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Towards a pseudoequational proof theory

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Abstract. A new scheme for proving pseudoidentities from a given set Σ of pseudoidentities, which is clearly sound, is also shown to be complete in many instances, such as when Σ defines a locally finite variety, a pseudovariety of groups, more generally, of completely simple semigroups, or of commutative monoids. Many further examples for which the scheme is complete are given when Σ defines a pseudovariety V which is σ-reducible for the equation $x = y$, provided Σ is enough to prove a basis of identities for the variety of σ-algebras generated by V. This gives ample evidence in support of the conjecture that the proof scheme is complete in general.

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1. Introduction

Pseudovarieties are classes of finite algebras closed under taking homomorphic images, subalgebras and finite direct products. They have been studied mostly in the context of finite semigroup theory due to the connections with automata and formal languages. In the framework of Eilenberg's correspondence [26], determining whether a regular language enjoys a suitable property of a certain kind is converted to the membership problem of its syntactic semigroup in the corresponding pseudovariety of semigroups. On the other hand, pseudovarieties are in many respects like the varieties of classical Universal Algebra, admitting relatively free algebras, albeit in general not finite, but rather profinite, and thus being defined by formal equations, where pseudoidentities play the role of identities [2]. While there is a natural proof scheme for identities that is sound and complete, which is provided by Birkhoff's completeness theorem for equational logic, the situation in the theory of pseudovarieties is not so simple.

Indeed, the first author has shown that there is no complete finite deduction system that is sufficient to prove a given pseudoidentity from a set of hypotheses (or basis) assuming that all models of the basis are also models of the pseudoidentity [2], Section 3.8. Some sort of topological closure operator seems to be required. While such an operator was also proposed by the first author (see [2], Section 3.8), it is very hard to handle and only one instance of its application has been found so far [12].

The main contribution of this paper is a new approach which consists in starting with all evaluation consequences of the basis, and completing them in the same term; then, transfinitely alternating transitive closure and topological closure. This proof scheme is clearly sound and, by definition, it is suitable for transfinite induction proofs. We show that it is complete in many familiar instances of concrete bases: whenever they define locally finite varieties (in their algebraic language), pseudovarieties of groups or, more generally, of completely simple semigroups, or pseudovarieties of commutative monoids. For the proof of some of these results, we need to show that concrete pseudoidentities that are valid in a pseudovariety with a given basis are provable from the basis. The main technique in proving such results is invoking excluded structures.

We also show that the proof scheme is complete when the pseudovariety defined by the basis Σ is σ-reducible with respect to the equation $x = y$, where σ is an implicit signature such that the variety of σ-algebras generated by the pseudovariety defined by Σ admits a basis whose elements can be obtained from Σ by our proof scheme. Combining with several reducibility results that can be found in the literature, an exercise that by no means we carry out exhaustively, this provides ample evidence in favor of the conjecture that our proof scheme is complete in general.

2. Preliminaries

We recall quickly in this section some basic notions from general algebra, which also serves to fix some notation. The reader is referred to [19] for basic notions on Universal Algebra and to [2], [5], [36] for an introduction to pseudovarieties and profinite structures.

By an *algebraic type* τ we mean a set of operation symbols, each of which has an associated finite arity. An algebra of type τ or a τ-*algebra* is a nonempty set endowed with an interpretation of each operation symbol of the type in question as an operation of the corresponding arity. In general, we fix an algebraic type and consider only algebras of that type. The (symbols of) operations of the type are sometimes called the *basic operations*.

By a *topological algebra* we mean an algebra endowed with a Hausdorff topology such that the interpretations of the basic operations are continuous functions. Such an algebra is *compact* if so is its topology. We endow finite algebras with the discrete topology. A topological algebra S is *residually* $\mathscr{C}$ for a class $\mathscr{C}$ of algebras if distinct points in S may be separated by continuous homomorphisms into members of $\mathscr{C}$.

Given a family $(S_i)_{i\in I}$ of topological algebras, where the index set is directed, and for each pair (i, j) of indices with $i \geq j$, a continuous homomorphism $\varphi_{i,j} : S_i \to S_j$ such that $\varphi_{i,i}$ is the identity mapping on S_i and, for $i \geq j \geq k$, $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$, we may consider the *inverse limit* $\varprojlim_{i\in I} S_i$, which may be described as the subset of the product $\prod_{i\in I} S_i$ consisting of all families $(s_i)_{i\in I}$ such that each $s_i \in S_i$ and, for $i \geq j$, $\varphi_{i,j}(s_i) = s_j$. In case each S_i is compact, $\varprojlim_{i\in I} S_i$ is nonempty and a closed subalgebra of $\prod_{i\in I} S_i$.

Recall that a *pseudovariety* is a (nonempty) class of finite algebras of a given type that is closed under taking homomorphic images, subalgebras and finite direct products. Let U be a pseudovariety. A *pro*-U *algebra* is an inverse limit of algebras from U. In other words, a pro-U algebra is a compact algebra that is residually U. A *profinite algebra* is a pro-U algebra for the class U of all finite algebras of the given type. In case a profinite algebra S is finitely generated as a topological algebra, meaning that a finitely generated subalgebra is dense, and if the signature is assumed to be finite, then there are, up to isomorphism, only countably many finite homomorphic images of S. Hence, such an S embeds in a countable product of finite algebras, which implies that its topology is metrizable; in particular, the topology of S is characterized by the convergence of sequences.

Given an arbitrary algebra S, one may consider all homomorphisms $S \to F$ onto algebras from a set of representatives of isomorphism classes of algebras from a given pseudovariety U. These homomorphisms form a directed set and so we may consider the inverse limit $\varprojlim_{S\to F} F$, which is called the *pro*-U *completion* of S and is denoted $\hat{S}_{\mathsf{U}}$. Note that the natural mapping $S \to \hat{S}_{\mathsf{U}}$ is injective if and only if S is residually U. In case U consists of all finite algebras of the given type, we drop the index U and talk about the *profinite completion* of S.

By an *alphabet* we simply mean a finite set. Its elements are called either *letters* or *variables*. Alphabets appear in this paper as free generating sets of various structures.

By a *term* (of a given algebraic type) we mean a formal expression on an alphabet A constructed using the basic operations according to their arities. In other words, it is an element of a free algebra $\mathscr{T}_A$ in the (Birkhoff) variety of all algebras of the given type.

A pro-U algebra S is said to be *freely generated by* A if it comes endowed with a function $\iota : A \to S$ such that, for every function $\varphi : A \to T$ into a pro-U algebra T, there is a unique continuous homomorphism $\hat{\varphi} : S \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & S \\ & \searrow^{\varphi} & \downarrow \hat{\varphi} \\ & & T \end{array}$$

Such a structure S is unique up to isomorphism of topological algebras and can be easily shown to be precisely the inverse limit of all A-generated members of U. It is denoted $\overline{\Omega}_A\mathsf{U}$. If $n = |A|$, then we may sometimes write $\overline{\Omega}_n\mathsf{U}$ instead of $\overline{\Omega}_A\mathsf{U}$ as it is easy to see that, up to isomorphism of topological algebras, $\overline{\Omega}_A\mathsf{U}$ only depends on the cardinality of the set A and not on the set itself. The subalgebra of $\overline{\Omega}_A\mathsf{U}$ generated by $\iota(A)$ is denoted $\Omega_A\mathsf{U}$; it is the algebra in the variety generated by U that is freely generated by the alphabet A. Note that $\overline{\Omega}_A\mathsf{U}$ may also be obtained as the profinite completion $\widehat{\Omega_A\mathsf{U}}$.

An element of $\overline{\Omega}_A\mathsf{U}$ is called a U-*pseudoword* or simply a *pseudoword* if the pseudovariety U is understood from the context;[1] those that lie in $\Omega_A\mathsf{U}$ are said to be *finite* whereas the remaining U-pseudowords are said to be *infinite*. Every pseudoword $w \in \overline{\Omega}_A\mathsf{U}$ has a natural interpretation as an operation of arity $|A|$ on a pro-U algebra T: $|A|$-tuples of elements of T may be identified with functions $\varphi : A \to T$ and so the interpretation of w becomes a function $w_T : T^A \to T$; the image $w_T(\varphi)$ is defined to be $\hat{\varphi}(w)$, where $\hat{\varphi}$ is given by the above commutative diagram. Viewed as operations, pseudowords are sometimes called *implicit operations* because their natural interpretations commute with continuous homomorphisms between pro-U algebras.

By a U-*pseudoidentity* we mean a formal equality $u = v$ with $u, v \in \overline{\Omega}_A\mathsf{U}$ for some alphabet A. A pseudoidentity $u = v$ holds in an algebra $T \in \mathsf{U}$ if $u_T = v_T$. For a set Σ of U-pseudoidentities the class $[\![\Sigma]\!]_{\mathsf{U}}$ consists of all algebras $T \in \mathsf{U}$ in which all pseudoidentities from Σ hold. When the ambient pseudovariety U is understood from the context, we may also write $[\![\Sigma]\!]$. This defines a pseudovariety contained in U, that is, a *subpseudovariety* of U, and Reiterman's theorem [35] states that every subpseudovariety of U is of this form.

In case $u, v \in \Omega_A\mathsf{U}$, the formal equality $u = v$ is called a U-*identity*. We may choose terms u' and v' which map respectively to u and v under the natural homomorphism $\mathscr{T}_A \to \Omega_A\mathsf{U}$ which fixes each generator. An algebra of U satisfies the U-identity $u = v$ if and only if it satisfies the identity $u' = v'$. We call the identity $u' = v'$ a *lifting* of $u = v$. For a set Σ of U-identities, we let Σ' be the union of the set consisting of an arbitrarily chosen lifting of each element of Σ together with a basis of identities for the variety generated by U. By $[\Sigma]$ we mean the variety $[\Sigma']$ consisting of all algebras that satisfy all identities from Σ'.

[1]The name pseudoword actually comes from the theory of semigroups since the elements of free semigroups are usually viewed as words. We would call it a *pseudoterm* in case U is the pseudovariety of all finite algebras of the given type, but this special case plays no role in this paper.

Let U be a pseudovariety of type τ. An *implicit signature* (over U) is a set σ of U-pseudowords including those of the form $o(a_1, \ldots, a_n) \in \Omega_{A_n}\mathsf{U}$, where o is a basic operation from τ of arity n and $A_n = \{a_1, \ldots, a_n\}$ is an n-letter alphabet. By the above, every pro-U algebra has a natural structure of σ-algebra. In particular, this is the case for the algebras from U which, as σ-algebras, generate a variety of σ-algebras denoted U^σ. It is also the case of $\overline{\Omega}_A\mathsf{U}$; the σ-subalgebra generated by $\iota(A)$ is denoted $\Omega_A^\sigma\mathsf{U}$ and it is easily shown to be the algebra of the variety U^σ freely generated by A.

Two pseudovarieties that have received a lot of attention are the pseudovariety S, of all finite semigroups, and M, of all finite monoids. For an element s of a finite semigroup S, there is a unique power s^n $(n \geq 1)$ of s which is an idempotent and it is denoted s^ω. The element s^{2n-1} is then also denoted $s^{\omega-1}$. In terms of the discrete topology, s^ω is the limit of the sequence $(s^{n!})_n$ while $s^{\omega-1}$ is the limit of the sequence $(s^{n!-1})_n$. It follows that if, instead of taking S finite we take S to be profinite, the sequences in question still converge and the notation for the limits is retained.

Consider the semiring $\mathbb{N}$ of all natural numbers (including zero) and the ring $\mathbb{Z}$ of integers, both under the usual addition and multiplication. These may be viewed as algebras of type consisting of two binary operation symbols, of addition and multiplication. Since both underlying additive structures (respectively a monoid and a group) are monogenic, so are their finite homomorphic images. Note also that a finite monogenic additive monoid M carries a natural structure of semiring via the natural additive homomorphism $\mathbb{N} \to M$. It follows that the profinite completion of each of $\mathbb{N}$ and $\mathbb{Z}$ as additive structures carries a multiplication which makes it isomorphic (as a topological algebra) with the semiring $\hat{\mathbb{N}}$, respectively with the ring $\hat{\mathbb{Z}}$.

Let G and G_p denote, respectively, the pseudovarieties of all finite groups and of all finite p-groups. The above allows us to describe the structure of the monoid $\overline{\Omega}_1\mathsf{M}$ and of the groups $\overline{\Omega}_1\mathsf{G}$ and $\overline{\Omega}_1\mathsf{G}_p$. Indeed we know that $\overline{\Omega}_1\mathsf{M}$ and $\overline{\Omega}_1\mathsf{G}$ are respectively the inverse limits of the finite monogenic monoids and of the finite cyclic groups, which we know that, as additive algebras, carry multiplicative structures which make them isomorphic to $\hat{\mathbb{N}}$ and $\hat{\mathbb{Z}}$, respectively. The isomorphisms $\hat{\mathbb{N}} \to \overline{\Omega}_1\mathsf{M}$ and $\hat{\mathbb{Z}} \to \overline{\Omega}_1\mathsf{G}$ are easy to describe: if the free generator is denoted x, they send each number n in $\mathbb{N}$ or in $\mathbb{Z}$ to x^n, respectively in $\overline{\Omega}_1\mathsf{M}$ and in $\overline{\Omega}_1\mathsf{G}$. For this reason, we also denote, for each α in $\hat{\mathbb{N}}$ or in $\hat{\mathbb{Z}}$ the image in $\overline{\Omega}_1\mathsf{M}$ or in $\overline{\Omega}_1\mathsf{G}$, respectively, by x^α. Thus the usual laws of exponents hold and $(x^\alpha)^\beta = x^{\alpha\beta}$ can be viewed as a composition of implicit operations.

Similarly, $\overline{\Omega}_1\mathsf{G}_p$ may be identified with the completion $\hat{\mathbb{Z}}_{\mathsf{G}_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, which is frequently denoted $\mathbb{Z}_p$. Moreover, thanks to the Chinese Remainder Theorem, it is easy to see that $\hat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$, where the index p runs over all primes. The structure of the ring $\mathbb{Z}_p$ is quite transparent: it is an integral domain and every ideal is both principal and closed. It follows that the principal ideals of the ring $\hat{\mathbb{Z}}$ are the closed ideals. In particular, every subset of $\hat{\mathbb{Z}}$ has a *greatest common divisor*, which is a generator of the closed ideal generated by the given set.

Since each of the (semi)rings $\mathbb{N}$ and $\mathbb{Z}$ is residually finite, and the latter is even residually G_p, there are natural embeddings $\mathbb{N} \to \hat{\mathbb{N}}$, $\mathbb{Z} \to \hat{\mathbb{Z}}$, and $\mathbb{Z} \to \mathbb{Z}_p$, which we view as inclusion mappings. It is easy to see that the invertible elements of $\mathbb{Z}_p$ are those that are not divisible by p. Hence, the invertible elements of $\hat{\mathbb{Z}}$ are those that are not divisible by any prime p.

The semiring $\hat{\mathbb{N}}$ has two additive idempotents, namely 0 and $\omega = \lim n!$. The maximal additive group H_ω containing ω is a closed ideal of $\hat{\mathbb{N}}$ which is generated by $\omega + 1$. The natural continuous homomorphism $\pi : \hat{\mathbb{N}} \to \hat{\mathbb{Z}}$, mapping 1 to 1, also maps $\omega + 1$ to 1 and therefore restricts to an isomorphism $H_\omega \to \hat{\mathbb{Z}}$. Thus, $\hat{\mathbb{Z}}$ may be identified with H_ω, which we do from hereon; it is a retract of $\hat{\mathbb{N}}$ under the mapping $\alpha \mapsto \omega + \alpha$. Also note that $\hat{\mathbb{N}}$ is the disjoint union of $\mathbb{N}$ with $\hat{\mathbb{Z}}$. We say that the elements of $\mathbb{N}$ are *finite* while those of $\hat{\mathbb{Z}}$ are *infinite*.

3. A proof scheme

Let U be a pseudovariety of a certain finite type of algebras involving only finitary operations. Let Σ be a set of pseudoidentities $u = v$ with u and v elements of the free pro-U algebra over some arbitrary finite alphabet, which may depend on u and v. We seek a complete proof scheme for pseudoidentities valid in the pseudovariety $[\![\Sigma]\!]$, that is, a deduction system that is capable of deducing from Σ exactly the pseudoidentities valid in $[\![\Sigma]\!]$.

Since we are concerned with proving one concrete pseudoidentity, we fix a finite alphabet A and consider only U-pseudoidentities over A. By transfinite recursion, we define, for each ordinal α, a set Σ_α of pseudoidentities over A as follows:

- Σ_0 consists of all *trivial* pseudoidentities $u = u$,[2] with $u \in \overline{\Omega}_A\mathsf{U}$ together with all pairs of the form
$$\big(\mathbf{t}\big(\varphi(u), w_1, \ldots, w_n\big), \mathbf{t}\big(\varphi(v), w_1, \ldots, w_n\big)\big)$$
such that either $u = v$ or $v = u$ is a pseudoidentity from Σ, say with $u, v \in \overline{\Omega}_B\mathsf{U}$, $\varphi : \overline{\Omega}_B\mathsf{U} \to \overline{\Omega}_A\mathsf{U}$ is a continuous homomorphism, $\mathbf{t}$ is a term (in the algebraic language of U), and $w_i \in \overline{\Omega}_A\mathsf{U}$ $(i = 1, \ldots, n)$;
- $\Sigma_{2\alpha+1}$ is the transitive closure of the binary relation $\Sigma_{2\alpha}$;
- $\Sigma_{2\alpha+2}$ is the topological closure of the relation $\Sigma_{2\alpha+1}$ in the space $\overline{\Omega}_A\mathsf{U} \times \overline{\Omega}_A\mathsf{U}$;
- if α is a limit ordinal, then $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$.

[2]Note that, in case Σ is nonempty, this is superfluous.

Finally, we let $\tilde{\Sigma} = \bigcup_\alpha \Sigma_\alpha$. The reader should keep in mind that, although the notation does not indicate it, the sets Σ_α and $\tilde{\Sigma}$ depend on the choice of finite alphabet A. Occasionally, this dependence will be made explicit by writing instead $\Sigma_{\alpha,A}$ and $\tilde{\Sigma}_A$.

Note that, $\Sigma_{\alpha+2} = \Sigma_\alpha$ if and only if Σ_α is both transitive and topologically closed, in which case $\Sigma_\beta = \Sigma_\alpha$ for every ordinal β with $\beta \geq \alpha$.

If α is a limit ordinal, then Σ_α is transitive for, given pseudoidentities $u = v$ and $v = w$ from Σ_α, there exists and ordinal $\beta < \alpha$ such that $u = v$ and $v = w$ both lie in Σ_β and, therefore, $u = w$ lies in the transitive closure of Σ_β, which is contained in $\Sigma_{\beta+2}$, whence also in Σ_α. Similarly, for the least uncountable ordinal ω_1, we claim that the set Σ_{ω_1} is topologically closed, which basically follows from the fact that the topological closure is captured by limits of sequences in the metric space $\overline{\Omega}_A\mathsf{U} \times \overline{\Omega}_A\mathsf{U}$. Indeed, suppose that $(u_n, v_n) \to (u, v)$ is a convergent sequence in $\overline{\Omega}_A\mathsf{U} \times \overline{\Omega}_A\mathsf{U}$ such that each pseudoidentity $u_n = v_n$ belongs to Σ_{ω_1}. Then, for each n, there exists a countable ordinal β_n such that $u_n = v_n$ belongs to Σ_{β_n}. It follows that $\beta = \bigcup_n \beta_n$ is again a countable ordinal and $u = v$ belongs to $\Sigma_{\beta+2}$, whence also to Σ_{ω_1}, which establishes the claim that Σ_{ω_1} is closed. The above proves the following result.

Proposition 3.1. *The least ordinal α such that Σ_α is both transitive and topologically closed satisfies $\alpha \leq \omega_1$. In particular, $\tilde{\Sigma}$ is a transitive closed binary relation on $\overline{\Omega}_A\mathsf{U}$.*

Consider a binary relation θ on $\overline{\Omega}_A\mathsf{U}$. We say that θ is *stable* if $(u, v) \in \theta$ implies

$$\big(\mathbf{t}(u, w_1, \ldots, w_n), \mathbf{t}(v, w_1, \ldots, w_n)\big) \in \theta$$

for every term $\mathbf{t}$ and $w_1, \ldots, w_n \in \overline{\Omega}_A\mathsf{U}$. We also say that θ is *fully invariant* if, for every continuous endomorphism φ of $\overline{\Omega}_A\mathsf{U}$ and $(u, v) \in \theta$, we have $\big(\varphi(u), \varphi(v)\big) \in \theta$. A stable equivalence relation is also called a *congruence*.

The following result establishes the soundness of the above proof scheme.

Proposition 3.2. *The relation $\tilde{\Sigma}$ is a fully invariant closed congruence on $\overline{\Omega}_A\mathsf{U}$. For every $(u, v) \in \tilde{\Sigma}$, the pseudoidentity $u = v$ is valid in $[\![\Sigma]\!]$.*

Proof. Note that Σ_0 is a reflexive and symmetric binary relation on $\overline{\Omega}_A\mathsf{U}$, whence so are all Σ_α as well as $\tilde{\Sigma}$. We prove by transfinite induction on α that each Σ_α is a stable fully invariant binary relation on $\overline{\Omega}_A\mathsf{U}$ whose elements, viewed as pseudoidentities, are valid in the pseudovariety $[\![\Sigma]\!]$.

For $\alpha = 0$, the claimed properties are immediate from the definition of Σ_0. Assuming the claim holds for all $\alpha < \beta$, then it clearly also holds for Σ_β in case β

is a limit ordinal. Otherwise, β is a successor ordinal, and we distinguish the cases where β is odd or even.

In case β is odd, that is, it is of the form $\beta = 2\gamma + 1$, Σ_β is the transitive closure of $\Sigma_{2\gamma}$ which, by the induction hypothesis, is a stable fully invariant binary relation on $\overline{\Omega}_A\mathsf{U}$ whose elements define pseudoidentities valid in the pseudovariety $[\![\Sigma]\!]$. The elements of Σ_β are pairs (u_0, u_n) such that there exist $u_1, \ldots, u_{n-1}$ with each $(u_i, u_{i+1}) \in \Sigma_{2\gamma}$ $(i = 0, \ldots, n-1)$. In particular, each pseudoidentity $u_i = u_{i+1}$ is valid in $[\![\Sigma]\!]$, whence so is $u_0 = u_n$. If $\mathbf{t}$ is a term and $w_1, \ldots, w_m$ are elements from $\overline{\Omega}_A\mathsf{U}$ then, by the induction hypothesis, each pair $\big(\mathbf{t}(u_i, w_1, \ldots, w_m), \mathbf{t}(u_{i+1}, w_1, \ldots, w_m)\big)$ belongs to $\Sigma_{2\gamma}$, and so the pair $\big(\mathbf{t}(u_0, w_1, \ldots, w_m), \mathbf{t}(u_n, w_1, \ldots, w_m)\big)$ also belongs to Σ_β. Finally, if φ is a continuous endomorphism of $\overline{\Omega}_A\mathsf{U}$, by the induction hypothesis each of the pairs $\big(\varphi(u_i), \varphi(u_{i+1})\big)$ belongs to $\Sigma_{2\gamma}$, which entails that $\big(\varphi(u_0), \varphi(u_n)\big)$ belongs to Σ_β.

Consider next the case where $\beta = 2\gamma + 2$, that is, a nonzero and non-limit even ordinal. Then, since $\overline{\Omega}_A\mathsf{U}$ is a metric space, every element from Σ_β is the limit (u, v) of a sequence $(u_n, v_n)_n$ of elements from $\Sigma_{2\gamma+1}$. By the induction hypothesis, as a pseudoidentity, every element of the sequence is valid in $[\![\Sigma]\!]$, whence so is $u = v$ since, under an evaluation of the elements of A in a finite algebra, the sequences $(u_n)_n$ and $(v_n)_n$ eventually stabilize, precisely at the values of u and v, respectively. Since $\Sigma_{2\gamma+1}$ is assumed to be stable, for a term $\mathbf{t}$ and $w_1, \ldots, w_m \in \overline{\Omega}_A\mathsf{U}$, the sequence of pairs $\big(\mathbf{t}(u_n, w_1, \ldots, w_m), \mathbf{t}(v_n, w_1, \ldots, w_m)\big)_n$ consists of elements of $\Sigma_{2\gamma+1}$, whence its limit $\big(\mathbf{t}(u, w_1, \ldots, w_m), \mathbf{t}(v, w_1, \ldots, w_m)\big)$ belongs to $\Sigma_\beta = \overline{\Sigma_{2\gamma+1}}$. Finally, if φ is a continuous endomorphism of $\overline{\Omega}_A\mathsf{U}$, then the elements of the sequence $\big(\varphi(u_n), \varphi(v_n)\big)_n$ belong to $\Sigma_{2\gamma+1}$ and, therefore, its limit $\big(\varphi(u), \varphi(v)\big)$ belongs to Σ_β. This completes the transfinite induction.

In view of the already established initial claim, we know that $\tilde{\Sigma}$ is a reflexive symmetric stable fully invariant binary relation consisting of pairs (u, v) such that the pseudoidentity $u = v$ is valid in $[\![\Sigma]\!]$. To complete the proof, it remains to recall that we already observed that $\tilde{\Sigma}$ is a closed transitive binary relation. □

We say that a U-pseudoidentity over a finite alphabet A is *provable* from Σ if it belongs to $\tilde{\Sigma}_A$. A set Γ of U-pseudoidentities is *provable* from Σ if every member of Γ is provable from Σ. Note that if Γ is provable from Σ and Δ is provable from Γ, then Δ is provable from Σ.

A *proof* of a pseudoidentity consists in a transfinite sequence of steps in which in step 0 we invoke pseudoidentities from Σ, suitably evaluated in $\overline{\Omega}_A\mathsf{U}$, in which both sides are plugged in the same place of an arbitrary term, and in later steps we either use transitivity of equality or take limits, in the latter two cases from already proved steps, or simply collect together all pseudoidentities in previous steps. In the last step in the proof we should have a set of pseudoidentities containing the one to be proved. Note that a pseudoidentity is provable from Σ if and only if it admits a proof from Σ.

An alternative but equivalent definition of proof, in the sense of capturing the same provable pseudoidentities, would be to take a transfinite sequence of pseudoidentities in which in each step we allow one of the pseudoidentities of the above step 0, we take $u = w$ if there are two previous steps of the form $u = v$ and $v = w$, or we take $u = v$ provided there is a sequence of earlier steps $(u_n = v_n)_n$ with $u = \lim u_n$ and $v = \lim v_n$. The latter steps are called *limiting steps*. The last step in such a proof should be the pseudoidentity to be proved.

If such a proof only involves a finite number of non-limiting steps and no limiting steps, then we say the proof is *algebraic*. Note that such a proof exists for a pseudoidentity over a given finite alphabet A precisely for the pseudoidentities of $\Sigma_{1,A}$.

Several examples of proofs in the above general sense can be found in the literature. In fact, all proofs that a pseudovariety defined by certain pseudoidentities satisfies a given pseudoidentity that we have been able to find in the literature seem to be expressible in this form. This suggests the following general conjecture which amounts to completeness of our proof scheme.

Conjecture 3.3. *A* U*-pseudoidentity* $u = v$ *is provable from a set* Σ *of* U*-pseudoidentities whenever* $[\![\Sigma]\!]$ *satisfies* $u = v$.

The above statement is to be interpreted as a logical clause depending on three parameters: U, Σ, and $u = v$, where each of the latter two determine the first one. Since we have been unable to establish the conjecture in full generality, particular instances of the conjecture, where one of the parameters is fixed may be of interest as they provide evidence towards the conjecture. If a parameter is fixed then those that are not determined by it are interpreted as being universally quantified.

We say that a set Σ of U-pseudoidentities is *h-strong* (*within* U*, or for* U) if the statement in the conjecture holds for the given fixed choice of Σ and an arbitrary choice of the U-pseudoidentity $u = v$. In case $\Sigma = \{w = z\}$ consists of a single pseudoidentity, we also say that $w = z$ is *h-strong* if so is Σ.

A U-pseudoidentity $u = v$ is said to be *t-strong* (*within* U*, or for* U) if the statement in the conjecture holds for the given fixed choice of $u = v$ and an arbitrary choice of Σ. The letters h and t in "h/t-strong" are meant to refer to whether the pseudoidentities appear as hypotheses or as thesis in the proofs.

The conjecture also involves an ambient pseudovariety U with respect to which pseudoidentities are taken. We say that the pseudovariety U is *strong* if the statement in the conjecture holds, that is, it holds for every set Σ of U-pseudoidentities and U-pseudoidentity $u = v$.

Taking into account the results of [2], Section 3.8, showing that Σ is h-strong is equivalent to showing that $\tilde{\Sigma}$ is a profinite congruence in the sense of [36], Page 139, that is, for every finite alphabet A, that $\tilde{\Sigma}$ is a closed congruence on

$\overline{\Omega}_A\mathsf{U}$ such that the quotient topological algebra $\overline{\Omega}_A\mathsf{U}/\tilde{\Sigma}$ is a profinite algebra. A further equivalent formulation of this property is that, given any two distinct $\tilde{\Sigma}$-classes, there is a clopen union of $\tilde{\Sigma}$-classes separating them. The difficulty in establishing this property in general is to obtain $\tilde{\Sigma}$-saturation of such clopen sets. In case all congruence classes are determined by a single class, as in the group case, this program is much easier to achieve (cf. Section 8).

We are able to prove the conjecture in several cases of interest. The remainder of the paper is concerned with gathering evidence for the conjecture.

4. A transfer result

The purpose of this section is to show that it is possible to extend the validity of the conjecture within a certain ambient pseudovariety V to a larger pseudovariety U provided that V admits an h-strong basis within U. We first prove that the converse is also true without the additional assumption on a basis of V.

We denote by $\pi_{A,\mathsf{U},\mathsf{V}}$ the natural continuous homomorphism $\overline{\Omega}_A\mathsf{U} \to \overline{\Omega}_A\mathsf{V}$, which fixes the free generators. Where it is clear from the context what some of A, U, and V are supposed to be, we may drop them from the preceding notation and, in particular, we may simply write π.

Proposition 4.1. *Let* U *be a pseudovariety and let* V *be a subpseudovariety of* U. *Suppose that the set* Γ *of* U*-pseudoidentities defines a subpseudovariety of* V *and* Γ *is h-strong within* U. *Then the set* $\Gamma' = \{\pi(u) = \pi(v) : (u = v) \in \Gamma\}$ *is h-strong within* V.

Proof. Let $u, v \in \overline{\Omega}_A\mathsf{U}$ be such that the V-pseudoidentity $\pi(u) = \pi(v)$ holds in the pseudovariety $[\![\Gamma']\!]_{\mathsf{V}} = [\![\Gamma]\!]_{\mathsf{U}}$. Then, the U-pseudoidentity $u = v$ holds in the pseudovariety $[\![\Gamma]\!]_{\mathsf{U}}$. Since we assume that Γ is h-strong within U, it follows that there is a proof of the pseudoidentity $u = v$ from Γ. Projecting by π into $\overline{\Omega}_A\mathsf{V}$ all steps in such a proof, we obtain a proof of $\pi(u) = \pi(v)$ from Γ'; more precisely, one may easily prove by induction on the ordinal α that $(\pi \times \pi)(\Gamma_\alpha) \subseteq \Gamma'_\alpha$. Hence, Γ' is h-strong within V. □

Going in the opposite direction is more interesting, but requires an additional assumption.

Proposition 4.2. *Let* U *be a pseudovariety and let* V *be a subpseudovariety of* U. *Suppose that* V *admits a basis* Σ *of* U*-pseudoidentities that is h-strong within* U *and consists of t-strong pseudoidentities. Let* Γ *be a set of* U*-pseudoidentities such that* $[\![\Gamma]\!]_{\mathsf{U}} \subseteq \mathsf{V}$. *If the set* $\Gamma' = \{\pi(u) = \pi(v) : (u = v) \in \Gamma\}$ *is h-strong within* V, *then the set* Γ *is h-strong within* U.

Proof. Let $u = v$ be a U-pseudoidentity valid in the pseudovariety $[\![\Gamma]\!]_{\mathsf{U}}$. This means that the V-pseudoidentity $\pi(u) = \pi(v)$ holds in $[\![\Gamma']\!]_{\mathsf{V}} = [\![\Gamma]\!]_{\mathsf{U}}$. Since Γ' is h-strong within V, $\pi(u) = \pi(v)$ is provable from Γ'. We need to show that $u = v$ is also provable from Γ. We first claim that there are u' and v' such that $u' = v'$ is provable from Γ, $\pi(u') = \pi(u)$, and $\pi(v') = \pi(v)$. Before proving the claim, we show how it allows us to conclude the proof of the proposition. Since Σ is h-strong within U, it follows that Σ proves $u' = u$ and $v' = v$. On the other hand, since the pseudovariety $[\![\Gamma]\!]$ is contained in V, it satisfies all pseudoidentities from Σ and, as these are assumed to be t-strong, they are provable from Γ. Hence, Γ proves $u' = u$, $v' = v$ and, assuming the claim, also $u' = v'$, which entails that Γ proves $u = v$. Thus, it remains to establish the claim.

Consider the sets $\Gamma'_\alpha = (\Gamma')_\alpha$ defined as in Section 3. The proof will be complete once we establish the above claim that every pseudoidentity in Γ'_α is of the form $\pi(w) = \pi(z)$ for some pseudoidentity $w = z$ provable from Γ. To prove the claim, we proceed by transfinite induction on α.

In case $\alpha = 0$, we have a pseudoidentity of the form

$$\mathbf{t}\big(\varphi\big(\pi(w)\big), \pi(s_1), \dots, \pi(s_n)\big) = \mathbf{t}\big(\varphi\big(\pi(z)\big), \pi(s_1), \dots, \pi(s_n)\big), \tag{1}$$

where $\mathbf{t}$ is a term, $w = z$ is a pseudoidentity from Γ, say over the set of variables B, $\varphi : \overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_A\mathsf{V}$ is a continuous homomorphism, and $s_1, \dots, s_n \in \overline{\Omega}_A\mathsf{U}$. By the universal property of relatively free profinite algebras, there is a continuous homomorphism $\psi : \overline{\Omega}_B\mathsf{U} \to \overline{\Omega}_A\mathsf{U}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\overline{\Omega}_B\mathsf{U} & \xrightarrow{\ \psi\ } & \overline{\Omega}_A\mathsf{U} \\
\big\downarrow{\scriptstyle \pi_B} & & \big\downarrow{\scriptstyle \pi_A} \\
\overline{\Omega}_B\mathsf{V} & \xrightarrow{\ \varphi\ } & \overline{\Omega}_A\mathsf{V}
\end{array}$$

It follows that the pseudoidentity (1) is obtained from

$$\mathbf{t}\big(\psi(w), s_1, \dots, s_n\big) = \mathbf{t}\big(\psi(z), s_1, \dots, s_n\big)$$

by applying π to each member, which completes the basic step $\alpha = 0$ of the induction.

Suppose next that $\alpha = 2\beta + 1$ and that the pseudoidentities $w_i = w_{i+1}$ $(i = 0, \dots, k-1)$ belong to $\Gamma'_{2\beta}$. By induction hypothesis, there exist $w''_i, w'_{i+1} \in \overline{\Omega}_A\mathsf{U}$ such that the pseudoidentity $w''_i = w'_{i+1}$ is provable from Γ, $\pi(w'_i) = w_i$, and $\pi(w''_{i+1}) = w_{i+1}$. In particular, we have $\pi(w'_i) = w_i = \pi(w''_i)$ for $i = 1, \dots, k-1$, and so each pseudoidentity $w'_i = w''_i$ is also provable from Γ. Hence, $w_0 = w_k$ is provable from Γ.

For the case $\alpha = 2\beta + 2$, consider a sequence $(w_n = z_n)_n$ of U-pseudoidentities such that each $\pi(w_n) = \pi(z_n)$ belongs to $\Gamma'_{2\beta+1}$ and suppose that the sequence $\big(\pi(w_n) = \pi(z_n)\big)_n$ converges to $\pi(w) = \pi(z)$. By compactness and continuity of π, we may as well assume that $(w_n = z_n)_n$ converges to $w = z$. Since each pseudoidentity $w_n = z_n$ is provable from Γ by induction hypothesis, $w = z$ is also provable from Γ.

Since the case of limit ordinals is trivial, the transfinite induction is complete and the claim is established. □

With essentially the same arguments, one may replace the t-strongness hypothesis by provability of Σ from Γ, which yields the following corollary.

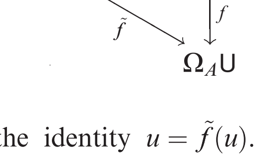
Corollary 4.3. *Let* U *be a pseudovariety and let* V *be a subpseudovariety of* U. *Suppose that* V *admits a basis of* U*-pseudoidentities* Σ *that is h-strong within* U *and that* V *is strong. Then every set of* U*-pseudoidentities from which* Σ *is provable is h-strong within* U.

5. Locally finite sets of identities

Recall that a variety is *locally finite* if all its finitely generated algebras are finite. We also say that a set of pseudoidentities is *locally finite* if it defines a pseudovariety which is contained in some locally finite variety. Note that, if Σ is a locally finite set of identities, then the variety $[\Sigma]$ is generated by $[\![\Sigma]\!]$ since every variety is generated by its finitely generated free members.

Theorem 5.1. *Every locally finite set of identities is h-strong.*

Proof. Consider a locally finite set Σ of identities. Let A be a finite set and let $\pi = \pi_{A,\mathsf{U},[\![\Sigma]\!]}$ be the natural continuous homomorphism. For each of the finitely many elements s of $\overline{\Omega}_A[\![\Sigma]\!]$ choose an element $f(s)$ of $\Omega_A\mathsf{U}$ such that $\pi\big(f(s)\big) = s$. Let $\tilde{f} = f \circ \pi$ and note that, since π is continuous, so is $\tilde{f}$.



For $u \in \Omega_A\mathsf{U}$, consider the identity $u = \tilde{f}(u)$. Since $\pi(u) = \pi \circ f \circ \pi(u) = \pi\big(\tilde{f}(u)\big)$, it is valid in the pseudovariety $[\![\Sigma]\!]$. As the basis Σ is locally finite, it is also valid in the variety $[\Sigma]$. By the completeness theorem for equational logic, it follows that the identity $u = \tilde{f}(u)$ is algebraically provable from Σ.

Finally, consider an arbitrary pseudoidentity $u = v$ with $u, v \in \overline{\Omega}_A \mathsf{U}$ and suppose that it is valid in $[\![\Sigma]\!]$, that is, $\pi(u) = \pi(v)$. Let $(u_n)_n$ and $(v_n)_n$ be sequences in $\Omega_A \mathsf{U}$ converging respectively to u and v. Since π is continuous and $\overline{\Omega}_A[\![\Sigma]\!]$ is a discrete space, we may as well assume that the sequences $\big(\pi(u_n)\big)_n$ and $\big(\pi(v_n)\big)_n$ are constant. It follows that each of the identities $u_n = v_n$ is valid in $[\![\Sigma]\!]$, whence the equality $\tilde{f}(u_n) = \tilde{f}(v_n)$ holds. By the preceding paragraph, we deduce that the identities $u_n = \tilde{f}(u_n) = v_n$ are algebraically provable from Σ. Hence, the pseudoidentity $u = v$ is, sidewise, the limit of the sequence $(u_n = v_n)_n$ of algebraically provable pseudoidentities, which shows that $u = v$ is provable from Σ, thereby establishing that Σ is h-strong. □

For a locally finite set of identities, we may take any basis of a variety generated by a single finite algebra. A classical example of locally finite identity for semigroups which is not of this type is $x^2 = x$ (see, for instance, [29], Theorems IV.4.7 and IV.4.9).

Along the same lines of the proof of Theorem 5.1, we may prove the following result.

Theorem 5.2. *Every locally finite set of identities defines a strong pseudovariety.*

Proof. Let V be the pseudovariety defined by a locally finite set Σ of identities and let $\mathscr{V}$ be the variety defined by Σ. Let Γ be a set of V-identities and let $u, v \in \Omega_A \mathsf{V} = \overline{\Omega}_A \mathsf{V}$ be such that the identity $u = v$ is valid in $[\![\Gamma]\!]_{\mathsf{V}}$. Let Γ' be a set of liftings of the elements of Γ. Since Σ is locally finite, the variety $[\Gamma] = [\Sigma \cup \Gamma']$ is generated by the pseudovariety $[\![\Gamma]\!]_{\mathsf{V}}$ and so the variety $[\Sigma \cup \Gamma']$ also satisfies the identity $u = v$. By the completeness theorem for equational logic, we deduce that $u = v$ is provable from $\Sigma \cup \Gamma'$. Since, in proofs within the ambient pseudovariety $\mathsf{V} = [\![\Sigma]\!]$, the identities from Σ are taken for granted, it follows that $u = v$ is provable from Γ within V. □

Combining Theorems 5.1 and 5.2 with Corollary 4.3, we obtain the following result.

Corollary 5.3. *Every set of pseudoidentities which proves a locally finite set of identities is h-strong.*

It would be nice to replace the provability assumption in Corollary 5.3 by the hypothesis that the given set of pseudoidentities is locally finite. This would follow from Corollary 4.3 if we could show that every identity is t-strong, which is a particular case of the conjecture that has not been established in general.

6. T-strongness

The main proposition of this section gathers the statement of t-strongness of several pseudoidentities. Some of these results play a role in the application of Proposition 4.2 in later sections.

We start with a simple lemma which is used in several points in the sequel.

Lemma 6.1. *If $x^\alpha = x^\beta$ is a nontrivial one-variable* M*-pseudoidentity, then it proves the pseudoidentity $x^\alpha = x^{\alpha+\omega}$.*

Proof. We start by noting that $x^\alpha = x^\beta$ proves $x^\alpha = x^{\alpha+\omega}$ if and only if it proves $x^\beta = x^{\beta+\omega}$. To establish it, by symmetry, we may assume that $x^\alpha = x^\beta$ proves $x^\beta = x^{\beta+\omega}$. Then, it also proves $x^\alpha = x^\beta = x^\beta x^\omega = x^\alpha x^\omega = x^{\alpha+\omega}$.

The cases where α or β are infinite are now immediate since ω is the neutral element of the additive group $\hat{\mathbb{Z}}$. Thus, it remains to consider the case where both α and β are finite and, by the equivalence in the preceding paragraph, we may as well assume that $\beta > \alpha$. We may then write the given pseudoidentity in the form $x^\alpha = x^\alpha x^{\beta-\alpha}$ which proves algebraically $x^\alpha = x^\alpha x^{(\beta-\alpha)n!}$ for every positive integer n. Taking the limit as n goes to infinity, we deduce that $x^\alpha = x^\beta$ proves $x^\alpha = x^\alpha x^{(\beta-\alpha)\omega} = x^{\alpha+\omega}$. □

The following finite semigroups play a role below:

- Sl_2 stands for the two-element semilattice;
- for positive integers m and n, let $B(m,n)$ denote the rectangular band $m \times n$, consisting of the pairs (i,j) with $1 \le i \le m$ and $1 \le j \le n$, where multiplication is described by $(i,j)(k,\ell) = (i,\ell)$;
- for positive integers m and n, let $C_{m,n} = \langle a : a^m = a^{m+n} \rangle$ be the monogenic semigroup with $m+n-1$ elements and maximal subgroup with n elements;
- for a positive integer n, let C_n be the cyclic group of order n;
- B_2 is the five-element aperiodic Brandt semigroup, which is given by the presentation

$$\langle a, b; aba = a, bab = b, a^2 = b^2 = 0 \rangle$$

as a semigroup with zero;
- N is the semigroup with zero given by the presentation

$$\langle a, b; a^2 = b^2 = ba = 0 \rangle;$$

- T is the semigroup with zero given by the presentation

$$\langle e, a; e^2 = e, ea = a, ae = 0 \rangle.$$

We say that two sets of U-pseudoidentities are *equivalent* if every pseudoidentity from each set is provable from the other set. Another simple result that is useful below is the following lemma.

Lemma 6.2. *Let* Γ *be a set of pseudoidentities and suppose that* Γ *is equivalent to a single pseudoidentity* ε. *If each pseudoidentity in* Γ *is t-strong, then so is* ε.

Proof. Let Σ be a set of pseudoidentities and suppose that $[\![\Sigma]\!]$ satisfies ε. From the equivalence hypothesis, it follows that $[\![\Sigma]\!]$ satisfies each pseudoidentity γ from Γ. Since γ is t-strong, we deduce that Σ proves γ. By the equivalence hypothesis again, we conclude that Σ proves ε. □

We are now ready for the announced proposition.

Proposition 6.3. *Each of the following pseudoidentities is t-strong within* M*:*

(i) $x^\omega = 1$;

(ii) $x^{\omega+1} = x$;

(iii) $x^{\omega+1} = x^\omega$;

(iv) $(x^\omega y)^\omega x^\omega = (x^\omega y)^\omega$;

(v) $(x^\omega y)^\omega = (y x^\omega)^\omega$;

(vi) $\big((xy)^\omega x (xy)^\omega\big)^\omega = (xy)^\omega$;

(vii) $(xy)^\omega x (xy)^\omega = (xy)^\omega$;

(viii) $\big((xy)^\omega x\big)^\omega = (xy)^\omega$;

(ix) $(xy)^\omega = (yx)^\omega$;

(x) $(xy)^\omega x = (xy)^\omega$;

(xi) $x^\omega y = y x^\omega$.

Proof. For simplicity, we may refer to one of (i)–(xi) as meaning either a part of the proposition or the pseudoidentity in it; which is the case should be clear from the context.

Let Σ be a set of M-pseudoidentities. For each of the pseudoidentities ε in the statement of the proposition, we assume that the pseudovariety $[\![\Sigma]\!]$ satisfies ε and show that Σ proves ε.

(i) Since all finite semigroups satisfying Σ are groups, in particular Sl_2 fails some pseudoidentity $u = v$ from Σ, which means that there is some variable that occurs in one of the sides but not in the other. Substituting x^ω for that variable and 1 for all others, we conclude that $x^\omega = 1$ is algebraically provable from Σ.

(ii) Since the monoid $C_{2,1}^1 = \{1, a, 0\}$ (with $a^2 = 0$) fails (ii), it must fail some pseudoidentity $u = v$ from Σ under some suitable evaluation. If such an evaluation gives the values 0 and 1 for u and v, then u and v do not involve the same variables, so that, substituting x^ω for one variable and 1 for all others, one gets $x^\omega = 1$, from which (ii) follows. Otherwise, one of the sides, say u, is evaluated to a and the other to either 0 or 1. Substituting 1 for all variables in u that are not evaluated to a and x for every other variable, we obtain a pseudoidentity of the form $x = x^\alpha$, with $\alpha \in \hat{\mathbb{N}} \setminus \{1\}$. By Lemma 6.1, each such pseudoidentity proves (ii).

(iii) Each of the cyclic groups of prime order fails the pseudoidentity (iii). For a pseudoword w and a variable x, let w_x be the pseudoword that is obtained from w by substituting 1 for every variable except x. Since the cyclic group C_p satisfies a pseudoidentity $u = v$ if and only if, for every variable x, it satisfies the pseudoidentity $u_x = v_x$, it follows that, if C_p fails Σ, then there is $\alpha_p \in \hat{\mathbb{N}} \setminus \mathbb{N} = \hat{\mathbb{Z}}$ such that p does not divide α_p and Σ proves $x^{\alpha_p} = x^\omega$.

Let α be a greatest common divisor of the α_p, which is a limit of a sequence of linear combinations of the α_p with nonnegative integer coefficients. Since each prime p does not divide α_p, it cannot divide α. Hence, α is invertible in the ring $\hat{\mathbb{Z}}$ and there exists $\beta \in \hat{\mathbb{Z}}$ such that $\alpha\beta = \omega + 1$. Now, if $\gamma = \sum_{i=1}^{n} \gamma_i \alpha_{p_i}$ with the coefficients γ_i in $\mathbb{N}$, then Σ proves $x^\gamma = x^\omega$ by raising both sides of each pseudoidentity $x^{\alpha_{p_i}} = x^\omega$ to the γ_i power and multiplying the results side by side. The pseudoidentity $x^\alpha = x^\omega$ is therefore a limit of pseudoidentities provable from Σ, and hence it is provable from Σ. Finally, raising both sides of $x^\alpha = x^\omega$ to the β power, we deduce that the pseudoidentity (iii) is provable from Σ.

(iv) As it is easy to see, and well known, $B(1,2)^1$ fails a pseudoidentity $u = v$ if and only if, from right to left, the order of first occurrences of variables in u and v is not the same. In particular, the monoid $B(1,2)^1$ fails (iv) and, therefore, it fails some pseudoidentity from Σ. Hence, there is a substitution that sends all variables but two to 1 that yields from some pseudoidentity in Σ a two-variable pseudoidentity of the form $ux = vy$, where x and y are distinct variables, or a nontrivial pseudoidentity $u = 1$ or $1 = u$. In the latter case, by substituting all variables by x^ω, we conclude that Σ proves $x^\omega = 1$ and, hence, every pseudoidentity in which both sides are products of ω powers, as is the case of (iv). Thus, it remains to consider the former case. Applying the substitution $x \mapsto x^\omega$ and $y \mapsto x^\omega y$ and raising both sides to the ω power, we conclude that Σ proves the pseudoidentity (iv).

(v) Noting that the sets of pseudoidentities $\{(x^\omega y)^\omega x^\omega = (x^\omega y)^\omega, x^\omega (y x^\omega)^\omega = (y x^\omega)^\omega\}$ and $\{(x^\omega y)^\omega = (y x^\omega)^\omega\}$ are equivalent, in view of (iv) and its dual, it suffices to apply Lemma 6.2.

(vi) The evaluation of x to a and y to b shows that the monoid B_2^1 fails the pseudoidentity (vi). Hence, there is some pseudoidentity $u = v$ from Σ and an evaluation of the variables in B_2^1 which yields different values for u and v. A variable being assigned the value 1 corresponds to deleting that variable. For all other values in B_2^1, since B_2 is generated by $\{a, b\}$ as a semigroup, we may first substitute the variable by a word in the variables x and y and then evaluate x to a and y to b. Thus, under the assumption that B_2^1 fails $u = v$, we conclude that there is a pseudoidentity $u' = v'$ in x and y that can be proved from Σ and which fails in B_2^1 under the evaluation $x \mapsto a$ and $y \mapsto b$. Under such an evaluation, not both sides are evaluated to 0. If one of the sides is 1 then, substituting x for y, we get a pseudoidentity of the form $x^\alpha = 1$, which proves $x^\omega = 1$ and, therefore, also the pseudoidentity (vi). Otherwise, one of the sides of the pseudoidentity $u' = v'$, say u', must be a factor of $(xy)^\omega$ while v' either admits x^2 or y^2 as a factor or does not start or end with the same letter as u'. By multiplying both sides of $u' = v'$ by suitable factors of $(xy)^\omega$, we may prove from $u' = v'$ a pseudoidentity $u'' = v''$ of the form $(xy)^\omega = w$ where w is a pseudoword that admits at least one of the words x^2 and y^2 as a factor, starts with x and ends with y. Substituting $(xy)^\omega x$ for x and $y(xy)^\omega$ for y, we obtain from $u'' = v''$ a pseudoidentity of the form

$$(xy)^\omega = (xy)^\alpha x (xy)^\beta t \tag{2}$$

or of the form

$$(xy)^\omega = (xy)^\alpha y (xy)^\beta t, \tag{3}$$

where α and β are infinite exponents and t is some pseudoword. From the pseudoidentity (2), we may prove, algebraically,

$$(xy)^\omega = (xy)^\alpha x \cdot (xy)^\omega \cdot (xy)^\beta t = \cdots = \big((xy)^\alpha x\big)^{n!} (xy)^\omega \big((xy)^\beta t\big)^{n!}$$

and so, taking limits, also

$$(xy)^\omega = \big((xy)^\alpha x\big)^\omega (xy)^\omega \big((xy)^\beta t\big)^\omega$$

which entails

$$\begin{aligned}(xy)^\omega &= \big((xy)^\alpha x\big)^\omega \big((xy)^\alpha x\big)^\omega (xy)^\omega \big((xy)^\beta t\big)^\omega \\ &= \big((xy)^\alpha x\big)^\omega (xy)^\omega = \big((xy)^\alpha x (xy)^\omega\big)^\omega.\end{aligned}$$

Similarly, from

$$(xy)^\omega = \big((xy)^\alpha x (xy)^\omega\big)^{\omega-1} (xy)^\alpha \cdot (xy)^\omega \cdot x (xy)^\omega,$$

concentrating on the rightmost factor, we may deduce the desired pseudoidentity (vi). If we start from the pseudoidentity (3) instead of (2), we reach similarly the pseudoidentity $(xy)^\omega = \big((xy)^\omega y(xy)^\omega\big)^\omega$. Interchanging x and y, we obtain $(yx)^\omega = \big((yx)^\omega x(yx)^\omega\big)^\omega$. Multiplying both sides on the left by x and on the right by $y(xy)^{\omega-1}$, and rearranging the right hand side (using equalities valid in M) yields the pseudoidentity (vi).

(vii) First note that (vii) proves (vi) by simply raising both sides to the ω power; it also proves (iii) by substituting y by x and using equalities that are valid in M. Conversely, from (vi), using (iii), we deduce that

$$\begin{aligned}(xy)^\omega x(xy)^\omega &= (xy)^\omega (xy)^\omega x(xy)^\omega = \big((xy)^\omega x(xy)^\omega\big)^\omega (xy)^\omega x(xy)^\omega \\ &= \big((xy)^\omega x(xy)^\omega\big)^{\omega+1} = \big((xy)^\omega x(xy)^\omega\big)^\omega = (xy)^\omega.\end{aligned}$$

It remains to apply Lemma 6.2 and the above.

(viii) By applying the substitution $x \mapsto x^\omega$ we transform (viii) into the pseudoidentity $\big((x^\omega y)^\omega x^\omega\big)^\omega = (x^\omega y)^\omega$. Since $(x^\omega y)^\omega x^\omega$ is an idempotent, this shows that (viii) proves (iv). Similarly, upon multiplication of both sides of (viii) on the right by $(xy)^\omega$, we conclude that (viii) proves (vi). In view of the above and Lemma 6.2, it remains to establish that, together, the pseudoidentities (iv) and (vi) prove (viii). Indeed, the substitution $x \mapsto (xy)^\omega x,\ y \mapsto (xy)^\omega$ in (iv) gives $\big((xy)^\omega x\big)^\omega = \big((xy)^\omega x(xy)^\omega\big)^\omega$. Combining the latter pseudoidentity with (vi), by transitivity we obtain (viii).

(ix) Note first that (ix) proves the pseudoidentities

$$(xy)^\omega = (yx)^\omega = y(xy)^{\omega-1} \cdot (xy)^\omega \cdot (xy)^\omega x.$$

Hence, (ix) proves $(xy)^\omega = \big(y(xy)^{\omega-1}\big)^{n!} \cdot (xy)^\omega \cdot \big((xy)^\omega x\big)^{n!}$; by taking limits, we get $(xy)^\omega = \big(y(xy)^{\omega-1}\big)^\omega \cdot (xy)^\omega \cdot \big((xy)^\omega x\big)^\omega$ and, therefore, (ix) proves (viii). Since (ix) is its own left right dual, (ix) also proves the dual of (viii), namely the pseudoidentity $\big(x(yx)^\omega\big)^\omega = (yx)^\omega$. Conversely, from (viii) and its dual, we may prove

$$(xy)^\omega = \big(y(xy)^\omega\big)^\omega = \big((yx)^\omega y\big)^\omega = (yx)^\omega.$$

In view of previous parts of the proposition, it suffices to invoke Lemma 6.2.

(x) Substituting y by x in the pseudoidentity (x) yields (iii) while, raising both sides to the ω power we obtain (viii). Once again, in view of Lemma 6.2, it suffices to show that, together, (iii) and (viii) also prove (x) which can be done as in proof of (vii):

$$(xy)^\omega = \big((xy)^\omega x\big)^\omega = \big((xy)^\omega x\big)^\omega (xy)^\omega x = (xy)^\omega (xy)^\omega x = (xy)^\omega x.$$

(xi) Raising both sides of (xi) to the ω power, we obtain (v) and so Σ proves (v). On the other hand, the monoid T^1 and its left right dual fails the pseudoidentity (xi). Hence, each of them fails some pseudoidentity from Σ.

Let us consider first the fact that T^1 fails some pseudoidentity from Σ. If there is such a pseudoidentity in which a variable appears on one side but not the other then, as in (i), Σ proves $x^\omega = 1$, which entails (xi). Hence, as $T^1 \setminus \{a\}$ is a semilattice, to fail a pseudoidentity from Σ, some variable must be evaluated by a and occur only as the last letter that is not evaluated to 1 in one of the sides of the pseudoidentity. Substituting $x^\omega y$ for that variable, 1 for every variable that is evaluated to 1, and x^ω for every other variable, we deduce that Σ proves a two-variable pseudoidentity of the form $x^\omega y = w$, where w is a pseudoword that admits yx as a factor. If y occurs more than once in w then, substituting x by 1, we get a nontrivial pseudoidentity $y = y^\alpha$, which yields $x = x^{\omega+1}$ by Lemma 6.1. Hence, Σ proves either $x = x^{\omega+1}$ or $x^\omega y = x^\omega y x^\omega$. Working instead with the dual of the monoid T^1, we deduce dually that Σ proves either $x = x^{\omega+1}$ or $yx^\omega = x^\omega y x^\omega$.

Suppose first that Σ proves $x = x^{\omega+1}$. Since Σ also proves (v), it proves the following pseudoidentities:

$$x^\omega y = (x^\omega y)^{\omega+1} = x^\omega (yx^\omega)^\omega y = x^\omega (x^\omega y)^\omega y = (x^\omega y)^\omega y = (yx^\omega)^\omega y.$$

Dually, Σ proves $yx^\omega = y(x^\omega y)^\omega$ and so also (xi). Thus, we may assume that Σ does not prove $x = x^{\omega+1}$. From the above, it follows that Σ proves $x^\omega y = x^\omega y x^\omega = yx^\omega$, as required. □

The choice of the monoids considered in the proof of Proposition 6.3 was guided by several results in the literature, even though such results are not explicitly used in the proof. For a pseudoidentity ε, we essentially take a *complete set of excluded monoids*, that is, a set of finite monoids such that a pseudovariety V satisfies ε if and only if it contains none of the monoids from the set. Such sets can be found in the literature for several pseudoidentities. See [2] for further details. It should be noted that the proof of Proposition 6.3 in fact implies that the sets in question are complete sets of excluded monoids.

We conclude with an example of a t-strong pseudoidentity for semigroups which is used in Section 9.

Proposition 6.4. *The following pseudoidentities are t-strong within* S*:*

(i) $x^{\omega+1} = x$;

(ii) $(xy)^\omega x = x$.

Proof. (i) This may be proved with the same argument as in the proof of Proposition 6.3(ii), using the semigroup $C_{2,1}$ instead of the monoid $C^1_{2,1}$.

(ii) Let Σ be a set of S-pseudoidentities such that $[\![\Sigma]\!]$ satisfies (ii). Substituting y by x in (ii), we obtain the pseudoidentity $x^{\omega+1} = x$. On the other hand, since Sl_2 fails (ii), it also fails some pseudoidentity from Σ, and so there is some variable z that occurs only on one side of that pseudoidentity. Substituting $x^\omega y x^\omega$ for z and x^ω for every other variable, we conclude that Σ proves a pseudoidentity of the form $(x^\omega y x^\omega)^\alpha = x^\omega$ and so also the special case where $\alpha = \omega$, which may be written in the form $(x^\omega y)^\omega x^\omega = x^\omega$. Substituting xy for y, multiplying both sides on the right by x, and using additionally the pseudoidentity $x^{\omega+1} = x$, which may be proved from Σ by (i), we obtain the required pseudoidentity (ii). □

7. H-strongness: the role of reducibility

In this section, we consider a method that allows us to give a class of examples of h-strong sets of pseudoidentities. In all of them, the key property of the pseudovariety $\mathsf{V} = [\![\Sigma]\!]$ is that every pseudoidentity $u = v$ valid in V is the limit of a sequence of identities in a suitable implicit signature σ which are also valid in V. In this case we say that the pseudovariety V is *σ-reducible for the equation $x = y$.*

Proposition 7.1. *Let σ be an implicit signature and let Σ be a set of U-pseudoidentities defining a σ-reducible pseudovariety V for the equation $x = y$. If the variety V^σ admits a basis whose identities are provable from Σ, then Σ is h-strong.*

Proof. Let $u = v$ be a pseudoidentity valid in V. Since V is σ-reducible, there exists a sequence of σ-identities $u_n = v_n$ valid in V that converges to $u = v$ in $\overline{\Omega}_A\mathsf{U} \times \overline{\Omega}_A\mathsf{U}$. In particular, the variety V^σ satisfies each of the identities $u_n = v_n$. If Σ' is a basis of the variety V^σ provable from Σ, then the completeness theorem of equational logic (which holds for an arbitrary algebraic type) guarantees that each σ-identity $u_n = v_n$ is provable from Σ' and, hence, also from Σ. □

We apply below Proposition 7.1 to the usual bases of pseudoidentities of several extensively studied pseudovarieties of semigroups or monoids. Before doing so, it is worth explaining why we use the above terminology introduced in [13] that has been widely adopted in the literature with an apparently different meaning. This connection needs to be clarified in order to justify invoking several published results that are required in our application of Proposition 7.1.

As above, fix an ambient pseudovariety U. We also consider a subpseudovariety V of U. By a *system of equations* we mean a set $\mathscr{S}$ whose elements $u = v$ are formal equalities (that is, pairs) of terms in the algebraic signature of U. Let X be the set of variables that occur in $\mathscr{S}$, a set which we assume to be finite. We are interested in V-*solutions* of $\mathscr{S}$ on a fixed but arbitrary finite set A of generators, which consist of a mapping φ assigning to each variable $x \in X$ an element $\varphi(x)$

of $\overline{\Omega}_A \mathsf{U}$ whose natural extension $\hat{\varphi}$ to terms is such that V satisfies each pseudoidentity $\hat{\varphi}(u) = \hat{\varphi}(v)$ with $u = v$ an equation from $\mathscr{S}$. Such systems are often constrained by assigning to each variable $x \in X$ a clopen subset K_x of $\overline{\Omega}_A \mathsf{U}$. The V-solution φ is said to *satisfy* the constraints if $\varphi(x) \in K_x$ for every $x \in X$.

The key property of the pseudovariety V introduced in [13] is the following. Let σ be an implicit signature, each of whose elements belongs to some $\overline{\Omega}_B \mathsf{U}$, where B is a finite set. We say that V is *σ-reducible for $\mathscr{S}$* if, for every choice of clopen constraints for X, if there is a V-solution of $\mathscr{S}$ satisfying the constraints, then there is such a solution taking its values in $\Omega_A^\sigma \mathsf{U}$, which we call a (V, σ)-*solution*. The following result is a topological reformulation of the definition of reducibility. Here, we view solutions of the system $\mathscr{S}$ as elements of the product space $(\overline{\Omega}_A \mathsf{U})^X$.

Proposition 7.2. *The pseudovariety V is σ-reducible for the system of equations $\mathscr{S}$ over the set of variables X if and only if the set of (V, σ)-solutions is dense in the set of all V-solutions.*

Proof. It is well known that the topological space $\overline{\Omega}_A \mathsf{U}$ is zero-dimensional, meaning that the clopen sets form a basis of the topology. Thus, in the product space $(\overline{\Omega}_A \mathsf{U})^X$, a basis is given by the set of all products of the form $\prod_{x \in X} K_x$ where the mapping $x \mapsto K_x$ is a choice of constraitns for the system $\mathscr{S}$. Hence, a V-solution φ of $\mathscr{S}$ can be arbitrarily approximated by (V, σ)-solutions if and only if, for any given choice of constraints which are satisfied by φ, there is a (V, σ)-solution of $\mathscr{S}$ which satisfies the same constraints. □

In particular, the terminology adopted at the beginning of the section is consistent with that from [13] provided the underlying algebraic type (but not necessarily σ) is finite since then topological closure is captured by taking limits of sequences.

The property of σ-reducibility of a pseudovariety V was conceived as part of a strong form of decidability called *σ-tameness*. The remaining requirements for σ-tameness are computability assumptions, namely: the pseudovariety V is assumed to be recursively enumerable, the signature σ is also assumed to be recursively enumerable and to consist of operations that are computable in elements of V, and the word problem in $\Omega_A^\sigma \mathsf{V}$ is supposed to be decidable. The computability assumptions on the pseudovariety V and the implicit signatures considered in all our examples are immediately verified.

It should be pointed out that the word problem in $\Omega_A^\sigma \mathsf{V}$ is algorithmically solvable if and only if the variety V^σ admits a recursive basis of identities. Besides σ-reducibility for the identity $x = y$, it is the knowledge of such a basis that underlies all our applications of Proposition 7.1. Thus, our results rely more properly on σ-tameness of V rather than just σ-reducibility for the identity $x = y$.

We are now ready to present our concrete examples of evidence for the conjecture which are obtained as applications of Proposition 7.1. In the following, the ambient pseudovariety U will be either the pseudovariety S of all finite semigroups or the pseudovariety M of all finite monoids. The implicit signatures involved are often either $\kappa = \{\_ \cdot \_, \_^{\omega-1}\}$ or $\{\_ \cdot \_, \_^{\omega}\}$ the latter of which, by abuse of notation, we also denote ω.

7.1. Some simple examples. Our first example is given by the pseudoidentity $x^{\omega+1} = x^{\omega}$, which defines the pseudovariety A of all finite aperiodic monoids. In view of Schützenberger's characterization of star-free languages [38], this is a very important pseudovariety.

Theorem 7.3. *For* $\mathsf{U} = \mathsf{M}$, *the pseudoidentity* $x^{\omega+1} = x^{\omega}$ *is h-strong.*

Proof. The first key ingredient here is that the pseudovariety A is ω-reducible for the equation $x = y$. This is proved in [10], Corollary 3.2 based on Henckell's computation of A-pointlike sets of finite monoids [27], [28].

The second key ingredient is a basis of identities for the variety A^{ω} obtained by McCammond [31], [9]. It consists of the following identities:

$$(xy)z = x(yz), \qquad x1 = 1x = x$$
$$(x^{\omega})^{\omega} = (x^{r})^{\omega} = x^{\omega}x^{\omega} = x^{\omega} \qquad (r \geq 2)$$
$$(xy)^{\omega}x = x(yx)^{\omega}$$
$$x^{\omega}x = xx^{\omega} = x^{\omega}$$

Except for the identities in the last line, which are immediately provable from $x^{\omega+1} = x^{\omega}$, all the other identities are valid in all finite monoids and so they require no proof in our proof setup within the ambient pseudovariety M. Applying Proposition 7.1, we conclude that the pseudoidentity $x^{\omega+1} = x^{\omega}$ is h-strong. □

Our next example is the usual basis of the pseudovariety R of all finite $\mathscr{R}$-trivial monoids.

Theorem 7.4. *For* $\mathsf{U} = \mathsf{M}$, *the pseudoidentity* $(xy)^{\omega}x = (xy)^{\omega}$ *is h-strong.*

Proof. The ω-reducibility of R for the equation $x = y$ was first proved in [7]. In fact, the same holds for arbitrary systems of ω-equations [8]. The following basis of identities for the variety R^{ω} was obtained in [17]:

$$(x^{r})^{\omega} = (x^{\omega})^{\omega} = x^{\omega} \qquad (r \geq 2)$$
$$(xy)z = x(yz), \qquad x1 = 1x = x$$
$$(xy)^{\omega} = (xy)^{\omega}x = (xy)^{\omega}x^{\omega} = x(yx)^{\omega}.$$

Of all the above identities the only one that is not obviously provable from $(xy)^\omega x = (xy)^\omega$ is $(xy)^\omega x^\omega = (xy)^\omega$. Yet, iterating the hypothesis, one gets $(xy)^\omega x^{n!} = (xy)^\omega$, whence also $(xy)^\omega x^\omega = (xy)^\omega$ by taking limits. □

While the two previous examples could be reformulated in the language of semigroups, as the pseudovariety of semigroups generated in each case has no additional monoids, for the next one the monoids in the pseudovariety constitute a much smaller class. Indeed, we now consider the pseudovariety LSl of all finite semigroups which are locally semilattices, that is, the pseudovariety defined by the set in the following result.

Theorem 7.5. *For* $\mathsf{U} = \mathsf{S}$, *the set of pseudoidentities*

$$\Sigma = \{x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega, x^\omega y x^\omega y x^\omega = x^\omega y x^\omega\}$$

is h*-strong.*

Proof. The ω-reducibility of LSl for the equation $x = y$ was first proved in [24], where graph systems of equations are also considered.[3] The following basis of identities for the variety LSl may be found in [21]:

$$\begin{aligned}
&(x^r)^\omega = x^\omega x^\omega = x^\omega x = x^\omega \qquad (r \geq 2)\\
&(xy)z = x(yz), \qquad (xy)^\omega x = x(yx)^\omega\\
&x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega, \qquad x^\omega y x^\omega y x^\omega = x^\omega y x^\omega\\
&(xy^\omega z)^\omega = (xy^\omega z)^2.
\end{aligned}$$

Of all the above identities, the only one that requires a proof from Σ is the last one. We first note that the second pseudoidentity from Σ immediately infers $(xy^\omega z)^3 = (xy^\omega z)^2$, which entails $(xy^\omega z)^{n+1} = (xy^\omega z)^n$ for every $n \geq 2$, whence also $(xy^\omega z)^{n!} = (xy^\omega z)^2$. Taking limits, we may further prove $(xy^\omega z)^\omega = (xy^\omega z)^2$. In view of Proposition 7.1, this concludes the proof of the theorem. □

7.2. Monoids in which regular elements are idempotents. The next example is given by two simple bases of pseudoidentities for the pseudovariety DA of all finite semigroups whose regular $\mathscr{D}$-classes are aperiodic subsemigroups, a property which is equivalent to all regular elements being idempotents. For the relevance of the pseudovariety DA, see [40].

[3] Later, reducibility of LSl was extended to arbitrary systems of κ-equations [22]. In view of the well-known decomposition $\mathsf{LSl} = \mathsf{Sl} * \mathsf{D}$ (see, for instance, [2], Section 10.8), a generalization in a different direction has been obtained in [23] where, in particular, it is proved that if V is κ-reducible for the equation $x = y$, then so is $\mathsf{V} * \mathsf{D}$. This was later extended to the case of graph systems of equations in [25]. Here, by a graph system of equations we mean the system associated with a finite directed graph which, for each edge $x \xrightarrow{y} z$ in the graph, includes the equation $xy = z$.

Theorem 7.6. *For* $\mathsf{U} = \mathsf{M}$*, the sets of pseudoidentities*

$$\Sigma = \{x^{\omega+1} = x^{\omega}, (xy)^{\omega}(yx)^{\omega}(xy)^{\omega} = (xy)^{\omega}\}$$
$$\Gamma = \{((xy)^{\omega}x)^{2} = (xy)^{\omega}x\}$$

are h-strong.

Proof. The ω-reducibility of DA for the equation $x = y$ has been proved in [10]. The following basis of identities for DA^{ω} has been recently obtained [11]:

$$(x^{r})^{\omega} = (x^{\omega})^{\omega} = x^{\omega} \qquad (r \geq 2)$$
$$(xy)^{\omega}x = x(yx)^{\omega}$$
$$(xy)z = x(yz), \qquad x1 = 1x = x$$
$$u^{\omega}vu^{\omega} = u^{\omega},$$

where $u, v \in \Omega_B\mathsf{U}$ for an arbitrary finite alphabet B and every variable that occurs in v also occurs in u. Only the family of identities in the last line requires a proof since the others are valid in every finite monoid.

We start by showing that Σ proves Γ. We may prove algebraically:

$$\begin{aligned}(xy)^{\omega} &= (xy)^{\omega}(yx)^{\omega}(xy)^{\omega} = (xy)^{\omega}(yx)^{\omega+1}(xy)^{\omega}\\ &= (xy)^{\omega}y \cdot (xy)^{\omega} \cdot x(xy)^{\omega} = \cdots = ((xy)^{\omega}y)^{n!}(xy)^{\omega}(x(xy)^{\omega})^{n!}.\end{aligned}$$

Taking limits, we get

$$(xy)^{\omega} = ((xy)^{\omega}y)^{\omega}(xy)^{\omega}(x(xy)^{\omega})^{\omega} \tag{4}$$

which, upon multiplication of both sides on the right by $x(xy)^{\omega}x$, yields

$$\begin{aligned}((xy)^{\omega}x)^{2} &= ((xy)^{\omega}y)^{\omega}(xy)^{\omega}(x(xy)^{\omega})^{\omega+1}x\\ &= ((xy)^{\omega}y)^{\omega}(xy)^{\omega}(x(xy)^{\omega})^{\omega}x \underset{(4)}{=} (xy)^{\omega}x.\end{aligned}$$

Conversely, we may algebraically prove Σ from Γ as follows. First, substituting x for y in the pseudoidentity $((xy)^{\omega}x)^{2} = (xy)^{\omega}x$ and multiplying both sides by $x^{\omega-1}$, we obtain

$$x^{\omega+1} = x^{\omega-1}((xx)^{\omega}x)^{2} = x^{\omega-1}(xx)^{\omega}x = x^{\omega}.$$

Also, multiplying both sides of the pseudoidentity from Γ on the right by $y(xy)^{\omega-1}$ yields

$$(xy)^{\omega}x(xy)^{\omega} = (xy)^{\omega}. \tag{5}$$

Next, we obtain

$$\begin{aligned}(xy)^{\omega} &= (xy)^{\omega-1}xy(xy)^{\omega} = (xy)^{\omega-1}x(yx)^{\omega}y \underset{(5)}{=} (xy)^{\omega-1}x(yx)^{\omega}y(yx)^{\omega}y \\ &= (xy)^{\omega}(yx)^{\omega}y = (xy)^{\omega}y(xy)^{\omega},\end{aligned}$$

and so

$$(xy)^{\omega}y(xy)^{\omega} = (xy)^{\omega}, \tag{6}$$

which entails

$$\begin{aligned}(xy)^{\omega} &\underset{(6)}{=} (xy)^{\omega}y(xy)^{\omega} \underset{(5)}{=} (xy)^{\omega}y(xy)^{\omega}x(xy)^{\omega} \\ &= (xy)^{\omega}(yx)^{\omega+1}(xy)^{\omega} = (xy)^{\omega}(yx)^{\omega}(xy)^{\omega}.\end{aligned}$$

From hereon, we are thus allowed to use $\Sigma \cup \Gamma$ as hypothesis in our proofs and we assume them without further mention. Recall that our objective is to prove every ω-identity of the form $u^{\omega}vu^{\omega} = u^{\omega}$ under the assumption that every variable in v also occurs in u.

We claim that the following statements hold, where e stands for an arbitrary idempotent in $\overline{\Omega}_A \mathsf{U}$:

(i) if $ewe = e$ may be proved, then $ew^2e = e$ may also be proved;

(ii) if $ew_1e = e = ew_2e$ may be proved, then $ew_1w_2e = e$ may also be proved;

(iii) the pseudoidentity $(xyz)^{\omega}y(xyz)^{\omega} = (xyz)^{\omega}$ is provable.

To establish (i), assume that we have proved $ewe = e$. Then we may also prove $ew = ewew = \cdots = (ew)^n$ and, taking limits, $ew = (ew)^{\omega} = (ew)^{\omega-1}$. Similarly, we can prove $we = (we)^{\omega}$. This yields the following equalities:

$$e = ewe = (ew)^{\omega}e = (ew)^{\omega}(we)^{\omega}(ew)^{\omega}e = ewweewe = ew^2e.$$

For (ii), suppose we have proved $ew_1e = e = ew_2e$. Then, in view of (i), we may prove

$$e = ee = ew_2eew_1e = e \cdot w_2ew_1 \cdot e = e \cdot w_2ew_1 \cdot w_2ew_1 \cdot e = ew_1w_2e.$$

To prove (iii), observe first that, from (5) and (6) we deduce that the pseudoidentities

$$(yzx)^{\omega} = (yzx)^{\omega}yz(yzx)^{\omega} = (yzx)^{\omega}y(yzx)^{\omega} = (yzx)^{\omega}x(yzx)^{\omega}$$

are provable, so that, by (ii), so are $(yzx)^{\omega} = (yzx)^{\omega}yx(yzx)^{\omega}$ and

$$\begin{aligned}(xyz)^{\omega} &= (xyz)^{\omega+1} = x(yzx)^{\omega}yz \underset{(ii)}{=} x(yzx)^{\omega}yzyx(yzx)^{\omega}yz \\ &= (xyz)^{\omega+1}y(xyz)^{\omega+1} = (xyz)^{\omega}y(xyz)^{\omega}.\end{aligned}$$

Let $v = v_1 \dots v_n$, where the v_i are letters. By (iii) since every letter appearing in v also appears in u, we may prove $u^{\omega}v_iu^{\omega} = u^{\omega}$ $(i = 1, \dots, n)$. Applying (ii) $n - 1$ times, we deduce that we may also prove $u^{\omega}vu^{\omega} = u^{\omega}$, as required. □

7.3. $\mathscr{J}$-trivial monoids. The next example consists of the two bases commonly used to describe the pseudovariety J of all finite $\mathscr{J}$-trivial monoids.

Theorem 7.7. *For $\mathsf{U} = \mathsf{M}$, the sets of pseudoidentities*

$$\Sigma = \{x^{\omega+1} = x^{\omega}, (xy)^{\omega} = (yx)^{\omega}\} \quad \textit{and} \quad \Gamma = \{(xy)^{\omega}x = (xy)^{\omega} = y(xy)^{\omega}\}$$

are h-strong.

Proof. It is not difficult to show that J is ω-reducible for the equation $x = y$. In fact, J is ω-reducible for all finite systems of equations, and even of κ-equations [4], Theorem 12.3. On the other hand, the following basis of identities for the variety J^{ω} is given in [2], Section 8.2:

$$\begin{aligned}&(xy)z = x(yz), \qquad x1 = 1x = x, \qquad (x^{\omega})^{\omega} = x^{\omega} \\ &x^{\omega}x = xx^{\omega} = x^{\omega} \\ &(xy)^{\omega} = (yx)^{\omega} = (x^{\omega}y^{\omega})^{\omega}.\end{aligned}$$

The identities in the first line are valid in all finite monoids and, therefore require no proof. In view of Proposition 7.1, to finish the proof it suffices to show that the remaining identities are provable from both Σ and Γ.

In the case of Σ, only the identity $(xy)^{\omega} = (x^{\omega}y^{\omega})^{\omega}$ needs to be considered. The following describes a proof from Σ. First, we do an algebraic proof:

$$(xy)^{\omega} = (xy)^{\omega+1} = x(yx)^{\omega}y = x(xy)^{\omega}y = \dots = x^m(xy)^{\omega}y^m. \tag{7}$$

Hence, we may also prove from Σ

$$(xy)^\omega = (yx)^\omega = y^m (yx)^\omega x^m = y^m (xy)^\omega x^m,$$

which, combined with (7), yields

$$(xy)^\omega = x^m y^m (xy)^\omega x^m y^m \tag{8}$$

Iterating (8), we get an algebraic proof of $(xy)^\omega = (x^m y^m)^m (xy)^\omega (x^m y^m)^m$. Letting $m = n!$ and taking limits, we obtain that Σ proves

$$(xy)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega (x^\omega y^\omega)^\omega. \tag{9}$$

Similarly, we may prove algebraically

$$(x^\omega y^\omega)^\omega = x^\omega y^\omega (x^\omega y^\omega)^\omega = x x^\omega y^\omega (x^\omega y^\omega)^\omega = x(x^\omega y^\omega)^\omega$$

and so also

$$(x^\omega y^\omega)^\omega = xy(x^\omega y^\omega)^\omega = \cdots = (xy)^{n!}(x^\omega y^\omega)^\omega.$$

Taking limits, we get $(x^\omega y^\omega)^\omega = (xy)^\omega (x^\omega y^\omega)^\omega$. Combining with (9) and taking into account that $(x^\omega y^\omega)^\omega$ is idempotent, we finally complete the proof of $(xy)^\omega = (x^\omega y^\omega)^\omega$ from Σ.

For Γ, we first note that, substituting x for y in $(xy)^\omega x = (xy)^\omega$, yields $x^{\omega+1} = x^\omega$. Hence, it suffices to show that the pseudoidentity $(xy)^\omega = (yx)^\omega$ is provable from Γ, which can be established algebraically:

$$(xy)^\omega = y(xy)^\omega = (yx)^\omega y = (yx)^\omega. \qquad \square$$

Note that, in the proof of Theorem 7.7, we alternated several times topological and transitive closure. More precisely, we actually proved that $\tilde{\Sigma} = \Sigma_4$. We do not know whether $\tilde{\Sigma} = \Sigma_3$ but show below that $\tilde{\Sigma} \neq \Sigma_2$.

We start with an auxiliary lemma involving equidivisibility. We say that a semigroup S is *equidivisible* if any two factorizations of the same element admit a common refinement [32]. We say that a pseudovariety (of semigroups or monoids) V is *equidivisible* if, for each finite set A, the semigroup $\overline{\Omega}_A\mathsf{V}$ is equidivisible. The equidivisible pseudovarieties of semigroups have been characterized in [6]. The characterization of equidivisible pseudovarieties of monoids can be derived from it by noting that, for a pseudovariety V of monoids, $\overline{\Omega}_A\mathsf{V} = (\overline{\Omega}_A\mathsf{W})^1$, where W is the pseudovariety of semigroups generated by V, which amounts to a simple exercise, together with the obvious observation that a semigroup S is equidivisible if and only if so is the monoid S^1. In particular, M is equidivisible.

The following lemma can surely be generalized but it is already sufficient for our purposes.

Lemma 7.8. *Let $w \in A^*$ and $u, v \in \overline{\Omega}_A\mathsf{M}$ be such that w^ω is a factor of the product uv. Then, w^ω is a factor of at least one of the factors u and v.*

Proof. Observe first that the result is trivial for $w = 1$ and assume $w \neq 1$. By equidivisibility, from the two factorizations $uv = xw^\omega y$ (for some $x, y \in \overline{\Omega}_A\mathsf{M}$), we know that there is a common refinement. Hence, if w^ω is a factor of neither u nor v, then there is a factorization $w^\omega = zt$ with $u = xz$ and $v = ty$. We reach a contradiction by showing that w^ω must be a factor of at least one of z and t.

First note that at least one of z and t is a not a finite word for, otherwise, so would be w^ω. By symmetry, we may as well assume that z is not a finite word. We claim that w^n is a prefix of z for every $n \geq 1$ and, therefore, so is w^ω, thereby reaching the desired contradiction. To prove the claim, consider the monoid M_k consisting of all words of A^* of length at most $k = |w^n|$ where the product is defined by $r \cdot s = rs$ if $|rs| \leq k$, while $r \cdot s$ is taken to be the prefix of rs of length k otherwise. Consider also the unique continuous homomorphism $\varphi_k : \overline{\Omega}_A\mathsf{M} \to M_k$ which maps each letter a from A to a as an element of M_k. Note that φ_k maps each word of length at most k to itself and every other finite word to its prefix of length k. It follows that $\varphi_k(s)$ is a prefix of s for every pseudoword $s \in \overline{\Omega}_A\mathsf{M}$.

Since z is not a finite word, there is a sequence of words $(z_m)_m$ converging to z with $|z_m| \geq k$ for every m. Then, from the equalities $\varphi_k(z) = \varphi_k(zt) = \varphi_k(w^\omega) = w^n$, we deduce that w^n is a prefix of z, as was claimed. □

Proposition 7.9. *For $\mathsf{U} = \mathsf{M}$ and $\Sigma = \{x^{\omega+1} = x^\omega, (xy)^\omega = (yx)^\omega\}$, we have $\tilde{\Sigma} \neq \Sigma_2$.*

Proof. For the purpose of the present proof, we take $A = \{x, y\}$.

We have shown in the proof of Theorem 7.7 that the ω-identity

$$(x^\omega y^\omega)^\omega = (xy)^\omega \tag{10}$$

belongs to Σ_3. We prove that it does belong to Σ_2. For that purpose, we claim that, if the pseudoidentity $w = (xy)^\omega$ is in Σ_2, then $(xy)^\omega$ is a factor of w. Since not even the word xyx is a factor of $(x^\omega y^\omega)^\omega$ (cf. [16], Lemma 8.2), we conclude that the pseudoidentity (10) cannot belong to Σ_2.

The hypothesis of the claim implies the existence of a sequence of pseudoidentities $(w_n = v_n)_n$ in Σ_1 which converges to $w = (xy)^\omega$. Now, by taking subsequences, we may as well assume that either the sequence $(v_n)_n$ consists only of finite words or only of infinite pseudowords. In the first case, the only Σ_1-

pseudoidentity of the form $u = v_n$ is the trivial pseudoidentity $v_n = v_n$. Since the pseudoidentity (10) is not trivial over M, the first case is excluded. In the second case, we note that $(xy)^\omega$ is a factor of v_n for every sufficiently large n. Indeed, the language $(xy)^*$ has an open closure in $\overline{\Omega}_A\mathsf{M}$ [2], Theorem 3.6.1, which consists of all powers of xy.

Thus, to establish our claim, it suffices to show that, if the pseudoidentity $u = v$ belongs to Σ_1 and the pseudoword $(xy)^\omega$ is a factor of v, then it is also a factor of u. Since Σ_1 is the transitive closure of Σ_0, by a straightforward induction argument it suffices to treat the case where $u = v$ belongs to Σ_0. Hence, there are a nonempty word $\mathbf{t}$, pseudowords $w_1, \ldots, w_n$, a pseudoidentity $u' = v'$ such that either it or $v' = u'$ belongs to Σ, and a continuous endomorphism φ of $\overline{\Omega}_A\mathsf{M}$ such that $u = \mathbf{t}(\varphi(u'), w_1, \ldots, w_n)$ and $v = \mathbf{t}(\varphi(v'), w_1, \ldots, w_n)$. Note that, since u' and v' are $\mathscr{J}$-equivalent, so are $\varphi(u')$ and $\varphi(v')$, which means that these two pseudowords have the same factors. Without loss of generality, we may assume that, writing $\mathbf{t} = \mathbf{t}(x_0, x_1, \ldots, x_n)$, all the letters x_i $(i = 1, \ldots, n)$ appear at least once in $\mathbf{t}$. By Lemma 7.8, we deduce from the assumption that $(xy)^\omega$ is a factor of v that it must also be a factor of either $\varphi(v')$ or one of the w_i. Hence, $(xy)^\omega$ is a factor of either $\varphi(u')$ or one of the w_i and, therefore, also of u. □

8. The group case

Let G be the pseudovariety of all finite groups. As far as subpseudovarieties of G are concerned, whether we view groups in the natural signatures for semigroups, monoids, or groups is irrelevant, since the identity element is the only idempotent x^ω and inversion is also captured by the semigroup pseudoword $x^{\omega-1}$. However, for the purpose of this section, we prefer to deal with the group signature, consisting of a binary multiplication, a constant symbol 1, for the identity element, and the unary operation of inversion.

Note that for $u, v \in \overline{\Omega}_A\mathsf{G}$, each of the pseudoidentities $u = v$ and $u^{-1}v = 1$ is provable from the other. Hence, in the language of groups, it suffices to deal with pseudoidentities of the form $w = 1$.

Theorem 8.1. *The pseudovariety* G *is strong.*

Proof. Let Σ be a set of G-pseudoidentities and let $u \in \overline{\Omega}_A\mathsf{G}$ be such that the group pseudovariety $[\![\Sigma]\!]$ satisfies the pseudoidentity $u = 1$. Consider the closed congruence $\tilde{\Sigma}$ on the profinite group $\overline{\Omega}_A\mathsf{G}$. The congruence class of 1 is a closed normal subgroup N of $\overline{\Omega}_A\mathsf{G}$. It is well known that N must be the intersection of the open normal subgroups of $\overline{\Omega}_A\mathsf{G}$ containing it (see [37], Proposition 2.1.4(d)), which means that the quotient topological group $\overline{\Omega}_A\mathsf{G}/N$ is profinite. Hence, if $u = 1$ is not provable from Σ, that is, if $u \notin N$, then there is a continuous homomorphism

$\varphi : \overline{\Omega}_A\mathsf{G} \to G$ onto a finite group such that $\varphi(N) = 1$ and $\varphi(u) \neq 1$. The condition $\varphi(N) = 1$ entails that G satisfies all pseudoidentities from Σ while the condition $\varphi(u) \neq 1$ implies that G fails the pseudoidentity $u = 1$, in contradiction with the assumption that $[\![\Sigma]\!]$ satisfies $u = 1$. Hence, $u = 1$ is provable from Σ. □

The key ingredient in the preceding proof is that congruences in groups are determined by a single class. So, for instance, the same method may be applied to show that the pseudovariety of all finite rings (not necessarily with identity element) is strong.

Theorem 8.2. *For* $\mathsf{U} = \mathsf{M}$, *the pseudoidentity* $x^{\omega} = 1$ *is h-strong.*

Proof. The key result upon which the proof is based is the κ-reducibility of $\mathsf{G} = [\![x^{\omega} = 1]\!]$ for the equation $x = y$. More generally, based on a celebrated theorem of Ash [18], which contains all the essential hard work, it has been observed in [13], Theorem 4.9 that G is κ-reducible for all finite graph systems of equations.[4] In view of Proposition 7.1, it remains to show that every κ-identity valid in G may be proved from $x^{\omega} = 1$ in the sense of Section 3. In fact, we show that an algebraic proof exists.

The essential observation is that the κ-identity

$$(xy)^{\omega-1} = y^{\omega-1}x^{\omega-1} \tag{11}$$

may be algebraically proved from $x^{\omega} = 1$. Indeed, here is a description of such a proof:

$$\begin{aligned}(xy)^{\omega-1} &= y^{\omega}(xy)^{\omega-1}x^{\omega} = y^{\omega-1}y(xy)^{\omega-1}xx^{\omega-1}\\ &= y^{\omega-1}(yx)^{\omega}x^{\omega-1} = y^{\omega-1}x^{\omega-1}.\end{aligned}$$

Note also that $(x^{\omega-1})^{\omega-1} = x^{\omega+1} = x$, where the latter equality is provable from $x^{\omega} = 1$. Applying repeatedly the identity (11), every κ-term may be rewritten in the form $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, where each x_i is a variable and each exponent ε_i is either 1 or $\omega - 1$. We say that a κ-term of this type is in *standard form*; a κ-identity whose sides are in standard form is also said to be in *standard form*. Thus, in the presence of the pseudoidentity $x^{\omega} = 1$, every κ-identity valid in G is algebraically provably equivalent to a κ-identity $u = v$ in standard form that is also valid in G. Since the free group is residually finite, $u = v$ may be viewed as a group identity (by removing the ω's from the exponents) that is satisfied by all groups, which means

[4]For G, every solution of the equation $y_1 = y_2$ is a solution of the graph system determined by the graph $x \overset{y_1}{\underset{y_2}{\rightrightarrows}} z$ and, conversely, every solution of the graph system involves a solution of the equation $y_1 = y_2$.

that, by applying the reduction rules $aa^{-1} \to 1$ and $a^{-1}a \to 1$ a finite number of times, both sides may be transformed to the same reduced group word. Since, in the context of finite monoids, both rules follow from the pseudoidentity $x^\omega = 1$, we conclude that $u = v$ is algebraically provable from $x^\omega = 1$. □

Although the next result is superseded by Corollary 9.4, it seems worthwhile to include it at this stage.

Corollary 8.3. *For* $\mathsf{U} = \mathsf{M}$, *every set* Σ *of pseudoidentities defining a group pseudovariety is h-strong.*

Proof. It suffices to apply Proposition 4.2 taking into account Theorems 8.2 and 8.1, and Proposition 6.3(i). □

9. The completely simple semigroup case

Let CS be the pseudovariety of all finite completely simple semigroups. It is defined, for instance, by the pseudoidentity $(xy)^\omega x = x$. By a well-known theorem of Rees, completely simple semigroups are precisely those that admit a Rees matrix representation $\mathscr{M}(I, G, \Lambda, P)$, where I and Λ are sets, G is a group, and $P : \Lambda \times I \to G$ is a function (the *sandwich matrix*, the image $P(\lambda, i)$ being usually denoted $p_{\lambda,i}$); as a set, it is the Cartesian product $I \times G \times \Lambda$, and multiplication is given by the formula

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda,j}h, \mu).$$

Assuming that 1 is a common element of I and Λ, the sandwich matrix may be supposed to be *normalized* in the sense that $p_{1,i} = p_{\lambda,1} = 1$ for all $i \in I$ and $\lambda \in \Lambda$.

The purpose of this section is to establish that CS is strong. The key ingredient is the following characterization of the congruences on a Rees matrix semigroup which may be extracted from [20], Theorem 10.48.

Theorem 9.1. *Let* $S = \mathscr{M}(I, G, \Lambda, P)$ *be a Rees matrix semigroup and let* ρ *be a congruence on* S. *Consider the relations*

$$\rho_1 = \{(i, j) \in I \times I : (i, 1, 1)\rho(j, 1, 1)\}$$
$$\rho_2 = \{(\lambda, \mu) \in \Lambda \times \Lambda : (1, 1, \lambda)\rho(1, 1, \mu)\}$$
$$N_\rho = \{g \in G : (1, g, 1)\rho(1, 1, 1)\}.$$

Then ρ_1 *(respectively* ρ_2*) is an equivalence relation on the set* I *(resp.* Λ*) and* N_ρ *is a normal subgroup of* G *such that*

$$(i,j) \in \rho_1 \Rightarrow p_{\lambda,i}N_\rho = p_{\lambda,j}N_\rho \tag{12}$$

$$(\lambda,\mu) \in \rho_2 \Rightarrow p_{\lambda,i}N_\rho = p_{\mu,i}N_\rho. \tag{13}$$

Conversely, for every triple $\tau = (\rho_1, \rho_2, N_\rho)$*, where* ρ_1 *(resp.* ρ_2*) is an equivalence relation on the set* I *(resp.* Λ*) and* N_ρ *is a normal subgroup of* G*, satisfying properties* (12) *and* (13)*, the relation*

$$\rho_\tau = \big\{\big((i,g,\lambda),(j,h,\mu)\big) \in S \times S : i\rho_1 j, \lambda\rho_2\mu, gN_\rho = hN_\rho\big\}$$

is a congruence on S *and every congruence on* S *is of this form.*

We may now proceed as in the proof of Theorem 8.1 to obtain our next result.

Theorem 9.2. *The pseudovariety* CS *is strong.*

Proof. For a set Σ of CS-pseudoidentities, consider the closed congruence $\rho = \tilde{\Sigma}$. Suppose that $u, v \in \overline{\Omega}_A\mathsf{CS}$ are such that the pseudoidentity $u = v$ is valid in $[\![\Sigma]\!]$. We claim that $u = v$ is provable from Σ, which amounts to the condition $u\rho v$.

Arguing by contradiction, suppose that $(u,v) \notin \rho$. By [1], there is an isomorphism $\psi : \overline{\Omega}_A\mathsf{CS} \to S = \mathscr{M}(A, \overline{\Omega}_X\mathsf{G}, A, P)$ where, choosing an element a_0 from A and letting $A' = (A\backslash\{a_0\})^2$ be the Cartesian square, we have $X = A \cup A'$, $p_{a_0,b} = p_{b,a_0} = 1$ for every $b \in A$, and $p_{a,b} = (a,b)$ for each $(a,b) \in A'$. Note that the letter a_0 plays the role of 1 in the normalization of the sandwich matrix.

Consider the normal subgroup N_ρ and the equivalence relations ρ_1 and ρ_2 as defined in Theorem 9.1. Note that N_ρ is a closed normal subgroup of $\overline{\Omega}_X\mathsf{G}$.

Let $\psi(u) = (a,g,b)$ and $\psi(v) = (c,h,d)$. Since we are assuming that $(u,v) \notin \rho$, at least one of the following conditions must hold:

$$(a,c) \notin \rho_1, \qquad (b,d) \notin \rho_2, \qquad \text{or} \qquad gN_\rho \neq hN_\rho.$$

In case one of the first two conditions holds, the mapping from S onto the rectangular band $T = A/\rho_1 \times A/\rho_2$ that maps each triple $(x,w,y) \in S$ to $(x/\rho_1, y/\rho_2)$ is a continuous homomorphism onto a semigroup from CS that distinguishes u and v. Moreover, its kernel congruence is contained in ρ, which implies that $T \in [\![\Sigma]\!]$, contradicting the assumption that $[\![\Sigma]\!]$ satisfies $u = v$. Hence, we may assume that $(a,c) \in \rho_1$ and $(b,d) \in \rho_2$, so that $g^{\omega-1}h$ does not belong to N_ρ. As in the proof of Theorem 8.1, we deduce that there is a clopen normal subgroup K of $\overline{\Omega}_X\mathsf{G}$ such that $N_\rho \subseteq K$ and $g^{\omega-1}h \notin K$. Since K contains N_ρ, the triple (ρ_1, ρ_2, K) still satisfies the analogues of conditions (12) and (13). Hence, by Theorem 9.1, it defines a congruence $\bar{\rho}$ on S. Since K has finite index in $\overline{\Omega}_X\mathsf{G}$, the congruence $\bar{\rho}$ has finite index in S. The reader may easily verify that, since ρ is a closed congruence on S and K is a closed subgroup of $\overline{\Omega}_X\mathsf{G}$, the congruence $\bar{\rho}$ is still closed, whence the natural mapping $S \to S/\bar{\rho}$ is a continuous homomorphism. As $\bar{\rho}$ contains ρ, the quotient semigroup $S/\bar{\rho}$ satisfies Σ but, by construction, fails the pseudoidentity $u = v$. This completes the proof. $\square$

We proceed with some observations on the variety $\mathscr{CS}$ of completely simple semigroups, consisting of algebras with a binary multiplication and unary "inversion" $\_^{-1}$ satisfying the following identities, where u^0 abbreviates uu^{-1}:

$$(xy)z = x(yz), \quad x^{-1}x = x^0, \quad x^0x = x, \quad (x^{-1})^{-1} = x \tag{14}$$

$$(xyx)^0 = x^0. \tag{15}$$

The identities (14) define the variety of completely regular semigroups [34]. In the presence of them, it is well known that the identity (15) is equivalent to

$$(xy)^0x = x. \tag{16}$$

Note that, when the inversion operation $\_^{-1}$ is interpreted as $\_^{\omega-1}$ in a finite semigroup, $\_^0$ becomes $\_^\omega$ and the first two identities in (14) are verified while the last two identities in (14) are valid in every completely regular finite semigroup.

Theorem 9.3. *For* $\mathsf{U} = \mathsf{S}$*, each of the sets of pseudoidentities*

$$\Sigma = \{(xyx)^\omega = x^\omega, x^{\omega+1} = x\} \quad \textit{and} \quad \Gamma = \{(xy)^\omega x = x\}$$

is h-strong.

Proof. As implied by the entry for the pseudovariety CS in [13], Table 2, the methods of [3] show that the pseudovariety CS is κ-reducible for graph systems of equations. In fact, in [3] vertices were allowed to be constrained by the clopen subset $\{1\} \subseteq (\overline{\Omega}_A\mathsf{S})^1$, which forces the corresponding variable to be evaluated by 1. It follows that CS is κ-reducible for the equation $x = y$.

Consider next the variety CS^κ. Since free completely simple semigroups are residually finite [33], Proposition 2.5, in view of the above remarks a basis of identities for CS^κ is given by

$$(xy)z = x(yz), \quad x^{\omega-1}x = xx^{\omega-1}, \quad xx^{\omega-1}x = x, \quad (x^{\omega-1})^{\omega-1} = x$$
$$xyx(xyx)^{\omega-1} = xx^{\omega-1}.$$

All the identities in the first line are obviously provable from Σ. On the other hand, substituting x for y gives a proof of $x^{\omega+1} = x$ from Γ and we already observed that in the presence of the identities in the above first line, the identities

$(xyx)^\omega = x^\omega$ and $(xy)^\omega x = x$ are provable from each other. Thus, the result follows from Proposition 7.1. □

Combining Theorems 9.2 and 9.3 with Proposition 6.4, and applying Proposition 4.2, we obtain the following result.

Corollary 9.4. *For* $\mathsf{U} = \mathsf{S}$, *every set* Σ *of pseudoidentities such that* $[\![\Sigma]\!] \subseteq \mathsf{CS}$ *is h-strong.*

Taking into account the κ-reducibility for the equation $x = y$ of the pseudovariety $\mathsf{CR} = [\![x^{\omega+1} = x]\!]$, of all finite completely regular semigroups, (cf. [15]) and the residual finiteness of free completely regular semigroups (viewed as algebras in the signature $\{\_ \cdot \_, \_^{-1}\}$), the following theorem can be proved in the same way as Theorem 9.3.

Theorem 9.5. *For* $\mathsf{U} = \mathsf{S}$, *the pseudoidentity* $x^{\omega+1} = x$ *is h-strong.*

10. The commutative case

Our next example is the usual basis of the pseudovariety Com of all finite commutative monoids.

Theorem 10.1. *For* $\mathsf{U} = \mathsf{M}$, *the identity* $xy = yx$ *is h-strong.*

Proof. Although [39] deals with the notion of hyperdecidability, which is weaker than σ-reducibility provided the implicit signature σ has suitable algorithmic properties (cf. [13], [14]), it is observed in [13] that the same methods show that Com is κ-reducible, in particular for the equation $x = y$. We claim that the following is a basis of identities for the variety Com^κ where, as usual, we write x^ω for the product $x^{\omega-1}x$:

$$x^{\omega-1}x^\omega = x^{\omega-1}, \qquad (x^{\omega-1})^{\omega-1} = x^{\omega+1}$$
$$(xy)z = x(yz), \qquad x1 = 1x = x$$
$$xy = yx$$
$$(xy)^{\omega-1} = x^{\omega-1}y^{\omega-1}.$$

Indeed, all such identities are clearly valid in Com. On the other hand, using the above identities, one may reduce every κ-term to one of the form $u = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, where each exponent ε_i is either 1 or $\omega - 1$ and the x_i are not necessarily distinct variables. Moreover, we may rearrange the factors so that the powers with the same base are not separated by other powers. On the other hand, powers with the same base may be collected together to powers of one of the forms x^n or $x^{\omega-1}x^n$, where n is a nonnegative integer, or $(x^{\omega-1})^n$, where $n \geq 2$. For convenience, we write $x^{\omega-1}x^n$ as $x^{\omega-1+n}$ and $(x^{\omega-1})^n$ as $x^{(\omega-1)n}$. In this form, we say that we have a κ-term in *completely reduced form*. Consider two κ-terms in completely reduced form $u = a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$ and $v = a_1^{\delta_1} \dots a_n^{\delta_n}$ over the alphabet $\{a_1, \dots, a_n\}$ and assume that the identity $u = v$ is valid in Com. Then, substituting 1 for every variable but one, we obtain an identity valid in Com of the form $x^\varepsilon = x^\delta$ with ε and δ exponents of one of the forms n, $\omega - 1 + n$ or $(\omega - 1)n$. By considering a suitable monogenic finite semigroup, one immediately verifies that $\varepsilon = \delta$. This proves the claim.

To conclude the proof, it remains to observe that the last identity in the above basis is provable from $xy = yx$. Indeed, each identity $(xy)^{n!-1} = x^{n!-1}y^{n!-1}$ can be easily proved and the result follows by taking limits. □

We next show that the pseudoidentity $xy = yx$ also defines a strong pseudovariety.

Theorem 10.2. *The pseudovariety* Com *is strong.*

Proof. First of all, we recall that elements of the free profinite monoid $\overline{\Omega}_A\mathsf{Com}$ over the set $A = \{a_1, \dots, a_n\}$ can be written in the form $a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$, where $\varepsilon_1, \dots, \varepsilon_n \in \hat{\mathbb{N}}$. Each pseudoidentity $a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n} = a_1^{\delta_1} \dots a_n^{\delta_n}$ is provably equivalent to the set of pseudoidentities $\{a_1^{\varepsilon_1} = a_1^{\delta_1}, \dots, a_n^{\varepsilon_n} = a_n^{\delta_n}\}$. Therefore, we may assume that all pseudoidentities which are under consideration are over a single variable.

Recall that $\overline{\Omega}_1\mathsf{Com}$ and $\overline{\Omega}_1\mathsf{M}$ are both isomorphic to $\hat{\mathbb{N}}$ via the mapping $\varepsilon \mapsto x^\varepsilon$ and that $\hat{\mathbb{Z}} = \hat{\mathbb{N}} \setminus \mathbb{N}$ is isomorphic to $\overline{\Omega}_1\mathsf{G}$. For the purpose of simplification of notation, we identify isomorphic structures, so, for example, we denote by π the continuous homomorphisms $\pi : \overline{\Omega}_1\mathsf{Com} \to \overline{\Omega}_1\mathsf{G}$ given by the rule $\pi(x^\varepsilon) = x^{\omega+\varepsilon}$, which is formally the projection $\pi_{1,\mathsf{M},\mathsf{G}} : \overline{\Omega}_1\mathsf{M} \to \overline{\Omega}_1\mathsf{G}$ after the identification $\overline{\Omega}_1\mathsf{Com} = \overline{\Omega}_1\mathsf{M}$. In this way, the group $\overline{\Omega}_1\mathsf{G}$ can be viewed as a retract of the monoid $\overline{\Omega}_1\mathsf{Com}$.

Let Σ be a set of pseudoidentities in the variable x and $x^\varepsilon = x^\delta$ be a nontrivial pseudoidentity satisfied by $[\![\Sigma]\!]$. We need to show that $x^\varepsilon = x^\delta$ belongs to $\tilde{\Sigma}$.

By the *finite index* $i(\Sigma)$ of Σ is meant the minimum natural number n such that a pseudoidentity of one of the forms $x^n = x^\mu$ or $x^\mu = x^n$ belongs to Σ where $\mu \neq n$. If such $n \in \mathbb{N}$ does not exist, we say that Σ has *infinite index*.

If n is the finite index of Σ, then in a nontrivial pseudoidentity $x^n = x^\mu$ such that it or its left-right dual lies in Σ, either μ is infinite or $n < \mu \in \mathbb{N}$. In both cases, in view of Lemma 6.1 the pseudoidentity $x^n = x^{\omega+n}$ is provable from Σ. On the other hand, the monogenic monoid $C_{n,1}^1$ belongs to $[\![\Sigma]\!]$, whence it satisfies $x^\varepsilon = x^\delta$. Hence, both ε and δ are infinite or greater than or equal to $i(\Sigma) = n$. From Lemma 6.1, it follows that $x^\varepsilon = x^{\omega+\varepsilon}$ and $x^\delta = x^{\omega+\delta}$ are provable from Σ.

Therefore, $x^\varepsilon = x^\delta$ is provable from Σ if and only if so is $x^{\omega+\varepsilon} = x^{\omega+\delta}$. On the other hand, if Σ has infinite index, then the monoid $C^1_{n,1}$ belongs to $[\![\Sigma]\!]$ for every $n \in \mathbb{N}$ and we see that $\varepsilon, \delta \notin \mathbb{N}$. Altogether, we can deal just with the case $\varepsilon, \delta \in \hat{\mathbb{Z}}$.

Before finishing the proof, we make one technical observation. Let ρ be the restriction of the relation $\tilde{\Sigma}$ to the set $\{x^\lambda : \lambda \in \hat{\mathbb{Z}}\}$ and τ denote the relation $\widetilde{\pi(\Sigma)}$ on $\overline{\Omega}_1\mathsf{G}$, where we abuse notation and write $\pi(\Sigma)$ instead of $(\pi \times \pi)(\Sigma)$. We claim that these relations are equal, under our identification of underlying sets with $\hat{\mathbb{Z}}$. Since $\hat{\mathbb{Z}}$ is a closed ideal in $\hat{\mathbb{N}}$ and $\tilde{\Sigma}$ is a closed congruence on $\hat{\mathbb{N}} = \overline{\Omega}_1\mathsf{Com}$, the relation ρ is a closed congruence on $\hat{\mathbb{Z}}$. We have $\pi(\Sigma) \subseteq \Sigma_0$, because each pseudoidentity $x^{\omega+\lambda} = x^{\omega+\mu}$ is provable from $x^\lambda = x^\mu$. Hence, $\pi(\Sigma) \subseteq \tilde{\Sigma}$, where $\pi(\Sigma)$ is a relation on $\hat{\mathbb{Z}}$. Therefore, $\pi(\Sigma) \subseteq \rho$ and the transitive-topological closure $\tau = \widetilde{\pi(\Sigma)}$ is also a subset of the closed congruence ρ. The reverse inclusion $\rho \subseteq \tau$ will be proved if we establish (by induction), for each α, the inclusion $\Sigma_\alpha|_{\hat{\mathbb{Z}}} \subseteq \big(\pi(\Sigma)\big)_\alpha$, where we let $\Sigma_\alpha|_{\hat{\mathbb{Z}}} = \Sigma_\alpha \cap (\hat{\mathbb{Z}} \times \hat{\mathbb{Z}})$.

Let $\alpha = 0$ and $(x^\lambda, x^\mu) \in \Sigma_0$, where $\lambda, \mu \in \hat{\mathbb{Z}}$. Taking into account commutativity, we may assume that $x^\lambda = x^{a+bk}$, $x^\mu = x^{a+b\ell}$, where $a, b, k, \ell \in \hat{\mathbb{N}}$ and $(x^k = x^\ell) \in \Sigma$. Then, $(x^{\omega+k} = x^{\omega+\ell}) \in \pi(\Sigma)$ and we have $(x^{a+b(\omega+k)} = x^{a+b(\omega+\ell)}) \in \big(\pi(\Sigma)\big)_0$. Since $\lambda, \mu \in \hat{\mathbb{Z}}$, we see that $a + b(\omega + k) = a + bk + \omega = \lambda + \omega = \lambda$ and, similarly, $a + b(\omega + \ell) = \mu$, which yields $\Sigma_0|_{\hat{\mathbb{Z}}} \subseteq \big(\pi(\Sigma)\big)_0$.

Now assume that $\Sigma_{2\gamma}|_{\hat{\mathbb{Z}}} \subseteq \big(\pi(\Sigma)\big)_{2\gamma}$ and let $(x^\lambda = x^\mu) \in \Sigma_{2\gamma+1}$, where $\lambda, \mu \in \hat{\mathbb{Z}}$. Then there is a finite sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \mu \in \hat{\mathbb{N}}$ such that $(x^{\lambda_{i-1}} = x^{\lambda_i}) \in \Sigma_{2\gamma}$, for each $i = 1, \ldots, m$. Since $\Sigma_{2\gamma}$ is stable under multiplication, we deduce that $(x^{\lambda_i+\omega} = x^{\lambda_{i+1}+\omega}) \in \Sigma_{2\gamma}|_{\hat{\mathbb{Z}}}$. By the induction assumption, these pairs also belong to $\big(\pi(\Sigma)\big)_{2\gamma}$ and since $\lambda + \omega = \lambda$ and $\mu + \omega = \mu$, we obtain $(x^\lambda = x^\mu) \in \big(\pi(\Sigma)\big)_{2\gamma+1}$.

Let $\Sigma_{2\gamma+1}|_{\hat{\mathbb{Z}}} \subseteq \big(\pi(\Sigma)\big)_{2\gamma+1}$ and assume that $(x^\lambda = x^\mu) \in \Sigma_{2\gamma+2}$, where $\lambda, \mu \in \hat{\mathbb{Z}}$. Then, there is an infinite sequence of pseudoidentities from $\Sigma_{2\gamma+1}$ converging to $x^\lambda = x^\mu$. Multiplying the pseudoidentities in the sequence by x^ω and using the fact that $\Sigma_{2\gamma+1}$ is a congruence, we obtain a sequence of pseudoidentities from $\Sigma_{2\gamma+1}|_{\hat{\mathbb{Z}}}$ converging to $x^\lambda = x^\mu$. Now, using the induction hypothesis we obtain $(x^\lambda = x^\mu) \in \big(\pi(\Sigma)\big)_{2\gamma+2}$.

To accomplish the proof of the claim by transfinite induction we need to consider a limit ordinal α, for which the inclusion follows immediately from the inclusions for smaller ordinals.

Now, we are ready to finish the proof of the theorem. We assumed that $[\![\Sigma]\!]$ satisfies the pseudoidentity $x^\varepsilon = x^\delta$, where $\varepsilon, \delta \in \hat{\mathbb{Z}}$. We need to show that $(x^\varepsilon = x^\delta) \in \tilde{\Sigma}$, or equivalently written $(\varepsilon, \delta) \in \rho$. Assume on the contrary that $(\varepsilon, \delta) \notin \rho$. We have proved the equality of two closed congruences ρ and τ. So, we can consider the monoid $\hat{\mathbb{Z}}/\rho = \hat{\mathbb{Z}}/\tau$, which is a profinite monoid, because it is a quotient of $\overline{\Omega}_1\mathsf{G}$ by the relation $\tau = \tilde{\Gamma}$ for $\Gamma = \pi(\Sigma)$ and G is strong by Theorem 8.1. Thus, there is a continuous homomorphism $f : \hat{\mathbb{Z}} \to G$ onto a finite (cyclic) group $G \in [\![\pi(\Sigma)]\!]_{\mathsf{G}} \subseteq [\![\Sigma]\!]$, such that $f(\varepsilon) \neq f(\delta)$. Now, we can consider the composition of f with π and we obtain the continuous homomorphism $g : \overline{\Omega}_1\mathsf{Com} \to G$, given by $g(x^\lambda) = f(\lambda + \omega)$, such that $g(x^\varepsilon) \neq g(x^\delta)$. This is a contradiction because the group $G \in [\![\Sigma]\!]$ must satisfy the pseudoidentity $x^\varepsilon = x^\delta$. Therefore, we have $(x^\varepsilon = x^\delta) \in \tilde{\Sigma}$ and the proof is finished. □

To complete the program already followed in Sections 8 and 9, we establish the missing strongness property for the pseudoidentity $xy = yx$. It was not established earlier, namely as part of Proposition 6.3, because our argument depends on Corollary 8.3.

Proposition 10.3. *The pseudoidentity of monoids $xy = yx$ is t-strong.*

Proof. Let Σ be a set of M-pseudoidentities such that all monoids in $[\![\Sigma]\!]$ are commutative. We need to show that Σ proves $xy = yx$.

In the terminology introduced at the end of Section 6, by a theorem of Margolis and Pin [30], a complete set of excluded monoids for the pseudoidentity $xy = yx$ is given by all non-Abelian groups together with N^1, $B(1,2)^1$, and $B(2,1)^1$. Hence, each such monoid fails some pseudoidentity from Σ. Actually, since $xy = yx$ entails $x^\omega y = yx^\omega$ and the latter pseudoidentity is t-strong by Proposition 6.3(xi), we already know that Σ proves $x^\omega y = yx^\omega$.

On the other hand, as N^1 fails some pseudoidentity $u = v$ from Σ, either Σ proves $x^\omega = 1$ or N^1 fails an at most two-variable pseudoidentity provable from Σ by evaluating x as a and y as b. Taking into account that the only nonzero product involving at least two factors using either a, b, or both is ab, we deduce that Σ proves either $x = x^{\omega+1}$ or a nontrivial pseudoidentity of the form $xy = w$. If w is not yx, then one may substitute one of the variables by 1 to get a nontrivial pseudoidentity of the from $x = x^\alpha$, which again entails $x = x^{\omega+1}$ by Lemma 6.1. Hence, Σ deduces either $x = x^{\omega+1}$ or $xy = yx$, which means that we may as well assume that Σ proves the former pseudoidentity. Thus, Σ proves the pseudoidentities

$$x^{\omega+1} = x \tag{17}$$

$$x^\omega y = yx^\omega, \tag{18}$$

which are known to define the join $\mathsf{Sl} \vee \mathsf{G}$, that is, the pseudovariety of all finite monoids that are semilattices of groups [2], Exercise 9.1.4. Since it is not so easy to deduce pseudoidentities from Σ from the knowledge that non-Abelian groups fail some pseudoidentity in Σ, we proceed instead to reduce our problem to the group case and apply Corollary 8.3 to draw the conclusion that Σ proves $xy = yx$.

First, we exhibit other provable consequences of Σ. Since we deal with the pseudoidentity $xy = yx$, we work over the alphabet $A = \{x, y\}$ only, even though more general pseudoidentities may be handled similarly. In fact we list some useful pseudoidentities that are provable from (17) and (18).

We have $(xy)^{\omega} = x^{\omega}(xy)^{\omega}y^{\omega} = y^{\omega}(xy)^{\omega}x^{\omega}$, where we used (17) first and (18) in the second step. Then we get $y^{\omega}(xy)^{\omega}x^{\omega} = y^{\omega-1}(yx)^{\omega+1}x^{\omega-1} = y^{\omega-1}yxx^{\omega-1} = y^{\omega}x^{\omega} = x^{\omega}y^{\omega}$, where the first and third equalities hold in $\overline{\Omega}_A\mathsf{M}$ and the second and fourth equalities follow respectively from (17) and (18). Now, for each word $w \in A^*$ containing both variables x and y, if we use the pseudoidentities $(xy)^{\omega} = x^{\omega}y^{\omega}$ and (18) repeatedly, we may prove $w^{\omega} = x^{\omega}y^{\omega}$. Hence, from (17) and (18) we may prove $w^{\omega}x^{\omega}y^{\omega} = x^{\omega}y^{\omega}$ for every pseudoword w over the alphabet $\{x, y\}$.

Recall that $\tilde{\Sigma}$ is a relation on $\overline{\Omega}_A\mathsf{M}$. Let $\Gamma = \Sigma \cup \{x^{\omega} = 1\}$ and consider $\tilde{\Gamma}$ on the same monoid $\overline{\Omega}_A\mathsf{M}$. Then $[\![\Gamma]\!] = [\![\Sigma]\!] \cap \mathsf{G}$. Applying Corollary 8.3 we obtain that $(xy = yx) \in \tilde{\Gamma}$. We claim, for any pseudoidentity $u = v$, that $(u = v) \in \tilde{\Gamma}$ implies $(ux^{\omega}y^{\omega} = vx^{\omega}y^{\omega}) \in \tilde{\Sigma}$. This gives the proof of the statement because the pseudoidentities $xy = xyx^{\omega}y^{\omega}$ and $yx = yxx^{\omega}y^{\omega}$ are provable from (17) and (18).

The claim will be proved if we show, for every ordinal α, the following implication:

$$(u = v) \in \Gamma_{\alpha} \Rightarrow (ux^{\omega}y^{\omega} = vx^{\omega}y^{\omega}) \in \tilde{\Sigma}.$$

Let $\alpha = 0$ and $(u = v) \in \Gamma_0$. If we use in the proof of $u = v$ a pseudoidentity from Σ, then $(u = v) \in \Sigma_0$ and therefore also $(ux^{\omega}y^{\omega} = vx^{\omega}y^{\omega}) \in \Sigma_0$. So, we may assume, without loss of generality, that $u = \mathbf{t}(\varphi(x^{\omega}), w_1, \ldots, w_n)$, $v = \mathbf{t}(\varphi(1), w_1, \ldots, w_n)$, where $\varphi : \overline{\Omega}_1\mathsf{M} \to \overline{\Omega}_A\mathsf{M}$ is a continuous homomorphism, $\mathbf{t}$ is a word and $w_i \in \overline{\Omega}_A\mathsf{M}$ $(i = 1, \ldots, n)$. Using (18) and the fact that $(\varphi(x))^{\omega}x^{\omega}y^{\omega} = x^{\omega}y^{\omega}$ is Σ-provable, we get that $(ux^{\omega}y^{\omega} = vx^{\omega}y^{\omega}) \in \tilde{\Sigma}$.

The other steps of the induction proof are easy to see as $\tilde{\Sigma}$ is a closed congruence. □

As in Sections 8 and 9, we may now apply Proposition 4.2 to obtain the following result.

Corollary 10.4. *For* $\mathsf{U} = \mathsf{M}$, *every set* Σ *of pseudoidentities defining a pseudovariety of commutative monoids is h-strong.*

11. Conclusion

We have introduced a natural and sound proof scheme for pseudoidentities. We have given ample evidence for the conjecture that our proof scheme is complete. There is a nice connection with the much studied notions of reducibility and tameness. In fact, in all our examples built on reducible pseudovarieties, we have $\Sigma_n = \tilde{\Sigma}$ for some integer n. For instance, for the set Σ of Theorem 7.7, we have observed that $\Sigma_4 = \tilde{\Sigma}$ and that $\Sigma_2 \neq \tilde{\Sigma}$. It is expectable that in general there may be no $n \geq 1$ such that $\Sigma_n = \tilde{\Sigma}$, or perhaps even $\Sigma_\omega \neq \tilde{\Sigma}$, but we have found no example in which that is the case.

We have not tried to treat exhaustively all cases of pseudovarieties which are known to be tame. Those that we have considered are perhaps those that appear more frequently in the literature. Of course, since we believe in our conjecture, more such results would only be adding evidence to our claim that we have identified a completely general phenomenon.

It is the knowledge of congruences on suitable relatively profinite monoids or semigroups that allowed us to prove that the pseudovarieties G, CS, and Com are strong. Much is also known about the congruences on $\overline{\Omega}_A\mathsf{CR}$ but we have not taken the natural path of trying to prove that the pseudovariety CR is strong.

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