ for all $t \in [0, T]$. Assume that all the cone angles are less than or equal to π. Then, the injectivity radius of the cone points $\operatorname{inj}(\Sigma) = \min_{p \in \{p_1, \dots, p_n\}} \operatorname{inj}(p)$ is uniformly bounded below along the flow. In particular, the distance between any two cone points is uniformly bounded below.*

Proof. By the κ-noncollapse and the bounds on the curvature, there is a lower bound on the volume and hence, by metric geometry, a lower bound on the injectivity radius, and this bounds are uniform in time as far as the curvature keeps uniformly bounded. Then, by Lemma 4.2, the cone points keep a uniformly bounded distance between them (the injectivity radius of the cone points is uniformly bounded below). $\square$

This theorem applies to the rescaled cone surfaces on the blow-up, and therefore there is a limit to a κ-solution which has the same cone structure as the original surface. Since the Harnack inequality holds for flows on cone surfaces, every κ-solution has an asymptotic shrinking soliton, complete and with bounded curvature. We classified all cone solitons in Section 2, and all the possible solitons are compact, namely the teardrop and the football solitons, or the constant curvature solitons.

Since we have a maximum principle for functions on cone surfaces, we can apply it on κ-solutions to the function $u = t^2|M|^2$, over $\mathcal{M}$ and nested compact time intervals $[t_1, t_2]$ with $t_1 \to -\infty$, as in the smooth case, and obtain that $|M| = 0$. This proves that every κ-solution on a cone surface is a soliton.

Theorem 3.4. *Let $\big(\mathcal{M}, (p_1, \dots, p_n), g(t)\big)$ be a κ-solution over a cone surface with cone angles less than or equal to π. Then it is a shrinking soliton.*

Since all κ solutions are therefore compact, this gives a strong restriction on which kind of surfaces with $\hat{\chi}(M) > 0$ can develop an infinite-curvature singularity. Note that by the definition of $\hat{\chi}$ and our restriction of angles $\leq \pi$, the underlying topological surface must be a sphere and at most three cone points can occur. Only in the case of two cone points there is an infinite-curvature singularity.

All together, this gives a uniformization of all closed cone surfaces with angles less than or equal to π, and in particular, a uniformization of all closed two-dimensional orbifolds.

Theorem 3.5. *Let $\left(\mathscr{M}, (p_1, \ldots, p_n), g_0\right)$ be a closed cone surface, and assume that the cone points are less than or equal to π. Then there exists an angle-preserving Ricci flow that converges, up to rescaling, to:*

- *a constant nonpositive curvature metric, if $\hat{\chi}(\mathscr{M}) \leq 0$.*
- *a spherical (constant positive curvature) metric, a teardrop soliton or a football soliton; if $\hat{\chi}(\mathscr{M}) > 0$.*

As a final remark, if some cone points are greater than π, then the nonlocal collapsing is not enough to guarantee that two cone points stay at a uniformly bounded distance, i.e. two cone points could approach each other asymptotically while maintaining bounded curvature and area on the surface, then colliding together. This phenomenon has been also observed and confirmed in [MRS15]. See also [PSSW14] and [PSSW15].

4. Compactness theorems for classes of cone surfaces

The theory of compactness of classes of manifolds traces back to Cheeger [Che70], Gromov [Gro07], Greene and Wu [GW88], and Peters [Pet87]. It studies the existence and regularity of a limit on sequences of manifolds, and in particular it is a key step in the Ricci flow theory (cf. [CCG+07], Ch 3, Ch 4). Hamilton adapted the existing theorems for manifolds, adding stronger hypothesis on the regularity of the curvature tensor, and proved a specific version for solutions to the Ricci flow that is the appropriate result needed to perform sequences of rescalings on the flow [Ham95a].

If our manifolds have cone-like singularities, the cone structure after passing to a limit a priori might be very different from the structure of the terms of the sequence. We will work on the class of pointed cone surfaces (that is, pairs (M, O) with M a cone surface and $O \in M$ a base point) with bounded curvatures on all points on the smooth part, and with bounded injectivity radius on the base point. Note that if the base point is on the smooth part, it must be away from a certain (uniform) distance of the cone points, or otherwise the injectivity radius could converge to zero as the base point approaches a cone point. Base points at exactly cone points are also allowed, with the natural definition via the exponential map from the tangent cone. This will ensure that all base points have standard neighbourhoods with a uniform radius. Additionally, we will need to impose some conditions about the magnitude of the cone angles to ensure stability of the cone structure.

Gromov's compactness theorem [Gro07], cf. [BBI01], Thm 7.4.15, Thm 10.7.2, ensures that a sequence of metric spaces with fixed dimension n, bounded diameter $\operatorname{diam} \leq D$ and curvature bounded below $\sec \geq \Lambda$ has a subsequence convergent in the Gromov–Hausdorff topology. This will ensure us a weak convergence of a sequence of cone surfaces to a limit which is, a priori, just a metric space.

Hamilton's compactness theorem [Ham95a], cf. [CCG+07], Thm 3.9, states that a sequence of complete pointed Riemannian manifolds (M_k, O_k, g_k) with $|\nabla^p Rm| \leq C_p$, $\forall k$, $\forall p$, and $\operatorname{inj}(O_k) \geq i_0$, $\forall k$ has a subsequence convergent in the $\mathscr{C}^\infty$ sense. We will apply this theorem to the smooth part of our surfaces. Our compactness result for cone surfaces is:

Theorem 4.1. *Let $\mathfrak{M}$ denote the class of pointed cone surfaces $(\mathscr{M}, O)$ satisfying*

(1) *cone points with angles $\leq \pi$,*

(2) $|\nabla^p Rm_x| \leq C_p$ $\forall x \notin \Sigma_{\mathscr{M}}$ *where $\Sigma_{\mathscr{M}}$ is the singular set of $\mathscr{M}$, for all $p \geq 0$,*

(3) $\operatorname{inj}(O) \geq i_0$, *if $O \notin \Sigma_{\mathscr{M}}$,*

(4) $\operatorname{inj}(O) \geq i_1$ *and $\alpha > \alpha_0 > 0$, if $O \in \Sigma_{\mathscr{M}}$.*

Then $\mathfrak{M}$ is compact in the topology of the (pointed) $\mathscr{C}^\infty$ convergence on the smooth part and Lipschitz on the singular points.

Lipschitz convergence is stronger than Gromov–Hausdorff convergence and, as we will see, the number of cone points at the limit is the same as the number of cone points on the terms of the approximating sequence, and the magnitude of the cone angles form convergent sequences for each cone point. Therefore, the cone structure is preserved at the limit (Lemma 4.1 below).

Note that the theorem uses two injectivity radius, one for smooth base points and another for singular base points. The bound on the first one allows a version of an "injectivity radius decay with distance" as in the case of smooth manifolds (Lemma 4.3 below) that is needed to apply Hamilton's compactness theorem. The bound on the second, together with the restriction *angles* $\leq \pi$ ensure that the cone points cannot get close together (Lemma 4.2 below).

We start by proving that two compact cone surfaces that are close enough in the Gromov–Hausdorff sense must have the same number of cone points, and their respective cone angles must also be close.

Lemma 4.1. *For all* $\varepsilon, i_1, i_0, \Lambda > 0$ *and* $\omega_0 < 2\pi$ *there exists* $\delta > 0$ *such that the following holds. Let* $\mathscr{M}$, $\bar{\mathscr{M}}$ *be two compact cone surfaces with* Σ, $\bar{\Sigma}$ *their singular sets, satisfying*

- *cone angles* $\leq \omega_0 < 2\pi$.
- $\operatorname{inj} x \geq i_1 \ \forall x \in \Sigma$ (*resp.* $\bar{\Sigma}$).
- $\operatorname{inj} x \geq i_0 \ \forall x \in \mathscr{M} \backslash \mathscr{U}_{i_1/2}(\Sigma)$ (*resp. for* $\bar{\mathscr{M}}$), *where* $\mathscr{U}_\eta(\Sigma) = \{y \in \mathscr{M} \mid d(y, \Sigma) < \eta\}$ *is an open neighbourhood of* Σ.
- $|\sec_x| \leq \Lambda$ *for all* $x \notin \Sigma$ (*resp.* $\bar{\Sigma}$).

If the Gromov–Hausdorff distance between them is $d_{GH}(\mathscr{M}, \bar{\mathscr{M}}) < \delta$, *then there exists a* 2δ*-isometry* $f : \mathscr{M} \to \bar{\mathscr{M}}$ *that sends cone points to cone points. Further, if* $p \in \Sigma$ *and* $f(p) \in \bar{\Sigma}$ *have cone angles* ω, $\bar{\omega}$, *respectively, then* $|\omega - \bar{\omega}| < \varepsilon$.

Proof. From the properties of Gromov–Hausdorff distance, we have [BBI01], Cor 7.3.28 that $d_{GH}(\mathscr{M}, \bar{\mathscr{M}}) \leq \delta$ implies that there exists $f : \mathscr{M} \to \bar{\mathscr{M}}$ a 2δ-isometry, that is, a possibly noncontinuous function such that $f(\mathscr{M})$ is a 2δ-net in $\bar{\mathscr{M}}$ and $\mathit{dis}\, f = \sup_{x,x'} \left| d_{\bar{\mathscr{M}}}\big(f(x), f(x')\big) - d_{\mathscr{M}}(x, x') \right| \leq 2\delta$. Our goal is to modify the map f so it sends cone points to cone points. Hence, we need to show that if p is a cone point in $\mathscr{M}$, then $f(p)$ is arbitrarily close to a cone point in $\bar{\mathscr{M}}$, choosing δ small enough.

Let $p \in \mathscr{M}$ be a cone point with angle $\omega < \omega_0 < 2\pi$. We launch four small geodesic rays spreading from p, of length $d < i_1$, on directions separated by an angle of $\alpha := \frac{\omega}{4}$. We join the four endpoints of the rays with geodesic paths to form a quadrilateral. The image by f of the vertices of this quadrilateral defines a new quadrilateral in $\bar{\mathscr{M}}$. We can compare the triangles formed by the rays and the sides of the quadrilaterals. Since the sides of the triangles are almost the same ($\pm 2\delta$), and the curvature is bounded, the angles also must be almost the same, and hence the angles around $f(p)$ will also add up less than 2π.

We proceed by contradiction, and we suppose that the image of the quadrilateral is contained in an open set of $\bar{\mathscr{M}}$ with no cone points. We consider the triangle in $\mathscr{M}$ defined by p and two rays of length d forming an angle α; with a third side of length l. The image of the vertices of this triangle by f defines a new triangle in $\bar{\mathscr{M}}$ of corresponding sides $\bar{d}_1$, $\bar{d}_2$ and $\bar{l}$, with an angle in $f(p)$ of α'. Since f is a 2δ-isometry, we have $|l - \bar{l}| < 2\delta$ and $|d - d_j| < 2\delta \ j = 1, 2$.

We look for an upper bound of α' in terms of α. We compare these triangles with their constant curvature models. We can assume without loss of generality (by a dilation) that the bound on the curvature is $\Lambda = 1$.

We compare the triangle in $\mathscr{M}$ with a hyperbolic triangle keeping the data side-angle-side fixed (a hinge). We compare the triangle in $\bar{\mathscr{M}}$ with a spherical triangle keeping the data side-side-side fixed (side lengths). By Toponogov's comparison theorems, we have $l < \tilde{l}$ and $\alpha' < \tilde{\alpha}'$, and (since $\alpha' < \frac{\pi}{2}$) we have

$$\cosh l < \cosh \tilde{l} \qquad \text{and} \qquad \cos \alpha' > \cos \tilde{\alpha}'.$$

On the other hand, by hyperbolic/spherical trigonometry

$$\cos \alpha = \frac{\cosh d \cosh d - \cosh \tilde{l}}{\sinh d \sinh d} \qquad \text{and} \qquad \cos \tilde{\alpha}' = -\frac{\cos \bar{d}_1 \cos \bar{d}_2 - \cos \bar{l}}{\sin \bar{d}_1 \sin \bar{d}_2},$$

hence,

$$\cos \alpha < \frac{\cosh d \cosh d - \cosh l}{\sinh d \sinh d} =: A \qquad \text{and} \qquad \cos \alpha' > \cos \tilde{\alpha}' =: B.$$

Although A and B cannot be ordered in general, it is easy to verify that $\lim_{d \to 0} \frac{A}{B} = 1$. Therefore, for all $\eta > 0$, we can assume $1 - \eta < \frac{A}{B} < 1 + \eta$ if we choose d small enough and $\delta < d^3$. Then, $|A - B| < \eta |B| < \eta$ and finally,

$$\cos \alpha' > B > A - \eta > \cos \alpha - \eta.$$

Given $0 < \alpha < \frac{\pi}{2}$ we have $\cos \alpha > 0$. Let η be such that $\cos \alpha - \eta > 0$. Then $\cos \alpha' > \cos \alpha - \eta > 0$ and therefore $\alpha' < \frac{\pi}{2}$. Applying this to the four triangles forming the quadrilateral in $\mathscr{M}$, the central point $f(p)$ of the image quadrilateral in $\bar{\mathscr{M}}$ must be conical.

Thus, we have seen that if δ is small enough, then $f(p)$ is arbitrarily close to a cone point on $\bar{\mathscr{M}}$. Only one cone point, since on $\bar{\mathscr{M}}$ there are no cone points arbitrarily close (the injectivity radius is bounded). We define $\tilde{f}$ as $\tilde{f}(x) = f(x)$ if x is a smooth point, and $\tilde{f}(p) = \bar{p}$ if p is a cone point, where $\bar{p}$ is the closest cone point on $\bar{\mathscr{M}}$ to $f(p)$. From our argument above, if f is a 2δ-isometry, then $\tilde{f}$ is a $(2\delta + \mu)$-isometry for any μ arbitrarily small.

Finally we compare the cone angles at p, $\bar{p}$. If $\alpha = \frac{\omega}{4}$ and $\bar{\alpha} = \frac{\bar{\omega}}{4}$, we have seen that $\cos \bar{\alpha} > \cos \alpha - \eta_1$. By symmetry, there is also a 2δ-isometry $g : \bar{\mathscr{M}} \to \mathscr{M}$ and hence $\cos \alpha > \cos \bar{\alpha} - \eta_2$. Therefore $|\cos \alpha - \cos \bar{\alpha}| < \max\{\eta_1, \eta_2\}$ and then $|\alpha - \bar{\alpha}| < \varepsilon$ if η_1, η_2 small enough. □

Now we prove that two cone points with cone angles less than or equal to π cannot be close together on a surface with bounded curvature. Heuristically, forcing two cone points to be close each other would force the curvature to descend towards $-\infty$ on the region near the geodesic joining the two cone points.

Lemma 4.2. *For all $i_0, \Lambda, D > 0$ there exists $C = C(i_0, \Lambda, D) > 0$ such that the following holds. Let $(\mathscr{M}, x_0)$ be a cone surface with smooth base point x_0, satisfying*

- *cone angles* $\leq \pi$.
- $\mathrm{inj}(x_0) > i_0$.
- $|\sec| < \Lambda$.

If $p, q \in B_D(x_0)$ *are two cone points, then* $d(p,q) > C$ *and* $\mathrm{inj}(p) > C$.

Proof. Let $p, q \in \mathscr{M}$ be two cone points, and let σ be the shortest geodesic arc joining them, and let $|\sigma|$ be its length. Let

$$B_R(\sigma) = \{x \in \mathscr{M} : d(x,\sigma) \leq R\}$$

be a neighbourhood of σ of radius R. Clearly, as $R \to \infty$, $B_R(\sigma)$ is exhausting all $\mathscr{M}$. Let

$$N_R(\sigma) = \{x \in \mathscr{M} : x = \exp_y(v),\ y \in \sigma,\ v \perp \pm\dot{\sigma}(y),\ \|v\| \leq R\}$$

be a normal neighbourhood of σ.

First step in the proof is that since the cone angles at p, q are $\leq \pi$, we have $B_R(\sigma) = N_R(\sigma)$. Indeed, if $x \in \mathscr{M}$ and $y \in \sigma$ is the point that realizes the distance, $d(x,\sigma) = d(x,y)$, then y must be attained perpendicularly, and hence y is the foot of the perpendicular from which the exponential emanates. To see this, if y is not an endpoint of σ, then if the angle θ between σ and the geodesic joining with x were less than $\frac{\pi}{2}$ on either side, the distance could be shortened towards that side. If y is one of the endpoints of σ, the angle θ must necessarily be $\leq \frac{\pi}{2}$, since the space of directions at p and q measures $\leq \pi$, and hence the angle forming two geodesics at a cone point must form an angle $\leq \frac{\pi}{2}$.

Let now V be the construction of $N_R(\sigma)$ ported to the hyperbolic space $\mathbb{H}^2_{-\Lambda}$ of constant negative curvature $-\Lambda$. That is: first draw a geodesic segment $\tilde{\sigma}(t)[0,|\sigma|] \to \mathbb{H}^2_{-\Lambda}$ of length $|\sigma|$; then for each $x = \exp_{\sigma(t)}(v) \in N_R(\sigma)$, add the point $\tilde{x} = \exp_{\tilde{\sigma}(t)} \tilde{v} \in \mathbb{H}^2_{-\Lambda}$, where $\tilde{v} \perp \dot{\tilde{\sigma}}(t)$, with the same orientation, and $|\tilde{v}| = |v|$.

Let $\tilde{N}_R(\tilde{\sigma}) \subset \mathbb{H}^2_{-\Lambda}$ be a normal neighbourhood of $\tilde{\sigma}$. Then it must contain V, that is, $V \subset \tilde{N}_R(\tilde{\sigma})$. Choosing appropriate coordinates on $\mathbb{H}^2_{-\Lambda}$, the hyperbolic metric (of curvature $-\Lambda$) can be written $dx^2 + \cosh^2(\sqrt{\Lambda}x)\,dy^2$ and $\tilde{\sigma}$ is the curve $\{y = 0\}$ with arc-parameter x. We can compute the area of $\tilde{N}_R(\tilde{\sigma})$,

$$\mathrm{Area}\big(\tilde{N}_R(\tilde{\sigma})\big) = \int_{-R}^{R}\int_0^{|\sigma|} \cosh(\sqrt{\Lambda}x)\,dx\,dy = 2|\sigma|\frac{\sinh(\sqrt{\Lambda}R)}{\sqrt{\Lambda}}.$$

Now we apply a comparison theorem. For $R = D + i_0$, $N_R(\sigma)$ contains $B(x_0, i_0)$ a smooth regular ball, whose area is bounded below by $C(i_0, \Lambda)$, the area of a ball with same radius in the spherical space of curvature $+\Lambda$. Then,

$$C(i_0,\Lambda) \le \operatorname{Area}\big(B(x_0,i_0)\big) \le \operatorname{Area}\big(N_R(\sigma)\big) \le \operatorname{Area}(V)$$
$$\le \operatorname{Area}\big(\tilde{N}_R(\tilde{\sigma})\big) = 2|\sigma|\frac{\sinh(\sqrt{\Lambda}R)}{\sqrt{\Lambda}} \le 2|\sigma|\frac{\sinh\big(\sqrt{\Lambda}(D+i_0)\big)}{\sqrt{\Lambda}}$$

Thus,

$$|\sigma| \ge \frac{C(i_0,\Lambda)\sqrt{\Lambda}}{2\sinh\big(\sqrt{\Lambda}(D+i_0)\big)}$$

and this bounds below the length of σ.

In order to bound $\operatorname{inj}(p)$, it suffices to consider $p=q$ and σ a geodesic loop based on p, and the argument above applies. □

We adapt now the Injectivity radius decay lemma [Che70], [CGT82], to the case of cone surfaces.

Lemma 4.3. *For all $i_0, \delta_0, R, \Lambda > 0$ there exists $C > 0$ such that the following holds. Let $(\mathscr{M}, x_0)$ be a cone surface with smooth base point x_0, and let $z \in \mathscr{M}$ smooth such that*

- *cone angles $\le \pi$.*
- $\operatorname{inj}(x_0) > i_0$.
- $|\sec| < \Lambda$ *on the smooth part.*
- $z \in B_R(x_0)$ *and* $d(z,\Sigma) > \delta_0 > 0$.

Then $\operatorname{inj}(z) > C$.

Proof. We start with a bound on the injectivity radius of the base point, $i_0 = \operatorname{inj}(x_0)$. By Günther–Bishop inequality [Cha06], Thm III.4.2 we can bound $\operatorname{vol} B(x_0,i_0)$,

$$\operatorname{vol} B(x_0,i_0) \ge \operatorname{vol}_\Lambda(i_0) = C_1(\Lambda,i_0) > 0$$

and this bounds below $\operatorname{vol} B(z,2R)$ since $R > d(x_0,z)$ and then $B(x_0,i_0) \subseteq B(z,2R)$,

$$\operatorname{vol} B(z,2R) \ge \operatorname{vol} B(x_0,i_0) > C_1(\Lambda,i_0).$$

Now let $\delta < \delta_0$. We can use Bishop–Gromov inequality [BBI01], Thm 10.6.6 to bound $\operatorname{vol} B(z,\delta)$,

$$\operatorname{vol} B(z,\delta) \ge C_2(\Lambda,R,\delta) \operatorname{vol} B(z,2R) > C_3(R,\delta,\Lambda,i_0).$$

Finally we apply Cheeger–Gromov–Taylor inequality [CGT82], Thm 4.3 to bound $\mathrm{inj}(z)$. In order to use this result, we need to reduce further $\delta < \frac{1}{4}\frac{\pi}{\sqrt{\Lambda}}$. Now the radius of the ball is under the conjugacy radius, and at the same time there are no cone points on that ball. Cheeger–Gromov–Taylor inequality bounds the radius of balls without conjugate points, in terms of the volume and curvature of the ball. A cone point would have here the same effect as a conjugate point (a pencil of geodesic rays through a cone point converges instantaneously). Thus,

$$\mathrm{inj}(z) > \frac{\delta}{2}\frac{1}{1+\frac{\mathrm{vol}_\Lambda(2\delta)}{\mathrm{vol}\,B(z,\delta)}} \geq \frac{\delta}{2}\frac{1}{1+\frac{\mathrm{vol}_\Lambda(2\delta)}{C_3(R,\delta,\Lambda,i_0)}} =: C(i_0,R,\Lambda,\delta_0). \qquad \square$$

We are now in position to prove the compactness theorem for cone surfaces.

Proof (*of Theorem* 4.1). Let $\{(\mathscr{M}_k, x_k)\}_{k=1}^{\infty}$ a sequence of pointed cone surfaces inside the class $\mathfrak{M}$. By Gromov's compactness theorem, there is a convergent subsequence to a pointed metric space $(\mathscr{X}_\infty, x_\infty)$ in the pointed Gromov–Hausdorff topology. Since we are not assuming compact surfaces, the pointed convergence is relevant. Gromov's theorem states that for all $R > 0$ there is a convergent subsequence

$$\mathscr{M}_k \cap B(x_k, R) \xrightarrow{GH} \mathscr{X}_\infty \cap B(x_\infty, R)$$

in the Gromov–Hausdorff topology.

To simplify the notation, we will use $M_k = \mathscr{M}_k \cap B(x_k, R)$ and $X_k = \mathscr{X}_k \cap B(x_k, R)$. If $\mathscr{M}_k$ are compact, we can take $\mathscr{M}_k = M_k$; otherwise, M_k are open noncomplete cone surfaces, and $M_k \to X_\infty$ in the Gromov–Hausdorff sense.

By Lemma 4.1, there is a subsequence such that M_k all have the same number of cone points, and their cone angles form convergent sequences. By Lemma 4.2, the cone points are separated by a uniform distance. Hence we can pick $\varepsilon = \varepsilon(R) > 0$ not depending on k, such that

$$\varepsilon < \frac{1}{10}\inf\{d(p,q) : p,q \in \Sigma_k,\ k = 1 \ldots \infty\}.$$

We remove on each surface a neighbourhood of radius ε of each cone point, and we form the sequence

$$M_k^\varepsilon := M_k \backslash \bigcup_{x\in\Sigma} B(x,\varepsilon)$$

of noncomplete smooth Riemannian surfaces. By Lemma 4.3 and Hamilton's compactness of manifolds, $M_k^\varepsilon \to M_\infty^\varepsilon$ in the $\mathscr{C}^\infty$ sense, and $M_\infty^\varepsilon \subset X_\infty$. If we reduce ε to $\varepsilon/2$, we obtain another $M_\infty^{\varepsilon/2}$. Since $M_k^\varepsilon \subset M_k^{\varepsilon/2}$, this passes to the limit as $M_\infty^\varepsilon \subset M_\infty^{\varepsilon/2}$, and both differential structures are compatible since an open covering of M_∞^ε is extended to an open covering of $M_\infty^{\varepsilon/2}$, where the radius of the covering balls only depends of the distance to the base point and the singular set. Therefore, $M_k^{\varepsilon/2^l} \to M_k$ in the $\mathscr{C}^\infty$ sense as $l \to \infty$, because is a nested sequence, and hence $M_k^{\varepsilon/2^l} \to M_\infty^{\varepsilon/2^l}$ in the $\mathscr{C}^\infty$ sense as $k \to \infty$, for all l. Since $\{M_\infty^{\varepsilon/2^l}\}_l$ is also a nested sequence, we can take a subsequence of the diagonal sequence $M_k^{\varepsilon/2^k} \to M_\infty^{(0)}$ in the $\mathscr{C}^\infty$ sense, where $M_\infty^{(0)} = \bigcup_{l>0} M_\infty^{\varepsilon/2^l}$. This limit $M_\infty^{(0)}$ is a smooth noncomplete surface inside the metric space X_∞. We define $\Sigma_\infty = X_\infty \backslash M_\infty^{(0)}$.

The only remaining issue is to check that Σ_∞ consist of cone points with a model cone metric. Let $p_k \in M_k$ be a sequence of cone points with a limit cone angle. In a neighbourhood of p_k we can write the metric of M_k as $g_k = dr^2 + h_k^2(r,\theta)\, d\theta^2$ with

$$h_k(0,\theta) = 0, \qquad \frac{\partial h_k}{\partial r}(0,\theta) = \frac{\alpha_k}{2\pi}, \qquad \frac{\partial^2 h_k}{\partial r^2}(0,\theta) = 0$$

where α_k is the cone angle. Last condition follows from bounded curvature, since $\frac{\partial^2 h_k}{\partial r^2} = -K_k(r,\theta) h_k$.

In other words, the function h_k can be written as $h_k = \frac{\alpha_k}{2\pi} r + O(r^3)$ and the function in $O(r^3)$ may depend on θ. By the convergence on the smooth part seen above, for any $r > 0$ we have $h_k \to h_\infty$ as $k \to \infty$, and the convergence is uniform on compact sets not containing $r = 0$. On the other hand, from Lemma 4.1 we have $\alpha_k \to \alpha$ as $k \to \infty$. We must check

$$1)\ \lim_{r\to 0} h_\infty(r,\theta) = 0, \qquad 2)\ \lim_{r\to 0} \frac{\partial h_\infty}{\partial r}(r,\theta) = \frac{\alpha}{2\pi}, \qquad 3)\ \lim_{r\to 0} \frac{\partial^2 h_\infty}{\partial r^2}(r,\theta) = 0.$$

For the third point, since the Riemannian curvature on the smooth part of the surfaces is uniformly bounded by hypothesis, also is the Riemannian curvature in the limit surface, and hence the second derivative on r is zero if and only if h_∞ is zero. Hence it follows from first point.

For the first point, the area element of the metrics g_k is $h_k(r,\theta)\, dr \wedge d\theta$. Since the curvature is bounded below by $-\Lambda$, by comparison the volume element is less than the volume element of the hyperbolic space of curvature $-\Lambda$, namely $\frac{\sinh(\sqrt{\Lambda} r)}{\sqrt{\Lambda}}\, dr \wedge d\theta$, thus, $h_k(r,\theta) \le \frac{\sinh(\sqrt{\Lambda} r)}{\sqrt{\Lambda}}$. This bound is uniform in k, and hence also applies in the limit h_∞. Therefore

$$\lim_{r\to 0} |h_\infty(r,\theta)| \le \lim_{r\to 0} \frac{\sinh(\sqrt{\Lambda} r)}{\sqrt{\Lambda}} = 0.$$

Finally we check the second point. By integration,

$$\frac{\partial h_k}{\partial r}(r,\theta) = \int_0^r \frac{\partial^2 h_k}{\partial t^2}(t,\theta)\,dt + \frac{\partial h_k}{\partial r}(0,\theta).$$

Then,

$$\begin{aligned}\left|\frac{\partial h_k}{\partial r}(r,\theta) - \frac{\alpha_k}{2\pi}\right| &\le \int_0^r \left|\frac{\partial^2 h_k}{\partial t^2}(t,\theta)\right| dt \le \int_0^r |K(t,\theta) h_k(t,\theta)|\,dt \le \Lambda \int_0^r |h_k(t,\theta)|\,dt \\ &\le \Lambda \int_0^r \frac{\sinh(\sqrt{\Lambda} r)}{\sqrt{\Lambda}}\,dt = \Lambda \frac{-1+\cosh(\sqrt{\Lambda} r)}{\Lambda} = \Lambda \frac{1}{2} r^2 + O(r^4) \le \Lambda r^2\end{aligned}$$

for r small. This bound is uniform in k. Thus,

$$\begin{aligned}\left|\frac{\partial h_\infty}{\partial r}(r,\theta) - \frac{\alpha}{2\pi}\right| &\le \left|\frac{\partial h_\infty}{\partial r}(r,\theta) - \frac{\partial h_k}{\partial r}(r,\theta)\right| + \left|\frac{\partial h_k}{\partial r}(r,\theta) - \frac{\alpha_k}{2\pi}\right| + \left|\frac{\alpha_k}{2\pi} - \frac{\alpha}{2\pi}\right| \\ &\le \varepsilon(r,k) + \Lambda r^2 + \varepsilon'(k)\end{aligned}$$

Now, as $k \to \infty$, we have $\varepsilon(r,k), \varepsilon'(k) \to 0$, and hence

$$\left|\frac{\partial h_\infty}{\partial r}(r,\theta) - \frac{\alpha}{2\pi}\right| \le \Lambda r^2$$

so

$$\lim_{k\to\infty} \left|\frac{\partial h_\infty}{\partial r}(r,\theta) - \frac{\alpha}{2\pi}\right| = 0.$$

This proves the convergence of the cone structure. The sequence h_k converges in the $\mathscr{C}^1$ sense by Arzelà–Ascoli theorem, so the metric tensors g_k converge in $\mathscr{C}^1$ on that coordinate chart, and this implies Lipschitz convergence of the sequence of (metric) surfaces. This finishes the proof of the theorem. □

We end the section by proving a compactness theorem for flows on cone surfaces. The smooth counterpart, Hamilton's compactness theorem for solutions of Ricci flow [Ham95a], is the actual result needed to analyse singularities of the flow.

Hamilton's compactness for flows states that given a sequence of complete pointed solutions to the Ricci flow $\big(M_k, O_k, g_k(t)\big)$ on a fixed time interval $t \in (a,b)$, with $|\mathrm{Rm}| \le C_0$ on $M_k \times (a,b)$ $\forall k$, and $\mathrm{inj}_{g_k(0)}(O_k) \ge i_0$ $\forall k$, there exists a converging subsequence to a pointed flow $\left(M_\infty, O_\infty, g_\infty(t)\right)$ with $t \in (a,b)$ in the $\mathscr{C}^\infty$ sense. The remarkable point is that the hypothesis can be relaxed from asking bounds on all the derivatives of the curvature to just asking a bound on the curvature. This is due to the Bernstein–Bando–Shi estimates, [Shi89], [CK04], Ch 7, i.e. the Ricci flow equation relates space and time derivatives, and derivating the equation the regularity propagates to higher derivatives bounds. These bounds, however, depend on time and get worse as $t \to 0$.

We now turn to flows on cone surfaces. Recall from Section 3 that the existence of the angle preserving flow is given by the work in [MRS15]. This flow preserves the cone angles and has, at least for short time, bounded curvature and derivative of the curvature. Our compactness theorem for flows is the following.

Theorem 4.2. *Let $\left(\mathscr{M}_k, g_k(t), O_k\right)$, with $t \in (a,b)$, be a sequence of pointed cone surfaces evolving according to the angle-preserving Ricci flow. Assume that*

(1) *all cone angles are less than or equal to π,*

(2) *$|Rm_x| \le C_0$ for all $x \in \mathscr{M}_k \times (a,b)$, with x not a cone point,*

(3) $\operatorname{inj}_{g_k(0)} O_k \ge i_0$, *if $O_k \notin \Sigma_k$,*

(4) $\operatorname{inj}_{g_k(0)} O_k \ge i_1$ *and $\alpha > \alpha_0 > 0$, if O_k is a cone point of angle α.*

Then there exists a convergent subsequence to a pointed limit flow $\left(\mathscr{M}_\infty, g_\infty(t), O_\infty\right)$ in the $\mathscr{C}^\infty$ sense on the smooth part, and Lipschitz on the singular points.

Proof. The same argument of Hamilton's compactness theorem for flows applies to neighbourhoods of smooth points (all bounds, such as Shi's estimates can be done locally). However, it remains to check that, in a neighbourhood of a cone point of fixed angle α, a sequence of angle-preserving flows subconverges to an angle-preserving flow.

From the short-time existence Theorem 3.3, for every initial metric $g_k(0)$ there exists an angle-preserving flow $g_k(t)$ for $t \in (a,b)$ a uniform time interval since the curvature is bounded by hypothesis. From Lemma 3.2, these metrics can be written in geodesic polar coordinates around a cone point as

$$g_k(t) = dr^2 + h^k(r,\theta,t)^2\, d\theta^2$$

and then the Ricci flow equation adopt the form of the system

$$\begin{cases} h^k_t = h^k_{rr} - h^k_r \int \frac{h^k_{rr}}{h^k} \\ h^k = 0 \qquad\qquad \text{on } r = 0. \end{cases} \tag{23}$$

We only need to apply the Arzelà–Ascoli theorem to the functions h^k to get uniform convergence over compact sets. Note that these compact sets may contain the cone point itself, since the functions are defined for $r \in [0, r_0)$. This is a very convenient property that we would not have in conformal coordinates.

From the proof of Theorem 4.1, h^k is uniformly bounded,

$$h^k \leq \frac{\sinh(\sqrt{C_0}r)}{\sqrt{C_0}}.$$

Also, the space derivatives of h^k are also uniformly bounded since

$$\left| h_r^k - \frac{\alpha}{2\pi} \right| \leq C_0 r^2.$$

Further, the second derivative h_{rr}^k is also uniformly bounded since the curvature $K = -\frac{h_{rr}^k}{h^k}$ is uniformly bounded by hypothesis.

Finally, the Ricci flow equation, written in the form of (23), ensures that the time derivative of h^k is uniformly bounded. Hence, we have uniform bounds on space and time derivatives and therefore, by Arzelà–Ascoli theorem there is a convergent subsequence of h^k to a limit h^∞. Up to now, we have that the limit function is $\mathscr{C}^1$ in space and $\mathscr{C}^0$ in time. The regularity can be improved by checking second derivatives. From [Ram14], Prop 4.2, we know that h_{rt}^k is uniformly bounded and vanishes for $r = 0$, hence the limit flow is also an angle-preserving flow and the cone angle is the same by the Gromov–Hausdorff (and stronger) convergence of the time-slice surfaces. Similarly, higher derivatives (in space and time) of h^k can be derived from expressions on the space derivatives of $K = -h_{rr}/h$ (each time derivative is translated into two space derivatives by the flow equation). But bounds on the derivatives of the curvature can be obtained from the bounds of the curvature itself by using Bernstein–Bando–Shi estimates, cf. [CK04], Thm 7.1. The proof of these estimates relies only on local computations in coordinates and in the application of a maximum principle for functions. The maximum principle holds on cone surfaces by Theorem 3.1, and hence the Bernstein–Bando–Shi estimates apply. This implies uniform bounds on the derivatives of h^k and therefore the convergence of the metric tensor on these coordinates is $\mathscr{C}^\infty$ both in space and time. From an intrinsic point of view, we can only claim Lipschitz convergence at the singular points, since no tangent vectors exist at these points in the smooth Riemannian sense. The convergence is $\mathscr{C}^\infty$ in the smooth part of the flow. $\square$

Appendix A. Gallery of embedded solitons

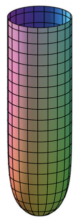

Figure 6. Hamilton's cigar soliton.
($\varepsilon = 0$, $a = 1$, $b = -1$)

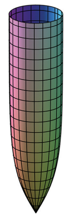

Figure 7. A cone-cigar soliton with cone angle 180°.
($\varepsilon = 0$, $a = 1$, $b = -0.5$)

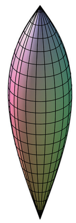

Figure 8. A football soliton with cone angles 108° and 183.38°.
($\varepsilon = -1$, $a = 1$, $b = 0.3$, $A = 4.56$)

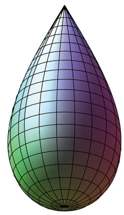

Figure 9. A teardrop soliton with cone angle 169.36°.
($\varepsilon = -1$, $a = 0.8$, $b = -1$, $A = 4.68$)

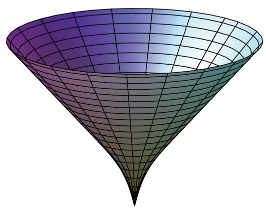

Figure 10. An $\alpha\beta$-cone soliton with asymptotic cone angle $\alpha = 240°$ and vertex cone angle $\beta = 90°$. Note that the curvature is negative since $\alpha > \beta$.
($\varepsilon = 1$, $a = 0.75$, $b = -0.25$)

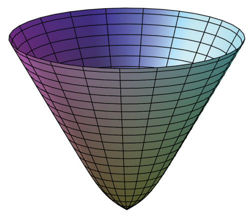

Figure 11. An $\alpha\beta$-cone soliton with asymptotic cone angle $\alpha = 180°$ and vertex cone angle $\beta = 306°$. Note that the curvature is positive since $\alpha < \beta$.
($\varepsilon = 1$, $a = 1$, $b = -0.85$)

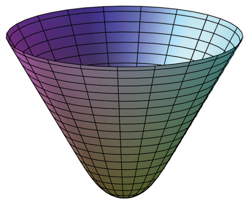

Figure 12. A blunt α-cone soliton with asymptotic cone angle $\alpha = 180°$.
$(\varepsilon = 1,\ a = 1,\ b = -1)$

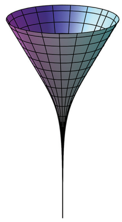

Figure 13. A cusped α-cone soliton with asymptotic cone angle $\alpha = 180°$.
$(\varepsilon = 1,\ a = 1$, separatrix $S\ (b \approx 0))$

Acknowledgments. This article is based on the author's PhD thesis [Ram14]. He wishes to thank his advisor, J. Porti, for his support and guidance during the realization of that thesis. He is very grateful to R. Mazzeo for useful conversations about the b-calculus used in [MRS15] and also about the long-time behavior of the flow in the cases not covered in this article and described in theirs. Also thanks to S. Maillot for the author's stay in Montpellier and to J. F. Rodrigues for his support and welcoming in Lisbon. Current financial support by project MAT/04561 of the Fundação para a Ciência e a Tecnologia, at CMAF-CIO, Universidade de Lisboa.

References

[BM15] Jacob Bernstein and Thomas Mettler. Two-dimensional gradient Ricci solitons revisited. *Int. Math. Res. Not. IMRN*, 1:78–98, 2015. Original preprint, arXiv:1303.6854 [math.DG], 2013.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[CCCY03] H. D. Cao, B. Chow, S. C. Chu, and S. T. Yau, editors. *Collected papers on Ricci flow*, volume 37 of *Series in Geometry and Topology*. International Press, Somerville, MA, 2003.

[Cao96] Huai-Dong Cao. Existence of gradient Kähler-Ricci solitons. In *Elliptic and parabolic methods in geometry* (Minneapolis, MN, 1994), pages 1–16. A K Peters, Wellesley, MA, 1996.

[Cao97] Huai-Dong Cao. Limits of solutions to the Kähler-Ricci flow. *J. Differential Geom.*, 45(2):257–272, 1997.

[CZ06] Huai-Dong Cao and Xi-Ping Zhu. A complete proof of the Poincaré and geometrization conjectures – application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.*, 10(2):165–492, 2006.

[Cha06] Isaac Chavel. *Riemannian geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2006. A modern introduction.

[Che70] Jeff Cheeger. Finiteness theorems for Riemannian manifolds. *Amer. J. Math.*, 92:61–74, 1970.

[CGT82] Jeff Cheeger, Mikhail Gromov, and Michael Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.

[CLT06] Xiuxiong Chen, Peng Lu, and Gang Tian. A note on uniformization of Riemann surfaces by Ricci flow. *Proc. Amer. Math. Soc.*, 134(11):3391–3393 (electronic), 2006.

[Cho91a] Bennett Chow. On the entropy estimate for the Ricci flow on compact 2-orbifolds. *J. Differential Geom.*, 33(2):597–600, 1991.

[Cho91b] Bennett Chow. The Ricci flow on the 2-sphere. *J. Differential Geom.*, 33(2):325–334, 1991.

[CCG+07] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. *The Ricci flow: techniques and applications.* Part I, volume 135 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007. Geometric aspects.

[CK04] Bennett Chow and Dan Knopf. *The Ricci flow: an introduction*, volume 110 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2004.

[CW91] Bennett Chow and Lang-Fang Wu. The Ricci flow on compact 2-orbifolds with curvature negative somewhere. *Comm. Pure Appl. Math.*, 44(3):275–286, 1991.

[Déb16] Clément Débin. A compactness theorem for surfaces with bounded integral curvature. Preprint, arXiv:1605.07755 [math.DG], 2016.

[GT11] Gregor Giesen and Peter M. Topping. Existence of Ricci flows of incomplete surfaces. *Comm. Partial Differential Equations*, 36(10):1860–1880, 2011.

[GW88] R. E. Greene and H. Wu. Lipschitz convergence of Riemannian manifolds. *Pacific J. Math.*, 131(1):119–141, 1988.

[Gro07] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, English edition, 2007. Based on the 1981 French original, with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by Sean Michael Bates.

[Ham82] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[Ham88] Richard S. Hamilton. The Ricci flow on surfaces. In *Mathematics and general relativity* (Santa Cruz, CA, 1986), volume 71 of *Contemp. Math.*, pages 237–262. Amer. Math. Soc., Providence, RI, 1988.

[Ham93] Richard S. Hamilton. The Harnack estimate for the Ricci flow. *J. Differential Geom.*, 37(1):225–243, 1993.

[Ham95a] Richard S. Hamilton. A compactness property for solutions of the Ricci flow. *Amer. J. Math.*, 117(3):545–572, 1995.

[Ham95b] Richard S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry*, Vol. II (Cambridge, MA, 1993), pages 7–136. Int. Press, Cambridge, MA, 1995.

[IMS11] James Isenberg, Rafe Mazzeo, and Natasa Sesum. Ricci flow in two dimensions. In *Surveys in geometric analysis and relativity*, volume 20 of Adv. Lect. Math. (ALM), pages 259–280. Int. Press, Somerville, MA, 2011. Preprint arXiv:1103.4669v1 [math.DG], 2011.

[Jef05] Thalia D. Jeffres. A maximum principle for parabolic equations on manifolds with cone singularities. *Adv. Geom.*, 5(2):319–323, 2005.

[KL08] Bruce Kleiner and John Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008.

[MRS15] Rafe Mazzeo, Yanir A. Rubinstein, and Natasa Sesum. Ricci flow on surfaces with conic singularities. *Anal. PDE*, 8(4):839–882, 2015.

[Mel93] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics.* A K Peters Ltd., Wellesley, MA, 1993.

[Per02] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159 [math.DG], 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:math/0307245 [math.DG], 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds. arXiv:math/0303109 [math.DG], 2003.

[Pet87] Stefan Peters. Convergence of Riemannian manifolds. *Compositio Math.*, 62(1):3–16, 1987.

[PSSW14] D. H. Phong, Jian Song, Jacob Sturm, and Xiaowei Wang. The Ricci flow on the sphere with marked points. Preprint, arXiv:1407.1118 [math.DG], 2014.

[PSSW15] D. H. Phong, Jian Song, Jacob Sturm, and Xiaowei Wang. Convergence of the conical Ricci flow on S^2 to a soliton. Preprint, arXiv:1503.04488 [math.DG], 2015.

[Ram13] Daniel Ramos. Gradient Ricci solitons on surfaces. Preprint, arXiv:1304.6391 [math.DG], 2013.

[Ram14] Daniel Ramos. Ricci flow on cone surfaces and a three-dimensional expanding soliton. PhD thesis, arXiv:1707.01486 [math.DG], 2014.

[Ram15] Daniel Ramos. Smoothening cone points with Ricci flow. *Bull. Soc. Math. France*, 143(4):619–633, 2015. Preprint, arXiv:1107.4591, 2011.

[Shi89] Wan-Xiong Shi. Deforming the metric on complete Riemannian manifolds. *J. Differential Geom.*, 30(1):223–301, 1989.

[Top06] Peter Topping. *Lectures on the Ricci flow*, volume 325 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2006.

[Wu91] Lang-Fang Wu. The Ricci flow on 2-orbifolds with positive curvature. *J. Differential Geom.*, 33(2):575–596, 1991.

[Ye04] Rugang Ye. On uniqueness of 2-dimensional κ-solutions. Preprint, 2004. http://www.math.ucsb.edu/~yer/2dkappa.pdf.

[Yin10] Hao Yin. Ricci flow on surfaces with conical singularities. *J. Geom. Anal.*, 20(4):970–995, 2010.

[Yin13] Hao Yin. Ricci flow on surfaces with conical singularities, II. Preprint, arXiv:1305.4355 [math.DG], 2013.

[Yin16] Hao Yin. Analysis aspects of Ricci flow on conical surfaces. Preprint, arXiv:1605.08836 [math.AP], 2016.

Received September 12, 2017; revision received March 1, 2018

D. Ramos, Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, Universidade de Lisboa, 1749-016 Lisbon, Portugal

E-mail: dramos@fc.ul.pt