

Nitsche's method for unilateral contact problems

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Abstract. We derive optimal a priori and a posteriori error estimates for Nitsche's method applied to unilateral contact problems. Our analysis is based on the interpretation of Nitsche's method as a stabilised finite element method for the mixed Lagrange multiplier formulation of the contact problem wherein the Lagrange multiplier has been eliminated elementwise. To simplify the presentation, we focus on the scalar Signorini problem and outline only the proofs of the main results since most of the auxiliary results can be traced to our previous works on the numerical approximation of variational inequalities. We end the paper by presenting results of our numerical computations which corroborate the efficiency and reliability of the a posteriori estimators.

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1. Introduction

Unilateral contact problems are of great engineering interest and they occur in numerous areas of physics and mechanics; cf. [26]. In mathematical terms, these problems are expressed as variational inequalities and most often approximated by the finite element method; cf. [19] and all the references therein.

One of the most common examples of unilateral contact problems is contact between two deformable elastic bodies. Here, we will consider the Signorini problem which consists in finding the equilibrium position of an elastic body resting on a rigid frictionless surface. For simplicity, we consider the scalar version of such problem, at times referred to as the Poisson–Signorini problem. However, our results carry over, with minor modifications, to the Signorini problem in linear elasticity.

The finite element treatment of the Signorini problem has shown to be more difficult than that of the obstacle problem (another archetypical variational inequality) due to the Signorini (no-penetration) condition at the boundary, and

has led to a number of papers over the past years; cf. [2–4], [6], [12], [20], [23], [29], [31] and the review papers [8], [36]. The above mentioned works address primal and mixed formulations and focus on obtaining optimal a priori estimates based on Falk’s Lemma [13]. As for the a posteriori error estimates for the Signorini problem, we refer to [5], [21], [22], [27], [30], [35].

Another approach that, at the same time, imposes the contact boundary conditions weakly, avoids the additional Lagrange multiplier of the mixed formulations and is consistent in contrast to the standard penalty formulations, is the Nitsche’s formulation, first proposed for the unilateral contact problems by Chouly and Hild [10], see also [7], [11] for further generalisations. In [7], [10], [11], optimal a priori estimates were derived but, to our knowledge, the only existing work on the a posteriori error estimation of unilateral contact problems approximated by Nitsche’s method is the recent work by Chouly et al. [9].

In this paper, we will continue to advocate (cf. [14–18]) that Nitsche’s method is most readily analysed as a stabilised finite element method, the relation which was first suggested in [32]. We will prove optimal a priori estimates for the stabilised mixed formulation and show the reliability and the efficiency for the a posteriori error estimators without additional saturation assumptions which are needed when Nitsche’s method is analysed directly (cf. [9]). We do emphasise though that the stabilised method is best implemented through the Nitsche’s formulation.

The paper is organised as follows. In Section 2, we introduce the Poisson–Signorini problem, write it in a mixed variational form and state a continuous stability result. In Section 3, we formulate the stabilised finite element method, state a discrete stability estimate and prove a priori error estimate. In Section 4, we introduce residual-based a posteriori error estimators and establish lower and upper bounds for the error in terms of the estimators. In Section 5, we deduce the Nitsche’s formulation from the stabilised one and in Section 6 report on our numerical computations.

We note that some of the proofs have been left out since they are formally similar to the ones proven in our previous works, in particular in [14], [16].

2. The continuous problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a polygonal (or polyhedral) domain and $\partial\Omega$ its boundary with $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$. We assume that the boundary parts Γ_D , Γ_N and Γ are all non-empty and non-overlapping and such that $\overline{\Gamma_D} \cap \overline{\Gamma} = \emptyset$, and that the part Γ where the contact may occur coincides with one of the sides of the polygon (or the polyhedron).

The Poisson–Signorini unilateral contact problem can be written as follows: find u such that

$$\begin{aligned}
-\Delta u &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma_D, \\
\frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N, \\
u \geq 0, \quad \frac{\partial u}{\partial n} \geq 0, \quad u \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma,
\end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$ is a given load function and n denotes the outer normal vector to Ω .

Remark 2.1. For ease of exposition, we consider here the simplest version of the Signorini problem although we could easily deal with a non-homogeneous Neumann condition $\frac{\partial u}{\partial n} = g$ on Γ_N , with a gap function ψ defined on Γ , such that $u \geq \psi$ and $(u - \psi) \frac{\partial u}{\partial n} = 0$ on Γ , and also with the linear elastic Signorini problem. Note also that by assuming that $\overline{\Gamma_D} \cap \overline{\Gamma} = \emptyset$ we avoid introducing the Lions–Magenes space $H_{00}^{1/2}(\Gamma)$; cf. [33].

To give a weak formulation for problem (2.1), we introduce the Hilbert spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}, \quad W := \{w|_{\Gamma} : w \in V\} = H^{1/2}(\Gamma),$$

endowed with the norms

$$\|v\|_V = \|\nabla v\|_0, \quad \|w\|_W = \inf_{v \in V, v|_{\Gamma} = w} \|v\|_V,$$

where $\|\cdot\|_0$ denotes the $L^2(\Omega)$ -norm. Now, defining a non-negative Lagrange multiplier by

$$\lambda = \frac{\partial u}{\partial n}, \tag{2.2}$$

we obtain the following (weak) mixed variational formulation of problem (2.1) (cf. [24]): find $(u, \lambda) \in V \times \Lambda$ such that

$$\begin{aligned}
(\nabla u, \nabla v) - \langle v, \lambda \rangle &= (f, v) \quad \forall v \in V, \\
\langle u, \mu - \lambda \rangle &\geq 0 \quad \forall \mu \in \Lambda.
\end{aligned} \tag{2.3}$$

Above, (\cdot, \cdot) is the standard $L^2(\Omega)$ -inner product,

$$\Lambda = \{\mu \in Q : \langle w, \mu \rangle \geq 0 \forall w \in W, w \geq 0 \text{ a.e. on } \Gamma\},$$

where $Q := W'$ is the topological dual of W , and $\langle \cdot, \cdot \rangle : W \times Q \rightarrow \mathbb{R}$ denotes the duality pairing. The norm in Q is defined as

$$\|\mu\|_Q = \sup_{w \in W} \frac{\langle w, \mu \rangle}{\|w\|_W}.$$

Remark 2.2. The corresponding primal weak formulation of problem (2.1) reads as follows: find $u \in K$ such that

$$(\nabla u, \nabla(v - u)) \geq (f, v - u) \quad \forall v \in K, \quad (2.4)$$

where

$$K = \{v \in V : v \geq 0 \text{ a.e. on } \Gamma\}.$$

Remark 2.3. It is well-known that problem (2.4), equivalently (2.3), admits a unique solution $u \in H^1(\Omega)$; cf. [25], [26]. However, even if the boundary is smooth and the data regular enough, the Signorini conditions limit the regularity of the solution. In fact, in the two-dimensional case the solution belongs only to H^r , $r < 5/2$, in the vicinity of points at Γ where the constraints change from binding to non-binding, cf. [28] and the discussion in [2]. If the boundary Γ has corners, further singularities may occur; cf. [1].

Defining the bilinear and linear forms, $\mathcal{B} : (V \times Q) \times (V \times Q) \rightarrow \mathbb{R}$ and $\mathcal{L} : V \rightarrow \mathbb{R}$ through

$$\mathcal{B}(w, \xi; v, \mu) = (\nabla w, \nabla v) - \langle w, \mu \rangle - \langle v, \xi \rangle, \quad \mathcal{L}(v) = (f, v),$$

the mixed variational formulation of (2.3) reads: find $(u, \lambda) \in V \times \Lambda$ such that

$$\mathcal{B}(u, \lambda; v, \mu - \lambda) \leq (v) \quad \forall (v, \mu) \in V \times \Lambda. \quad (2.5)$$

The proof of the following result is straightforward (see, e.g., [16]):

Theorem 2.4 (Continuous stability). *For all $(v, \xi) \in V \times Q$ there exists $w \in V$ such that*

$$\mathcal{B}(v, \xi; w, -\xi) \gtrsim (\|v\|_V + \|\xi\|_Q)^2 \quad (2.6)$$

and

$$\|w\|_V \lesssim \|v\|_V + \|\xi\|_Q. \quad (2.7)$$

Note that above and in the following we write $a \gtrsim b$ (or $a \lesssim b$) when $a \geq Cb$ (or $a \leq Cb$) for some positive constant C independent of the finite element mesh.

3. The stabilised method

Our discretisation is based on a conforming shape-regular partitioning \mathcal{C}_h of Ω into non-overlapping triangles or tetrahedra, with $h > 0$ denoting the mesh parameter. We denote the interior edges or facets of \mathcal{C}_h by \mathcal{E}_h and let \mathcal{G}_h and \mathcal{N}_h be the partitioning of the boundary parts Γ and Γ_N , respectively, corresponding to the partitioning \mathcal{C}_h , into line segments ($d = 2$) or triangles ($d = 3$). The finite element spaces

$$V_h \subset V, \quad Q_h \subset Q,$$

are finite dimensional and consist of piecewise polynomial functions. We also define

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \text{ on } \Gamma\} \subset \Lambda,$$

and introduce the discrete bilinear form \mathcal{B}_h as

$$\mathcal{B}_h(w, \xi; v, \mu) = \mathcal{B}(w, \xi; v, \mu) - \alpha \mathcal{S}_h(w, \xi; v, \mu),$$

where $\alpha > 0$ is a stabilisation parameter and the stabilising term \mathcal{S}_h is defined as

$$\mathcal{S}_h(w, \xi; v, \mu) = \sum_{E \in \mathcal{G}_h} h_E \left(\xi - \frac{\partial w}{\partial n}, \mu - \frac{\partial v}{\partial n} \right)_E,$$

with h_E denoting the diameter of $E \in \mathcal{G}_h$ and $(\cdot, \cdot)_E$ the $L^2(E)$ -inner product; similarly $(\cdot, \cdot)_K$ denotes the $L^2(K)$ -inner product and $\|\cdot\|_{0,E}$ and $\|\cdot\|_{0,K}$ stand for the $L^2(E)$ and $L^2(K)$ -norms, respectively.

The stabilised finite element method is now formulated as follows: find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\mathcal{B}_h(u_h, \lambda_h; v_h, \mu_h - \lambda_h) \leq \mathcal{L}(v_h) \quad \forall (v_h, \mu_h) \in V_h \times \Lambda_h. \quad (3.1)$$

The proof of stability for method (3.1) is very similar to the one given for the obstacle problem in [16], Theorem 4.1, see also [14], and is thus omitted. We only note that to estimate the stabilising term we use the following inverse estimate, proven by a scaling argument.

Lemma 3.1. *There exists $C_I > 0$, independent of h , such that*

$$C_I \sum_{E \in \mathcal{G}_h} h_E \left\| \frac{\partial v_h}{\partial n} \right\|_{0,E}^2 \leq \|v_h\|_V^2 \quad \forall v_h \in V_h. \quad (3.2)$$

Theorem 3.2 (Discrete stability). *Let $0 < \alpha < C_I$. For all $(v_h, \xi_h) \in V_h \times Q_h$, there exists $w_h \in V_h$, such that*

$$\mathcal{B}_h(v_h, \xi_h; w_h, -\xi_h) \gtrsim \left(\|v_h\|_V + \|\xi_h\|_Q + \left(\sum_{E \in \mathcal{G}_h} h_E \|\xi_h\|_{0,E}^2 \right)^{1/2} \right)^2 \quad (3.3)$$

and

$$\|w_h\|_V \lesssim \|v_h\|_V + \|\xi_h\|_Q. \quad (3.4)$$

Remark 3.3. The additional stability for ξ_h in a mesh-dependent norm in Theorem 3.2 will be needed in the proof of Theorem 3.5.

We will need the following Lemma in establishing the a priori error estimate. Its detailed proof can be found in [14].

Lemma 3.4. *Let $f_h \in V_h$ be the L^2 projection of f , define*

$$\text{osc}_K(f) = h_K \|f - f_h\|_{0,K}$$

and, for each $E \in \mathcal{G}_h$, let $K(E) \in \mathcal{C}_h$ denote the element satisfying $\partial K(E) \cap E = E$. For any $(v_h, \mu_h) \in V_h \times Q_h$ it holds that

$$\begin{aligned} & \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \mu_h - \frac{\partial v_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \\ & \lesssim \|u - v_h\|_V + \|\lambda - \mu_h\|_Q + \left(\sum_{E \in \mathcal{G}_h} \text{osc}_{K(E)}(f)^2 \right)^{1/2}. \end{aligned} \quad (3.5)$$

As usual, the a priori estimate follows from the discrete stability estimate.

Theorem 3.5 (A priori error estimate). *Let $(u, \lambda) \in V \times \Lambda$ be the solution to the continuous problem (2.5) and let $(u_h, \lambda_h) \in V_h \times \Lambda_h$ be its approximation obtained by solving the discrete problem (3.1). Then the following estimate holds*

$$\begin{aligned} \|u - u_h\|_V + \|\lambda - \lambda_h\|_Q & \lesssim \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{\mu_h \in \Lambda_h} (\|\lambda - \mu_h\|_Q + \sqrt{\langle u, \mu_h \rangle}) \\ & \quad + \left(\sum_{E \in \mathcal{G}_h} \text{osc}_{K(E)}(f)^2 \right)^{1/2}. \end{aligned}$$

Proof. In view of Theorem 3.2, it holds that

$$\begin{aligned} & \left(\left(\sum_{E \in \mathcal{G}_h} h_E \|\mu_h - \lambda_h\|_{0,E}^2 \right)^{1/2} + \|u_h - v_h\|_V + \|\lambda_h - \mu_h\|_Q \right)^2 \\ & \lesssim \mathcal{B}_h(u_h - v_h, \lambda_h - \mu_h; w_h, \mu_h - \lambda_h), \end{aligned}$$

with some $w_h \in V_h$ satisfying

$$\|w_h\|_V \lesssim \|u_h - v_h\|_V + \|\lambda_h - \mu_h\|_Q. \tag{3.6}$$

It follows that

$$\begin{aligned} & \mathcal{B}_h(u_h - v_h, \lambda_h - \mu_h; w_h, \mu_h - \lambda_h) \\ & = \mathcal{B}_h(u_h, \lambda_h; w_h, \mu_h - \lambda_h) - \mathcal{B}_h(v_h, \mu_h; w_h, \mu_h - \lambda_h) \\ & \leq \mathcal{L}(w_h) + \mathcal{B}(u - v_h, \lambda - \mu_h; w_h, \mu_h - \lambda_h) - \mathcal{B}(u, \lambda; w_h, \mu_h - \lambda_h) \\ & \quad + \alpha \sum_{E \in \mathcal{G}_h} h_E \left(\mu_h - \frac{\partial v_h}{\partial n}, \mu_h - \lambda_h - \frac{\partial w_h}{\partial n} \right)_E, \end{aligned}$$

where we have used the bilinearity of \mathcal{B} and \mathcal{B}_h , and the discrete problem statement (3.1). Observe that

$$\begin{aligned} \mathcal{L}(w_h) - \mathcal{B}(u, \lambda; w_h, \mu_h - \lambda_h) & = (f, w_h) - (\nabla u, \nabla w_h) + \langle w_h, \lambda \rangle + \langle u, \mu_h - \lambda_h \rangle \\ & \leq \langle u, \mu_h - \lambda_h \rangle + \langle u, \lambda_h - \lambda \rangle = \langle u, \mu_h - \lambda \rangle = \langle u, \mu_h \rangle, \end{aligned}$$

where we have used the problem (2.3) and recalled that $0 \leq \langle u, \mu - \lambda \rangle, \forall \mu \in \Lambda$. Moreover

$$\begin{aligned} & \sum_{E \in \mathcal{G}_h} h_E \left(\mu_h - \frac{\partial v_h}{\partial n}, \mu_h - \lambda_h - \frac{\partial w_h}{\partial n} \right)_E \\ & \leq \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \mu_h - \frac{\partial v_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{G}_h} h_E \|\mu_h - \lambda_h\|_{0,E}^2 \right)^{1/2} \\ & \quad + \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \mu_h - \frac{\partial v_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \frac{\partial w_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2}. \end{aligned}$$

The a priori estimate now follows from the continuity of the bilinear form \mathcal{B} , the inverse estimate (3.2) and bound (3.6), Lemma 3.4, and from the triangle inequality. □

4. A posteriori error analysis

We will define the local error estimators corresponding to the finite element solution (u_h, λ_h) as

$$\begin{aligned}\eta_K^2 &= h_K^2 \|\Delta u_h + f\|_{0,K}^2, & K \in \mathcal{C}_h, \\ \eta_{E,\Omega}^2 &= h_E \|\llbracket \nabla u_h \cdot n \rrbracket\|_{0,E}^2, & E \in \mathcal{E}_h, \\ \eta_{E,\Gamma}^2 &= h_E \left\| \lambda_h - \frac{\partial u_h}{\partial n} \right\|_{0,E}^2, & E \in \mathcal{G}_h, \\ \eta_{E,\Gamma_N}^2 &= h_E \left\| \frac{\partial u_h}{\partial n} \right\|_{0,E}^2, & E \in \mathcal{N}_h,\end{aligned}$$

where $\llbracket \nabla u_h \cdot n \rrbracket$ denotes the jump in the normal derivative across the inter-element boundaries.

We also define the global error estimators η and S through

$$\begin{aligned}\eta^2 &= \sum_{K \in \mathcal{C}_h} \eta_K^2 + \sum_{E \in \mathcal{E}_h} \eta_{E,\Omega}^2 + \sum_{E \in \mathcal{G}_h} \eta_{E,\Gamma}^2 + \sum_{E \in \mathcal{N}_h} \eta_{E,\Gamma_N}^2, \\ S &= \|u_h^-\|_W + \sqrt{(\lambda_h, u_h^+)}.\end{aligned}$$

where $w^+ = \max\{w, 0\}$ denotes the positive and $w^- = \min\{w, 0\}$ the negative part of w . We will now establish both the efficiency and the reliability of the error estimators.

Theorem 4.1 (A posteriori estimate). *It holds that*

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_Q \lesssim \eta + S \quad (4.1)$$

and

$$\eta \lesssim \|u - u_h\|_V + \|\lambda - \lambda_h\|_Q + \left(\sum_{K \in \mathcal{C}_h} \text{osc}_K(f)^2 \right)^{1/2}. \quad (4.2)$$

Proof. The continuous stability condition (cf. Theorem 2.4) guarantees the existence of $v \in V$ such that

$$\|v\|_V \lesssim \|u - u_h\|_V + \|\lambda - \lambda_h\|_Q \quad (4.3)$$

and

$$(\|u - u_h\|_V + \|\lambda - \lambda_h\|_Q)^2 \lesssim \mathcal{B}(u - u_h, \lambda - \lambda_h; v, \lambda_h - \lambda).$$

On the other hand, testing in (3.1) with $(-\tilde{v}, \lambda_h)$, where \tilde{v} is the Clément interpolant of v , we obtain

$$0 \leq -\mathcal{B}_h(u_h, \lambda_h, -\tilde{v}, 0) + \mathcal{L}(-\tilde{v}).$$

It follows that

$$\begin{aligned} & (\|u - u_h\|_V + \|\lambda - \lambda_h\|_Q)^2 \\ & \lesssim \mathcal{B}(u, \lambda; v, \lambda_h - \lambda) - \mathcal{B}(u_h, \lambda_h; v, \lambda_h - \lambda) \\ & \quad - \mathcal{B}(u_h, \lambda_h, -\tilde{v}, 0) + \mathcal{L}(-\tilde{v}) + \alpha \mathcal{S}_h(u_h, \lambda_h, -\tilde{v}, 0) \\ & \lesssim \mathcal{L}(v - \tilde{v}) - \mathcal{B}(u_h, \lambda_h; v - \tilde{v}, \lambda_h - \lambda) + \alpha \mathcal{S}_h(u_h, \lambda_h, -\tilde{v}, 0), \end{aligned}$$

where we have used the bilinearity of \mathcal{B} and the problem statement (2.5).

Observe first that

$$\begin{aligned} & \mathcal{L}(v - \tilde{v}) - \mathcal{B}(u_h, \lambda_h; v - \tilde{v}, \lambda_h - \lambda) \\ & = (f, v - \tilde{v}) - (\nabla u_h, \nabla(v - \tilde{v})) + \langle v - \tilde{v}, \lambda_h \rangle + \langle u_h, \lambda_h - \lambda \rangle. \end{aligned} \quad (4.4)$$

After elementwise integration by parts, the first four terms on the right-hand side of (4.4) read

$$\begin{aligned} & \sum_{K \in \mathcal{G}_h} (\Delta u_h + f, v - \tilde{v})_K - \sum_{E \in \mathcal{E}_h} ([\nabla u_h \cdot n], v - \tilde{v})_E \\ & \quad + \sum_{E \in \mathcal{G}_h} \left(\lambda_h - \frac{\partial u_h}{\partial n}, v - \tilde{v} \right)_E - \sum_{E \in \mathcal{N}_h} \left(\frac{\partial u_h}{\partial n}, v - \tilde{v} \right)_E. \end{aligned}$$

The last term in (4.4) can be estimated as follows

$$\langle u_h, \lambda_h - \lambda \rangle \leq (\lambda_h, u_h^+) + \langle u_h^-, \lambda_h - \lambda \rangle \leq (\lambda_h, u_h^+) + \|\lambda - \lambda_h\|_Q \|u_h^-\|_W,$$

given that $\langle u_h^+, \lambda \rangle \geq 0$. For the stabilisation term $\mathcal{S}_h(u_h, \lambda_h, -\tilde{v}, 0)$, we obtain

$$\begin{aligned} & \sum_{E \in \mathcal{G}_h} h_E \left(\lambda_h - \frac{\partial u_h}{\partial n}, \frac{\partial \tilde{v}}{\partial n} \right)_E \\ & \leq \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \lambda_h - \frac{\partial u_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \frac{\partial \tilde{v}}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \\ & \lesssim \left(\sum_{E \in \mathcal{G}_h} h_E \left\| \lambda_h - \frac{\partial u_h}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \|\tilde{v}\|_V, \end{aligned}$$

where in the last step we have used the discrete inverse estimate (3.2).

For the Clément interpolant \tilde{v} , it holds that

$$\|\tilde{v}\|_V^2 + \sum_{K \in \mathcal{G}_h} h_K^{-2} \|v - \tilde{v}\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h \cup \mathcal{G}_h \cup \mathcal{A}_h} h_E^{-1} \|v - \tilde{v}\|_{0,E}^2 \lesssim \|v\|_V^2. \quad (4.5)$$

The reliability estimate (4.1) can now be established using the Cauchy–Schwarz inequality together with (4.3) and (4.5).

The efficiency follows from standard lower bounds [34] and from Lemma 3.2. \square

5. Nitsche’s method

Nitsche’s formulation can be elegantly derived from the stabilised method following the reasoning suggested in [32]. In fact, testing with $(0, \mu_h)$ in (3.1), yields

$$(u_h, \mu_h - \lambda_h) + \alpha \sum_{E \in \mathcal{G}_h} h_E \left(\lambda_h - \frac{\partial u_h}{\partial n}, \mu_h - \lambda_h \right)_E \geq 0 \quad \forall \mu_h \in \Lambda_h.$$

In particular,

$$(u_h, \lambda_h) + \alpha \sum_{E \in \mathcal{G}_h} h_E \left(\lambda_h - \frac{\partial u_h}{\partial n}, \lambda_h \right)_E = 0.$$

By a standard argument, we thus obtain locally

$$\lambda_h|_E = \Pi_h \frac{\partial u_h}{\partial n} \Big|_E - (\alpha h_E)^{-1} \Pi_h u_h|_E \quad \forall E \in \Gamma_C, \quad (5.1)$$

where Π_h is the L^2 projection onto Λ_h . Assuming equal polynomial order for both variables, say $k \geq 1$, and recalling that the discrete Lagrange multiplier is discontinuous from element to element, it follows that $\Pi_h = I$ and we have

$$Q_h = \{\mu_h \in Q : \mu_h \in P_k(E) \forall E \in \mathcal{G}_h\}.$$

Therefore, using the test function $(v_h, 0)$ in (3.1) and substituting (5.1) into the resulting equation, leads to

$$\begin{aligned}
 & (\nabla u_h, \nabla v_h) + \sum_{E \in \Gamma_C} (\alpha h_E)^{-1} (u_h, v_h)_E - \sum_{E \in \Gamma_C} \left\{ \left(\frac{\partial u_h}{\partial n}, v_h \right)_E + \left(u_h, \frac{\partial v_h}{\partial n} \right)_E \right\} \\
 & - \alpha \sum_{E \in \Gamma \setminus \Gamma_C} h_E \left(\frac{\partial u_h}{\partial n}, \frac{\partial v_h}{\partial n} \right)_E = (f, v_h) \quad \forall v_h \in V_h.
 \end{aligned}$$

Defining an $L^2(\Gamma)$ function \hat{h} through

$$\hat{h}|_E = h_E \quad \forall E \in \mathcal{G}_h,$$

the discrete Lagrange multiplier can be written globally as

$$\lambda_h = \left(\frac{\partial u_h}{\partial n} \Big|_{\Gamma} - (\alpha \hat{h})^{-1} u_h \Big|_{\Gamma} \right)^+ \tag{5.2}$$

and the discrete contact region becomes

$$\Gamma_C^h = \{(x, y) \in \Gamma : \lambda_h(x, y) > 0\}.$$

Nitsche's method thus reads as follows: find $u_h \in V_h$ and $\Gamma_C^h = \Gamma_C^h(u_h)$ such that

$$\begin{aligned}
 & (\nabla u_h, \nabla v_h)_{\Omega} + \alpha^{-1} (\hat{h}^{-1} u_h, v_h)_{\Gamma_C^h} - \left(\frac{\partial u_h}{\partial n}, v_h \right)_{\Gamma_C^h} - \left(u_h, \frac{\partial v_h}{\partial n} \right)_{\Gamma_C^h} \\
 & - \alpha \left(\hat{h} \frac{\partial u_h}{\partial n}, \frac{\partial v_h}{\partial n} \right)_{\Gamma \setminus \Gamma_C^h} = (f, v_h) \quad \forall v_h \in V_h.
 \end{aligned} \tag{5.3}$$

Remark 5.1. Note that formulation (5.3) is equivalent to the Nitsche's method introduced in [10], with $\gamma = \alpha \hat{h}$, and to the symmetric variant ($\theta_1 = -1$) of the Nitsche's method proposed in [11].

For the implementational aspects of the Nitsche's method (5.3), we refer to [17] where a similar method was applied to the obstacle problem.

6. Numerical verification

Let us consider problem (5.3) in $\Omega = (0, 1)^2$ with the loading function $f(x, y) = x \cos(2\pi y)$ and the boundary $\partial\Omega$ divided into the parts

$$\begin{aligned}
 \Gamma_D &= \{(x, y) \in \mathbb{R}^2 : x = 0, 0 < y < 1\}, \\
 \Gamma_N &= \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = 1\}, \\
 \Gamma &= \{(x, y) \in \mathbb{R}^2 : x = 1, 0 < y < 1\}.
 \end{aligned}$$

The contact region is now a proper subset of Γ . There are two points on Γ where the constraints change from binding to non-binding and the exact solution belongs to H^r , $r < 5/2$, see the comments in Remark 2.3. We employ Nitsche's method with the quadratic finite element space

$$V_h = \{w \in V : w|_K \in P_2(K) \forall K \in \mathcal{C}_h\}.$$

Thus, in view of the regularity of u , the convergence rate of the error $\|u - u_h\|_V$ with uniform mesh refinement is limited to $\mathcal{O}(N^{-3/4})$ where N is the number of degrees of freedom. Note that for a uniform mesh the relationship between the number of degrees of freedom and the mesh parameter is $N \sim h^{-2}$ so the optimal convergence rate, $\mathcal{O}(h^2)$, is not attained.

To overcome the limited regularity, we consider also a sequence of adaptive meshes. Starting with an initial mesh, we compute the discrete solution u_h , the corresponding Lagrange multiplier through (5.2), and the respective error indicator

$$\begin{aligned} \mathcal{E}_K^2 &= h_K^2 \|\Delta u_h + f\|_{0,K}^2 + h_K \|\llbracket \nabla u_h \cdot n \rrbracket\|_{0,\partial K \setminus \partial \Omega}^2 \\ &\quad + h_K \left\| \lambda_h - \frac{\partial u_h}{\partial n} \right\|_{0,\partial K \cap \Gamma}^2 + h_K \left\| \frac{\partial u_h}{\partial n} \right\|_{0,\partial K \cap \Gamma_N}^2 \end{aligned}$$

for every K in the initial mesh. The mesh is then improved by splitting the elements that have large error indicators. This process is performed repeatedly until a predetermined value of N is reached. Please refer to [14] or [18] for more details on how to choose and split the elements.

The discrete solution is visualised in Figure 1. The resulting sequence of adaptive meshes is given in Figure 2 and the convergence of the global error estimator $\eta + S$ is compared between the uniform and adaptive mesh sequences in Figure 3. As expected, the adaptive meshes are more refined near the singular points on the contact boundary. Moreover, the observed convergence rate of the uniform refinement strategy is suboptimal, due to the limited regularity of the exact solution, whereas the adaptive method successfully recovers the optimal rate of convergence, $\mathcal{O}(N^{-1})$.

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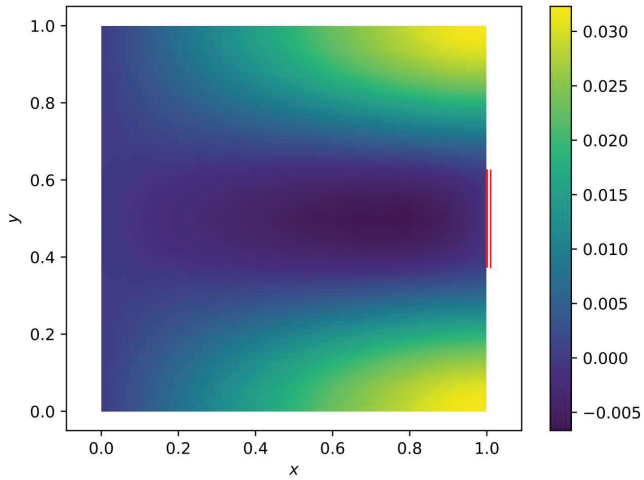


Figure 1. The discrete solution after 8 adaptive refinements. The contact region on the rightmost boundary is highlighted in red.

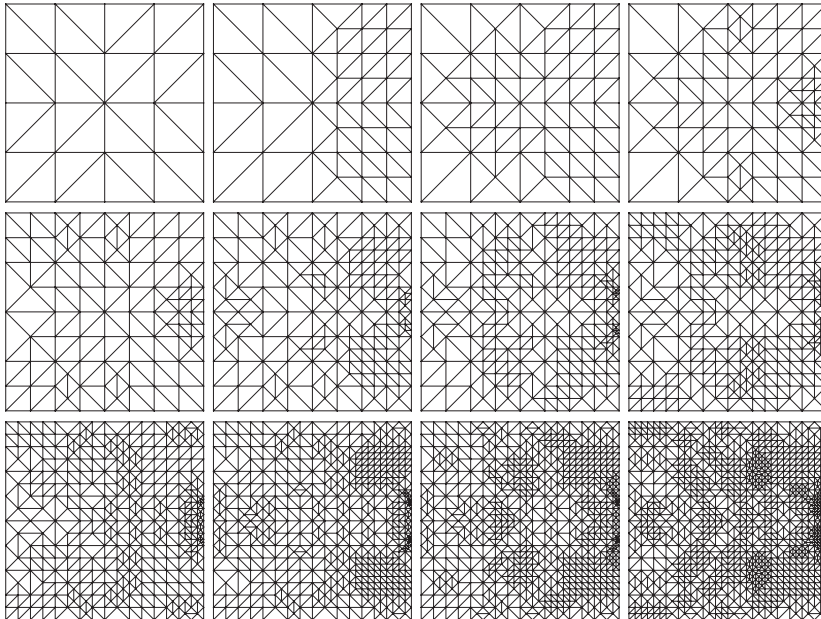


Figure 2. The initial mesh and the resulting sequence of adaptively refined meshes.

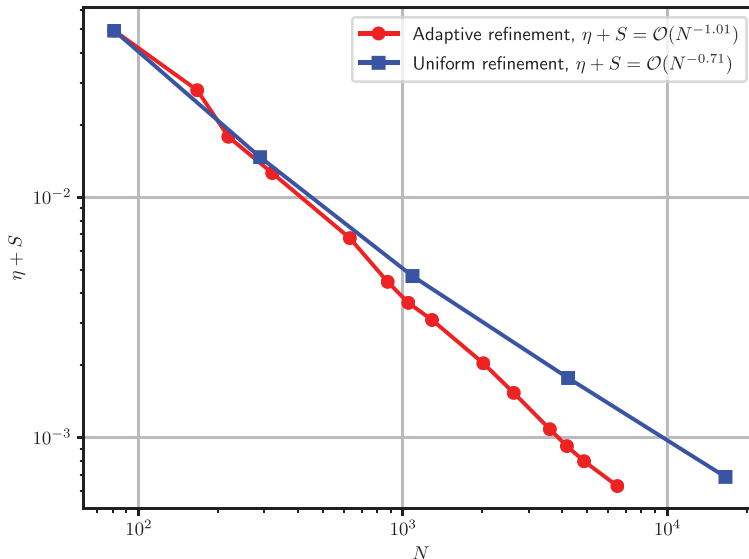


Figure 3. Convergence of the error estimator $\eta + S$ as a function of the number of degrees of freedom N for the uniform and the adaptive mesh sequences. The observed approximate convergence rates are given in the text box window.

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