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An extremal property of lattice polygons

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Abstract. We find the critical number of vertices of a convex lattice polygon that guarantees that the polygon contains at least one point of a given square sublattice. As a tool, we prove Diophantine inequalities relating the number of edges of a broken line and the coordinates of its endpoints.

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1. Introduction

Lattice points in convex sets is a classical subject. It started with Minkowski's Convex Body Theorem, which became the foundation of the geometry of numbers. The theorem states that if a compact set in $\mathbb{R}^d$ is symmetric with respect to origin and has volume at least 2^d, then it contains a point of the integral lattice $\mathbb{Z}^d$. The constant 2^d cannot be improved. This theorem has quite a few modificatioins and generalizations, see e.g. the nice short survey [23].

There are numerous results concerning lattice points in various regions, see e.g. [7], [9], [11], [13]. The regions at issue can be either general convex and nonconvex or polyhedral sets. Among more recent works we note the following that are close to ours. The papers [2], [6], [18], [19] deal with the largest possible number of facets of maximal lattice-free polytopes. The papers [3], [15], [17], [20] study properties of lattice polytopes having a specified (positive) number of interior lattice points such as upper bounds for the volume and the number of sublattice points and a classification of such polytopes. The papers [21], [22] deal with similar issues for polygons.

Besides, there are other interesting results about lattice polygons, such as [1], [23], [24], not to mention Pick's well-known theorem.

In this paper we consider the natural problem of relating the existence of sublattice points in a convex lattice polygon to the number of vertices (or edges) of the polygon.

In higher dimensions, a large number of faces cannot guarantee that the polytope contains a point of a given sublattice. For instance, there is no upper bound for the number of vertices and facets of polytopes in $\mathbb{R}^3$ free of points of $(2\mathbb{Z})^3$.

Surprisingly, things are different in two dimensions. It follows from known results (e.g. [16]) that given an integer n, any convex lattice polygon with enough vertices contains a point of the lattice $(n\mathbb{Z})^2$. It was noticed in [8] that any convex integral pentagon on the plain contains a point of $(2\mathbb{Z})^2$.

In the spirit of the Minkowski Convex Body Theorem, our main goal is to explicitly state the critical number of vertices that ensures that the convex lattice polygon contains a point of $(n\mathbb{Z})^2$, where $n \geq 3$ is a given integer. The answer is

Theorem 1.1. *Given an integer $n \geq 3$, any convex integral polygon with $2n+3$ vertices contains a point of $(n\mathbb{Z})^2$.*

Clearly, the constant $2n+3$ is optimal, since it is easy to construct a polygon with $2n+2$ vertices lying in the slab $\{(x_1, x_2) : 0 \leq x_2 \leq n\}$ and free from points of $(n\mathbb{Z})^2$.

Curiously, in the case $n = 2$ the critical value is $5 < 2n + 3$. This apparent exception is accomodated by the formula $v(\delta, n) = 2n + 2\min(\delta, 3) - 3$ for the critical number of vertices ensuring that the lattice polygon contains a point of the 'rectangular' lattice $\delta\mathbb{Z} \times n\mathbb{Z}$, where δ divides n. The proof of this generalization of Theorem 1.1 is out of scope of this paper.

Theorem 1.1 can be alternatively cast as a sharp bound on the number of vertices of sublattice-free polygons:

Theorem 1.2. *Given an integer $n \geq 3$, any convex integral polygon free from points of $(n\mathbb{Z})^2$ has at most $2n+2$ vertices.*

We prefer to prove it in this formulation. Due to the optimality of the result, the proof of Theorem 1.2 requires a subtle geometric analysis. It is broken down into estimating the number of vertices for particular classes of polygons free from points of $(n\mathbb{Z})^2$. The general idea is to break up the boundary of a polygon into several broken lines and to translate geometrical constraints imposed on the polygon into Diaphantine inequalities. This translation is carried out by means of tools developed in Sections 3 and 4. Resulting inequalities relate the numbers of edges of the broken lines and the coordinates of their endpoints, which are also the overall dimensions of the polygon. Subsequent analysis of the inequalities can prove rather technical owing to the number of parameters and nonlinearities.

The rest of the paper is organized as follows.

In Section 2 we briefly discuss lattices and lattice polygons. In particular, we define the classes of polygons the proof of Theorem 1.2 can be restricted to.

In Sections 3 and 4 we study a class of broken lines we call slopes. Section 3 gathers core tools used to translate geometric constraints on broken lines into Diaphantine inequalities. Section 4 contains the proofs missing from the previous section and is not used in subsequent sections.

In Section 5 we prove the reduction of the general case to several particular types of polygons. This section uses notation introduced in Section 3 but does not require more advanced the tools developed there.

In Sections 6–9 we consider particular classes of polygons obtaining bounds on the number of verties.

Theorem 1.2 is proved in Section 10.

2. Preliminaries

In what follows we use standard notations $\lfloor \cdot \rfloor$ for the floor function, $\lceil \cdot \rceil$ for the ceiling function, $^+$ for the positive part, and $|\cdot|$ for the cardinality of a finite set. By $[\mathbf{a}, \mathbf{b}]$ we denote the segment with the endpoints $\mathbf{a}$ and $\mathbf{b}$. We always denote the vectors of the standard basis of $\mathbb{R}^2$ by $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ and the standard coordinates in $\mathbb{R}^2$ by x_1, x_2.

Now we list a few familiar properties of lattices. The proofs can be found in [7], [9], [12], [13].

Suppose the system of vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ is linearly independent; then the set

$$\{u_1\mathbf{a}_1 + u_2\mathbf{a}_2 : u_1, u_2 \in \mathbb{Z}\}$$

is called *a lattice* spanned by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_1$, $\mathbf{a}_2$ are called the *basis* of the lattice.

Example 2.1. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$ span the *integral lattice* denoted by $\mathbb{Z}^2$. It is the set of points with both coordinates integral. Those are called *integral points*.

Example 2.2. Given $n \in \mathbb{Z}$, the vectors $n\mathbf{e}_1$, $n\mathbf{e}_2$ span the lattice $n\mathbb{Z}^2$, alternatively designated $(n\mathbb{Z})^2$.

Note that any lattice is a subgroup of the additive group of the linear space $\mathbb{R}^2$ and a free abelian group of rank 2.

A matrix $A \in M_2(\mathbb{Z})$ is called *unimodular*, if $\det A = \pm 1$.

Proposition 2.3. *Let* $(\mathbf{f}_1, \mathbf{f}_2)$ *be a basis of a lattice* Λ*; then the vectors* $a_{i1}\mathbf{f}_j + a_{i2}\mathbf{f}_2$, *where* $i = 1, 2$, *form a basis of* Λ *if and only if the matrix* (a_{ij}) *is unimodular.*

We use the term 'Λ-point' as a synonym of 'point of Λ'.

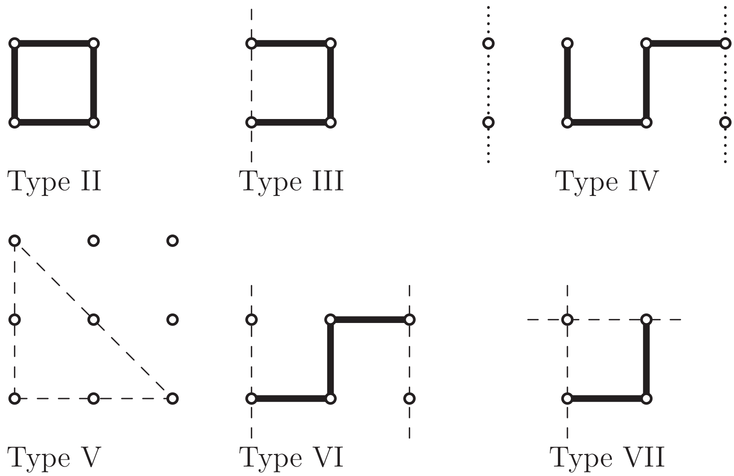

Figure 1. Definitions 2.5 and 5.1 (see next section) introduce the types of polygons in terms of intersection with segments and lines. Here thick segments split polygons of the specified type, dashed lines do not split them, and dotted lines have no common points with them.

A linear transformation of the plane is called a (linear) automorphism of a lattice if it maps the lattice onto itself. It is easily seen that a linear transformation is an automorphism of a lattice if and only if it maps some (hence, any) basis of the lattice onto another basis. Consequently, given a matrix $A \in M_2(\mathbb{R})$, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of $\mathbb{Z}^2$ if and only if the matrix A is unimodular. We call such transformation *unimodular*. For any positive integer n, the automorphisms of $n\mathbb{Z}^2$ are exactly unimodular transformations.

Clearly, linear automorphisms of a lattice form a group.

Let Λ be a lattice. A vector $\mathbf{f} \in \Lambda$ is called Λ-*primitive*, if any representation $\mathbf{f} = u\mathbf{g}$ with $g \in \Lambda$ and $u \in \mathbb{Z}$ implies $u = \pm 1$.

Proposition 2.4. *Suppose that* Λ *is a lattice and* $\mathbf{f}$ *and* $\mathbf{g}$ *are* Λ-*primitive vectors; then there exists an automorphism* A *of* Λ *such that* $A\mathbf{f} = \mathbf{g}$.

An *affine automorphism* of a lattice Λ is an affine transformation of $\mathbb{R}^2$ mapping Λ onto itself. It is not hard to see that given $A \in M_2(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^2$, the mapping $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ is an affine automorphism of Λ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of Λ and $\mathbf{b} \in \Lambda$. In particular, affine automorphisms of $n\mathbb{Z}^2$, where n is a positive integer, are exactly the transformations of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, where A is unimodular and $\mathbf{b} \in n\mathbb{Z}^2$.

An *affine frame* of a lattice Λ is a pair $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ consisting of a point $\mathbf{o} \in \Lambda$ and a basis $(\mathbf{f}_1, \mathbf{f}_2)$ of Λ. An *integral frame* is an affine frame of $\mathbb{Z}^2$.

The *convex polygon* is a two-dimensional polytope, i.e. the convex hull of a finite set of points that has nonempty interior. In what follows we only consider convex polygons, so we often drop the word 'convex'. We assume that the reader is familiar with basic terminology such as vertex and edge, see [14], [25] for details. A polygon with N vertices, $N \geq 3$, is called an N-gon. The vertices of an *integral polygon* belong to $\mathbb{Z}^2$. Integral polygons are also called *lattice polygons*.

Of course, if P is a convex integral N-gon and φ is an affine automorphism of $\mathbb{Z}^2$, the image $\varphi(P)$ is still a convex integral N-gon. Obviously, if P is free from points of a lattice Λ, then so is its image under any affine automorphism of Λ.

We say that a line or a segment *splits* a polygon, if it divides the polygon into two parts with nonempty interior.

It is convenient to introduce the following classes of polygons.

Definition 2.5. Let $n \geq 3$ be an integer, and P be an integral polygon free of points of $n\mathbb{Z}^2$. We say that P is a

- *type I_n polygon*, if no line of the form $x_1 = jn$ or $x_2 = jn$ where $j \in \mathbb{Z}$, splits P, or, equivalently, if P lies in a slab of the form $jn \leq x_1 \leq (j+1)n$ or $jn \leq x_2 \leq (j+1)n$, where $j \in \mathbb{Z}$;
- *type II_n polygon*, if each of the segments $[\mathbf{0}, (n,0)]$, $[(n,0),(n,n)]$, $[(0,n),(n,n)]$, and $[\mathbf{0},(0,n)]$ splits P;
- *type III_n polygon*, if each of the segments $[\mathbf{0},(n,0)]$, $[(n,0),(n,n)]$, and $[(n,n),(0,n)]$ splits P, and the line $x_1 = 0$ does not split P;
- *type IV_n polygon*, if each of the segments $[\mathbf{0},(0,n)]$, $[\mathbf{0},(n,0)]$, $[(n,0),(n,n)]$, and $[(n,n),(2n,n)]$ splits P and P has no common points with the lines $x_1 = -n$ and $x_1 = 2n$;
- *type V_n polygon*, if it lies in the triangle

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_1 + x_2 \leq 2n. \tag{2.1}$$

The types of polygons are illustrated on Figure 1. The following theorem is proved in Section 5:

Theorem 2.6. *Suppose that an integral polygon P is free of points of the lattice $n\mathbb{Z}^2$, where $n \in \mathbb{Z}$, $n \geq 2$; then there exists an affine automorphism φ of $n\mathbb{Z}^2$ such that $\varphi(P)$ is a polygon of one of the types I_n–V_n.*

Remark 2.7. It follows from Proposition 2.6 that it suffices to prove Theorem 1.2 for polygons of types I–V.

Remark 2.8. Theorem 1.2 is obvious for type I_n polygons, since all the vertices of any such polygon lie on one of $n+1$ lines, and each line can contain at most two vertices. The rest types require specific tools introduced in Section 3.

3. Slopes

3.1. Slopes. Let $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{R}^2$, and let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_N$ $(N \geq 0)$ be a finite sequence of points on the plane. If $N \geq 1$, set

$$\mathbf{v}_i - \mathbf{v}_{i-1} = \mathbf{a}_i = a_{i1}\mathbf{f}_1 + a_{i2}\mathbf{f}_2 \qquad (i = 1, \ldots, N). \tag{3.1}$$

If

$$a_{i1} > 0, \qquad a_{i2} < 0 \qquad (i = 1, \ldots, N) \tag{3.2}$$

and

$$\begin{vmatrix} a_{i1} & a_{i+1,1} \\ a_{i2} & a_{i+1,2} \end{vmatrix} > 0 \qquad (i = 1, \ldots, N-1), \tag{3.3}$$

we say that the union Q of the segments $[\mathbf{v}_0, \mathbf{v}_1], [\mathbf{v}_1, \mathbf{v}_2], \ldots, [\mathbf{v}_{N-1}, \mathbf{v}_N]$ is a *slope* with respect to the basis $(\mathbf{f}_1, \mathbf{f}_2)$. These segments are called the *edges* of the slope, and the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_N$, its *vertices*, $\mathbf{v}_0$ and $\mathbf{v}_N$ being the *endpoints*. If $N = 1$, we call the segment $[\mathbf{v}_0, \mathbf{v}_1]$ a slope if (3.2) holds, and if $N = 0$, we still call the one-point set $\{\mathbf{v}_0\}$ a slope. If all the vertices of Q belong to a lattice Γ, we call it a Γ-*slope*. A $\mathbb{Z}^2$-slope is called *integral*, and it is the only kind of slopes we are interested in.

It is not hard to prove that the vertices and edges of a slope are uniquely defined, and that the basis induces a unique ordering of vertices.

Figure 2 illustrates the concepts of a slope and of an affine frame splitting a slope, to be considered below.

Remark 3.1. If Q is a slope with respect to a basis $(\mathbf{f}_1, \mathbf{f}_2)$, then it is a slope with respect to the basis $(\mathbf{f}_2, \mathbf{f}_1)$, too.

Although the following statement is simple, it provides an important tool for estimating the number of edges of a slope. We are interested in comparing the doubled number of edges with the 'width' of the slope, i.e. its projection on the axis spanned by $\mathbf{f}_1$.

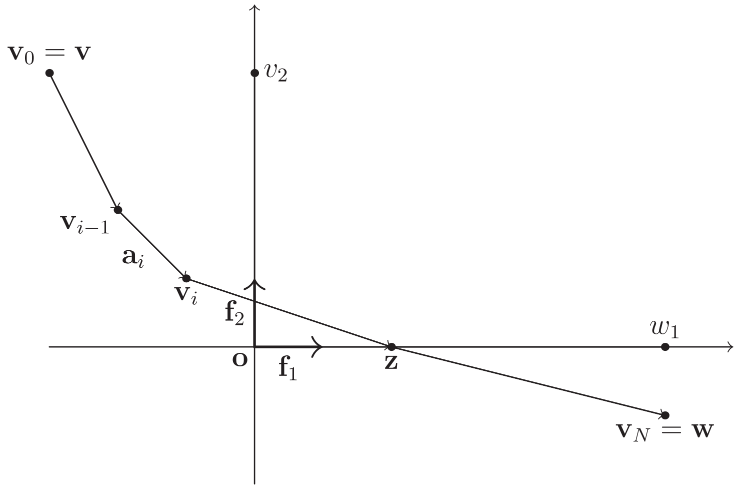

Figure 2. The broken line is a slope with respect to the basis $(\mathbf{f}_1, \mathbf{f}_2)$. It is convex and the vectors associated with its edges point down and to the right. The frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits the slope and forms small angle with it, since there is a supporting line passing through the point $\mathbf{z}$ and forming an angle $\leq \pi/4$ with the axis.

Proposition 3.2. *Let $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$ and $\mathbf{v}$ and $\mathbf{w}$ be the endpoints of an integral slope (with respect to $(\mathbf{f}_1, \mathbf{f}_2)$) having N edges. Let*

$$\mathbf{w} - \mathbf{v} = b_1\mathbf{f}_1 + b_2\mathbf{f}_2.$$

Then there exists an integer s such that

$$2N \leq |b_1| + s, \tag{3.4}$$

$$|b_2| \geq \frac{s(s+1)}{2}, \tag{3.5}$$

$$0 \leq s \leq N. \tag{3.6}$$

Proof. Let $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_1, \ldots, \mathbf{v}_N = \mathbf{w}$ be the vertices of the slope and assume that (3.1)–(3.3) hold. It follows from (3.2) and (3.3) that $\mathbf{a}_i \neq \mathbf{a}_j$ for $i \neq j$. Set $A = \{\mathbf{a}_i : a_{i1} = 1\}$ and $s = |A|$. Observe that s satisfies (3.6).

Let us prove (3.4). If $\mathbf{a}_i \notin A$, we have $a_{i1} \geq 2$, so

$$|b_1| = \sum_{i=1}^{N} a_{i1} = \sum_{\mathbf{a} \in A}^{N} a_{i1} + \sum_{\mathbf{a} \notin A}^{N} a_{i1} \geq |A| + 2(N - |A|) = 2N - s,$$

and (3.4) follows.

Let us prove (3.5). It is easily seen that the vectors belonging to A are of the form $\mathbf{f}_1 - u\mathbf{f}_2$, where $u \in \mathbb{Z}$, $u \geq 0$. Thus,

$$|b_2| = \sum_{i=1}^{N}(-a_{i2}) \geq \sum_{\mathbf{a}_i \in A}(-a_{i2}) \geq 1 + 2 + \cdots + s = \frac{s(s+1)}{2},$$

as claimed. □

3.2. Splitting frames. Let $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an integral frame and Q be a slope with respect to $(\mathbf{f}_1, \mathbf{f}_2)$.

Definition 3.3. We say that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *splits* the slope Q if

(1) one endpoint $\mathbf{v} = \mathbf{o} + v_1\mathbf{f}_1 + v_2\mathbf{f}_2$ of Q satisfies

$$v_1 < 0, \qquad v_2 > 0, \tag{3.7}$$

while the other endpoint $\mathbf{w} = \mathbf{o} + w_1\mathbf{f}_1 + w_2\mathbf{f}_2$ satisfies

$$w_1 > 0, \qquad w_2 < 0; \tag{3.8}$$

(2) there exists a point on Q having both positive coordinates in the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$.

Remark 3.4. Obviously, a frame can only split a slope if the slope has at least one edge.

Remark 3.5. If an integral frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope Q, it is obvious that Q has no points in the quadrant $\{\mathbf{o} + \lambda_1\mathbf{f}_1 + \lambda_2\mathbf{f}_2 : \lambda_1, \lambda_2 \leq 0\}$.

Suppose that a frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope Q and let $\mathbf{z}$ be the point where Q meets the ray $\{\mathbf{o} + \lambda\mathbf{f}_1 : \lambda \geq 0\}$. If there is a supporting line for Q passing through $\mathbf{z}$ that forms an angle $\leq \pi/4$ with the ray, we say that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *forms small angle* with the slope Q.

Proposition 3.6. *Suppose that an integral frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope Q; then the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ splits it as well, and at least one of the frames forms small angle with Q. If there exists a point $\mathbf{y} = \mathbf{o} + y_1\mathbf{f}_1 + y_2\mathbf{f}_2 \in Q$ such that $y_2 > 0$ and $y_1 + y_2 \leq 0$, then $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with Q.*

The proof is left to the reader.

The following theorem provides a more sophisticated bound of the number of edges of a slope than Proposition 3.2. This time we are comparing the doubled number of edges with the length of the projection of the slope on the positive half-axes of the frame, where by the projection on a half-axis we mean the intersection of the projection on the axis with the half-axis. It turns out that the doubled number of edges is always less then or equal to the total length of the projection.

Theorem 3.7. *Suppose that an integral frame* $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *splits an integral slope* Q *having* N *edges and the endpoints* $\mathbf{v} = \mathbf{o} + v_1\mathbf{f}_1 + v_2\mathbf{f}_2$ *and* $\mathbf{w} = \mathbf{o} + w_1\mathbf{f}_1 + w_2\mathbf{f}_2$ *satisfying* (3.7) *and* (3.8). *Then there exist* $s \in \mathbb{Z}$ *and* $t \in \mathbb{Z}$ *such that*

$$0 \le s \le t, \tag{3.9}$$

$$v_2 - s \ge 0, \tag{3.10}$$

$$-v_1 < ts - \frac{s^2 - s}{2} + (v_2 - s)(t + 1), \tag{3.11}$$

$$2N \le v_2 + w_1 - t + s. \tag{3.12}$$

Moreover, if $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *forms small angle with* Q, *we have*

$$2N \le v_2 + w_1 - t + s - \left\lceil \frac{-w_2}{2} \right\rceil + 1. \tag{3.13}$$

Corollary 3.8. *Under the hypotheses of Theorem* 3.7,

$$2N \le v_2 + w_1,$$

and if $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *forms small angle with* Q, *then*

$$2N \le v_2 + w_1 - \left\lceil \frac{-w_2}{2} \right\rceil + 1.$$

The proof of Theorem 3.7 is rather technical and is deferred to Section 4.

3.3. The boundary of a convex polygon. Let P be a convex integral polygon. Define

$$\begin{aligned}
\mathscr{N} &= \max\{x_2 : (x_1, x_2) \in P\}, & \mathscr{S} &= \min\{x_2 : (x_1, x_2) \in P\},\\
\mathscr{N}_- &= \min\{x_1 : (x_1, \mathscr{N}) \in P\}, & \mathscr{S}_- &= \min\{x_1 : (x_1, \mathscr{S}) \in P\},\\
\mathscr{N}_+ &= \max\{x_1 : (x_1, \mathscr{N}) \in P\}, & \mathscr{S}_+ &= \max\{x_1 : (x_1, \mathscr{S}) \in P\},\\
\mathscr{W} &= \min\{x_1 : (x_1, x_2) \in P\}, & \mathscr{E} &= \max\{x_1 : (x_1, x_2) \in P\},\\
\mathscr{W}_- &= \min\{x_2 : (\mathscr{W}, x_2) \in P\}, & \mathscr{E}_- &= \min\{x_2 : (\mathscr{E}, x_2) \in P\},\\
\mathscr{W}_+ &= \max\{x_2 : (\mathscr{W}, x_2) \in P\}, & \mathscr{E}_+ &= \max\{x_2 : (\mathscr{E}, x_2) \in P\}.
\end{aligned}$$

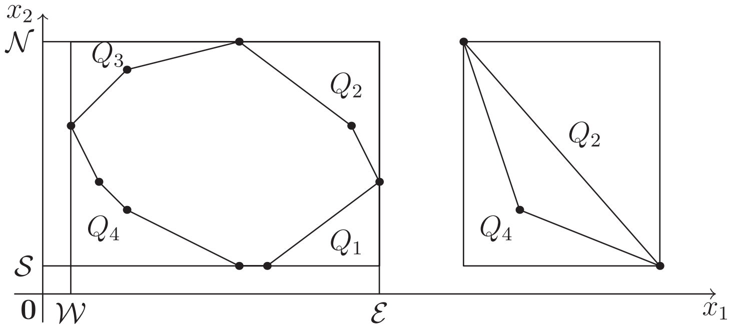

Figure 3. The edges of a polygon not belonging to the bounding box form four maximal slopes Q_k. These slopes may degenerate into a point, as is the case for the triangle on the right, which has only two nontrivial maximal slopes.

All these are integers. Note that $(\mathscr{S}_-, \mathscr{S})$, $(\mathscr{S}_+, \mathscr{S})$, $(\mathscr{N}_-, \mathscr{N})$, $(\mathscr{N}_+, \mathscr{N})$, $(\mathscr{W}, \mathscr{W}_-)$, $(\mathscr{W}, \mathscr{W}_+)$, $(\mathscr{E}, \mathscr{E}_-)$, and $(\mathscr{E}, \mathscr{E}_+)$ are (not necessarily distinct) vertices of P.

There are four slopes naturally associated with a given polygon P.

Let us enumerate the vertices of P starting from $\mathbf{v}_0 = (\mathscr{W}, \mathscr{W}_-)$ and going in the positive direction until we come to $v_{N_4} = (\mathscr{S}_-, \mathscr{S})$. Clearly, the sequence $\mathbf{v}_0, \dots, \mathbf{v}_{N_4}$ gives rise to a slope with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$. We denote it by Q_4. Obviously, Q_4 is an inclusion-wise maximal slope with respect to $(\mathbf{e}_1, \mathbf{e}_2)$ contained in the boundary of P. Likewise, we define the slope Q_1 with respect to $(\mathbf{e}_2, -\mathbf{e}_1)$ having the endpoints $(\mathscr{S}_+, \mathscr{S})$ and $(\mathscr{E}, \mathscr{E}_-)$, the slope Q_2 with respect to $(-\mathbf{e}_1, -\mathbf{e}_2)$ having the endpoints $(\mathscr{E}, \mathscr{E}_+)$ and $(\mathscr{N}_+, \mathscr{N})$, and the slope Q_3 with respect to $(-\mathbf{e}_2, \mathbf{e}_1)$ having the endpoints $(\mathscr{N}_-, \mathscr{N})$ and $(\mathscr{W}, \mathscr{W}_+)$. We call those *maximal slopes* of the polygon P and denote by N_k the number of edges of Q_k.

Remark 3.9. In general, Q_k is not the only maximal slope with respect to the correspondent basis, because there are other single-point maximal slopes. However, we single Q_k out by explicitly indicating its endpoints. For a given polygon, some of the maximal slopes Q_k may have but one vertex.

Define

$$M_1 = \begin{cases} 0, & \text{if } \mathscr{S}_- = \mathscr{S}_+, \\ 1, & \text{otherwise;} \end{cases} \qquad M_2 = \begin{cases} 0, & \text{if } \mathscr{E}_- = \mathscr{E}_+, \\ 1, & \text{otherwise;} \end{cases}$$

$$M_3 = \begin{cases} 0, & \text{if } \mathscr{N}_- = \mathscr{N}_+, \\ 1, & \text{otherwise;} \end{cases} \qquad M_4 = \begin{cases} 0, & \text{if } \mathscr{W}_- = \mathscr{W}_+, \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 3.10. *Let P be an N-gon; then each edge of P either lies on a horizontal or a vertical line or it is the edge of exactly one of the maximal slopes* *of P; thus,*

$$N = \sum_{k=1}^{4} N_k + \sum_{k=1}^{4} M_k.$$

The point of Proposition 3.10 is that if we want to estimate the number of edges of a polygon, we can do so by considering its maximal slopes and applying the techniques presented above. The following statement is a helpful sufficient condition for a frame to split a maximal slope.

Proposition 3.11. *Let P be a convex integral polygon and $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an integral frame such that $\mathbf{f}_1, \mathbf{f}_2 \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. Suppose that $\mathbf{o}$ does not belong to P and the rays $\{\mathbf{c} + \lambda\mathbf{f}_j : \lambda \geq 0\}$ $(j = 1, 2)$ split P; then $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits Q_k, where*

$$k = \begin{cases} 1, & \textit{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, \mathbf{e}_2) \quad \textit{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, -\mathbf{e}_1), \\ 2, & \textit{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, -\mathbf{e}_1) \textit{ or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, -\mathbf{e}_2), \\ 3, & \textit{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, -\mathbf{e}_2) \quad \textit{or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, \mathbf{e}_1), \\ 4, & \textit{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, \mathbf{e}_1) \qquad \textit{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, \mathbf{e}_2). \end{cases}$$

The following simple statement also proves useful.

Proposition 3.12. *Let P be an integral polygon; then*

$$\mathscr{S}_+ - \mathscr{S}_- \geq M_1, \qquad \mathscr{E}_+ - \mathscr{E}_- \geq M_2,$$
$$\mathscr{N}_+ - \mathscr{N}_- \geq M_3, \qquad \mathscr{W}_+ - \mathscr{W}_- \geq M_4.$$

The proofs of Propositions 3.10, 3.11, and 3.12 are left to the reader.

4. Properties of slopes

The aim of this section is to prove Theorem 3.7.

4.1. Preliminaries. Throughout this section $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ is an integral frame splitting an integral slope Q. Let $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_1, \ldots, \mathbf{v}_N = \mathbf{w}$ be the vertices of Q. We define $\mathbf{a}_i$ by (3.1) and assume that (3.2) and (3.3) hold. By $\varepsilon_i = [\mathbf{v}_{i-1}, \mathbf{v}_i]$ $(i = 1, \ldots, N)$ denote the edges of Q and by E, the set of edges. Set

$$\mathbf{v}_i - \mathbf{o} = v_{i1}\mathbf{f}_1 + v_{i2}\mathbf{f}_2 \qquad (i = 0, \ldots, N).$$

Note that v_{ij} and a_{ij} are integers.

Further, set

$$k = \min\{i : v_{i2} < 0\}, \qquad \alpha = -\frac{a_{k1}}{a_{k2}}, \qquad t = \lceil \alpha \rceil - 1,$$
$$S = \{\varepsilon_i \in E : i < k,\ a_{i2} = -1\}, \qquad s = |S|.$$

All these are well-defined.

Remark 4.1. It is easily seen that $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with Q if and only if $\alpha \geq 1$.

Remark 4.2. Let us define $\tilde{\alpha} = -a_{\tilde{k}2}/a_{\tilde{k}1}$, where $\tilde{k} = \min\{i : v_{i1} \geq 0\}$. The coefficient $\tilde{\alpha}$ is related to the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ in the same way as α is to $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$. The statement of Proposition 3.6 saying that at least one of those frames forms small angle with Q can be equivalently expressed in the form of the inequality

$$\min\{\alpha, \tilde{\alpha}\} \geq 1.$$

Moreover, it is not hard to prove that equality holds if and only if $\alpha = \tilde{\alpha} = 1$, in which case $k = \tilde{k}$, and consequently $v_{k-1,1} < 0$ and $v_{k-1,2} > 0$.

Lemma 4.3. *The cardinality s of S satisfies*

$$s \leq t, \tag{4.1}$$

and, moreover,

$$\sum_{\varepsilon_i \in S} a_{i1} \leq (t - s)s + \frac{s(s+1)}{2}. \tag{4.2}$$

Proof. Let $S = \{\varepsilon_{i_1}, \ldots, \varepsilon_{i_s}\}$, where $i_1 < \cdots < i_s < k$. It follows from (3.2) and (3.3) that

$$\frac{a_{11}}{-a_{12}} < \frac{a_{21}}{-a_{22}} < \cdots < \frac{a_{k1}}{-a_{k2}} = \alpha.$$

Hence, as $a_{i_p 2} = -1$, we see that

$$0 < a_{i_1 1} < a_{i_2 1} < \cdots < a_{i_s 1} \leq \lceil \alpha \rceil - 1 = t. \tag{4.3}$$

This implies (4.1). Moreover, (4.3) implies that $a_{i_p 1} \leq t - s + p$, where $p = 1, \ldots, s$, and upon summation, we recover (4.2). $\square$

Given a segment $\varepsilon = [\mathbf{p}, \mathbf{q}]$, where

$$\mathbf{p} = p_1\mathbf{f}_1 + p_2\mathbf{f}_2, \qquad \mathbf{q} = q_1\mathbf{f}_1 + q_2\mathbf{f}_2,$$

define

$$\begin{aligned} \pi_1(\varepsilon) &= |p_1^+ - q_1^+|, \\ \pi_2(\varepsilon) &= |p_2^+ - q_2^+|, \\ \hat{\pi}(\varepsilon_i) &= \pi_1(\varepsilon_i) + \pi_2(\varepsilon_i) - 2. \end{aligned}$$

Observe that these functions take integral values provided that ε is an integral segment. We extend the functions π_1, π_2, and $\hat{\pi}$ to the collection of finite sets of segments.

Remark 4.4. If F is a finite set of segments, we obviously have

$$\pi_j(F) \geq 0 \qquad (j = 1, 2), \tag{4.4}$$

$$\hat{\pi}(F) = \pi(F) - 2|F|. \tag{4.5}$$

If F is the set of edges of a (not necessarily integral) slope having the endpoints $\tilde{\mathbf{v}} = \mathbf{o} + \tilde{v}_1\mathbf{f}_1 + \tilde{v}_2\mathbf{f}_2$ and $\tilde{\mathbf{w}} = \mathbf{o} + \tilde{w}_1\mathbf{f}_1 + \tilde{w}_2\mathbf{f}_2$, then

$$\pi_1(F) = |\tilde{v}_1^+ - \tilde{w}_1^+|, \qquad \pi_2(F) = |\tilde{v}_2^+ - \tilde{w}_2^+|.$$

Set

$$E_1 = \{\varepsilon_1, \ldots, \varepsilon_k\}, \qquad E_2 = \{\varepsilon_{k+1}, \ldots, \varepsilon_N\}.$$

Clearly, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$.

4.2. Auxiliary statements.

Lemma 4.5. *We have*

$$\hat{\pi}(E_1) \geq (v_{k-1,1}^+ + v_{k-1,2} - 1) + \delta + (t - s) + \lfloor(-v_{k2} - 1)\alpha\rfloor, \tag{4.6}$$

where

$$\delta = \begin{cases} 1, & \textit{if } v_{k-1,2} > 0 \textit{ and } \alpha \in \mathbb{Z}, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.7}$$

Proof. Let us show the inequality

$$\pi_1(\varepsilon_k) \geq \delta + 1 + t + \lfloor(-v_{k2} - 1)\alpha\rfloor. \tag{4.8}$$

Since the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits the slope Q, it follows from the definition that the ray $\{\mathbf{o} + \lambda \mathbf{f}_1 : \lambda \geq 0\}$ meets Q at a point $\mathbf{z} = \mathbf{o} + z_1 \mathbf{f}_1$, where $z_1 > 0$. As $v_{k2} < 0 \leq v_{k-1,2}$, it is easily seen that $\mathbf{z}$ belongs to the edge ε_k and either coincides with $\mathbf{v}_{k-1}$ or is an inner point of the edge. We consider these cases separately.

If $\mathbf{z} = \mathbf{v}_{k-1}$, then $v_{k-1,2} = 0$ and $v_{k-1,1} > 0$, so

$$\pi_1(\varepsilon_k) = v_{k1} - v_{k-1,1} = a_{k1} = -\alpha a_{k2} = \alpha(-v_{k2}) = \lceil \alpha \rceil + \lfloor (-v_{k2} - 1)\alpha \rfloor,$$

and as in this case $\delta = 0$, (4.8) follows. The inequality (4.8) is proved.

Assume that $\mathbf{z}$ is an interior point of ε_k, then $v_{k-1,2} > 0$. As in this case $\mathbf{z}$ is not the leftmost point of ε_k, it is clear that $\pi_1(\varepsilon_k)$ is strictly greater than $v_{k1} - z_1 = \alpha(-v_{k2})$. Thus,

$$\pi_1(\varepsilon_k) \geq \lfloor \alpha(-v_{k2}) \rfloor + 1 \geq \lfloor \alpha \rfloor + \lfloor (-v_{k2} - 1)\alpha \rfloor + 1 = \lceil \alpha \rceil + \delta + \lfloor (-v_{k2} - 1)\alpha \rfloor,$$

and (4.8) follows. The inequality is proved.

Since $v_{k-1,2} \geq 0$ and $v_{k2} < 0$, we have $\pi_2(\varepsilon_k) = v_{k-1,2}$. This and (4.8) implies

$$\hat{\pi}(\varepsilon_k) \geq v_{k-1,2} + \delta - 1 + t + \lfloor (-v_{k2} - 1)\alpha \rfloor. \tag{4.9}$$

Let $\tilde{Q}$ be the subslope of Q with the vertices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{k-1}$ and $\tilde{E}$ be the set of its edges, then $\varepsilon_k \notin \tilde{E}$, and $\tilde{E} \cup \{\varepsilon_k\} = E_1$. As $\tilde{Q}$ lies in the upper half-plane, for any $\varepsilon_i \in \tilde{E}$ we have $\pi_2(\varepsilon_i) = v_{i-1,2} - v_{i2} = -a_{i2}$. Consequently, $\pi_2(\varepsilon_i) = 1$ if $\varepsilon_i \in S$ and $\pi_2(\varepsilon_i) \geq 2$ otherwise, whence

$$\pi_2(\tilde{E}) \geq 2|\tilde{E}| - s. \tag{4.10}$$

Further, by Remark 4.4, we have $\pi_1(\tilde{E}) = v^+_{k-1,1} - v^+_{01} = v^+_{k-1,1}$. Combining this with (4.10) and using (4.5), we obtain

$$\hat{\pi}(\tilde{E}) \geq v^+_{k-1,1} - s. \tag{4.11}$$

As $\hat{\pi}(E_1) = \hat{\pi}(\varepsilon_k) + \hat{\pi}(\tilde{E})$, we sum (4.9) and (4.11) and obtain (4.6). □

Lemma 4.6. *Suppose that* $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *forms small angle with* Q*; then*

$$\hat{\pi}(E_2) \geq \frac{1}{2}(v_{k2} - w_2 - 1). \tag{4.12}$$

Proof. If $E_2 = \emptyset$, we have $\mathbf{v}_k = \mathbf{w}$ and (4.12) is obvious.

Assume that $E_2 \neq \emptyset$ and take $\varepsilon_i \in E_2$. It is easy to see that all the edges belonging to E_2 lie in the right half-plane, so $\pi_1(\varepsilon_i) = v_{i1} - v_{i-1,1} = a_{i1}$, and since

$\pi_2(\varepsilon_i) \geq 0$, we obtain

$$\hat{\pi}(\varepsilon_i) \geq a_{i1} - 2$$

(actually, the equality holds here). Write the last inequality in form

$$\hat{\pi}(\varepsilon_i) \geq \frac{-a_{i2} + g(a_{i1}, a_{i2})}{2}, \tag{4.13}$$

where

$$g(m_1, m_2) = 2m_1 + m_2 - 4.$$

Using (3.3), we see that

$$\alpha = \frac{a_{k1}}{-a_{k2}} < \frac{a_{k+1,1}}{-a_{k+1,2}} < \cdots < \frac{a_{N1}}{-a_{N2}};$$

besides, as $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with Q, we have $\alpha \geq 1$ (Remark 4.1); consequently, as $i > k$, we have

$$a_{i1} + a_{i2} > 0.$$

Now it is not hard to check that g takes nonnegative values in all the points of the set

$$\{(m_1, m_2) \in \mathbb{Z}^2 : m_1 > 0, m_2 < 0, m_1 + m_2 > 0\}$$

except $(2, -1)$, and $g(2, -1) = -1$. Consequently, (4.13) gives

$$\hat{\pi}(\varepsilon_i) \geq \frac{-a_{i2} - \delta_i}{2},$$

where

$$\delta_i = \begin{cases} 1, & \text{if } \mathbf{a}_i = 2\mathbf{f}_1 - \mathbf{f}_2, \\ 0, & \text{otherwise.} \end{cases}$$

The vectors $\mathbf{a}_i$ are distinct, so at most one δ_i is nonzero. Thus, we have:

$$\hat{\pi}(E_2) = \sum_{i=k+1}^{N} \hat{\pi}(\varepsilon_i) \geq \frac{1}{2}\Big(-\sum_{i=k+1}^{N} a_{i2} - \sum_{i=k+1}^{N} \delta_i\Big) \geq \frac{1}{2}(v_{k2} - v_{N2} - 1),$$

and (4.12) is proved. □

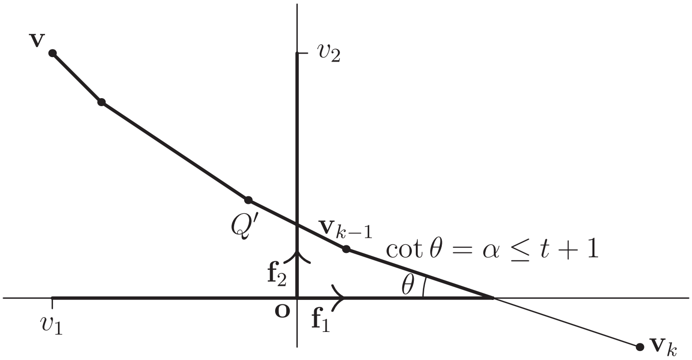

Figure 4. The intersection of Q with the upper half-space is a (possibly non-integral) slope Q'. The length of the vertical projection of Q' is v_2 and that of its horizontal projection is strictly greater than $-v_1$. All the edges belonging to S are edges of Q'. They all contribute s to the length of the vertical projection of Q', whence (3.10). Also, they all contribute at most $t+(t-1)+\cdots+(t-s+1) = ts - s(s-1)/2$ to the horizontal projection. The horizontal contribution of any other edge is less than $t+1$ times its vertical contribution, totalling at most $(t+1)(v_2-s)$ for all the edges not in S, and (3.11) follows.

4.3. Proof of Theorem 3.7.

Proof of Theorem 3.7. First assume that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with Q, or, equivalently, $\alpha \ge 1$ (Remark 4.1). We show that in these case inequalities (3.9)–(3.13) hold with t and s defined in Section 4.1.

Inequalities (3.9)–(3.11) follow from Lemma 4.3 and simple combinatorial arguments, see Figure 4.

Let us prove (3.12). Since the sets E_1 and E_2 are disjoint and their union is E, we have $\hat{\pi}(E) = \hat{\pi}(E_1) + \hat{\pi}(E_2)$. Evoking Lemmas 4.5 and 4.6, we obtain

$$\hat{\pi}(E) = (v_{k-1,1}^{+} + v_{k-1,2} - 1) + \delta + (t-s) + \lfloor(-v_{k2}-1)\alpha\rfloor + \frac{1}{2}(v_{k2} - w_2 - 1), \tag{4.14}$$

where δ is defined by (4.7). By definition, $\delta \ge 0$. The first term on the right-hand side of (4.14) is nonnegative, since at least one coordinate of $\mathbf{v}_{k-1}$ must be positive (Remark 3.5). Let us estimate the fourth term on the right-hand side of (4.14). By the definition of k we have $v_{k2} < 0$, so $-v_{k2} - 1 \ge 0$, and by assumption, $\alpha \ge 1$; thus, we have:

$$\lfloor(-v_{k2}-1)\alpha\rfloor \ge -v_{k2} - 1 \ge \frac{1}{2}(-v_{k2} - 1).$$

Consequently, from (4.14) we obtain

$$\hat{\pi}(E) \geq t - s + \frac{-w_2}{2} - 1.$$

As $\hat{\pi}(E)$ is an integer, this yields

$$\hat{\pi}(E) \geq t - s + \left\lceil \frac{-w_2}{2} \right\rceil - 1.$$

As $\pi_1(E) + \pi_2(E) = v_2 + w_1$ by Remark 4.4, now it remains to use (4.5) in order to obtain (3.12).

Now assume that the frame does not form small angle with Q. Let us check that in this case (3.9)–(3.12) hold with $t = s = 0$.

Inequality (3.9) becomes trivial, and (3.10) follows from the definition of a splitting frame. Inequality (3.11) becomes

$$-v_{N1} < v_{N2}.$$

It is true, since otherwise by Proposition 3.6 the frame would form small angle with Q. To prove (3.12), it suffices to apply the proved part of the theorem to Q and the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$, which forms small angle with Q by Proposition 3.6. Indeed, with certain $\tilde{t}$ and $\tilde{s}$ by virtue of (3.12) and (3.9) we have:

$$2N \leq w_1 + v_2 - \tilde{t} + \tilde{s} \leq v_2 + w_1,$$

so (3.12) holds for $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ with $t = s = 0$ as claimed. □

5. Types of polygons

5.1. Auxiliary types of polygons. This section is devoted to the proof of Theorem 2.6. Throughout the section, $n \geq 3$ is a fixed integer.

First, we introduce two more types of polygons (see Figure 1).

Definition 5.1. Let $n \geq 3$ be an integer, and P be an integral polygon free of points of $n\mathbb{Z}^2$. We say that P is a

- *type VI_n polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$, $[\mathbf{0}, (0, n)]$, and $[(0, n), (n, n)]$ split P, and the lines $x_1 = \pm n$ do not split P;
- *type VII_n polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$ splits P, and the lines $x_1 = -n$ and $x_2 = n$ do not split P.

We start with the following weakened version of Theorem 2.6:

Lemma 5.2. *Suppose that an integral polygon P is free of points of the lattice $n\mathbb{Z}^2$, where $n \in \mathbb{Z}$, $n \geq 2$; then there exists an affine automorphism φ of $n\mathbb{Z}^2$ such that $\varphi(P)$ is a polygon of one of the types I_n–VII_n.*

The lemma follows from known results on maximal lattice-free sets (see, e.g. [4], Theorem 3, cf. also [5], [10], [16]).

To complete the proof of Theorem 2.6, it remains to make sure that polygons of types VI and VII can be mapped onto polygons of types I–V. This is established in Lemmas 5.5 and 5.8.

5.2. The lift. Here we introduce the lift transformation for a class of polygons.

Let P be an integral polygon free of $n\mathbb{Z}^2$-points, where $n \geq 3$ is an integer. Assume that the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$ split P. Given $a \in \mathbb{Z}$, consider the unimodular transformation

$$A_a = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

and the polygon $P_a = A_a P$.

Lemma 5.3. *The set of such $a \in \mathbb{Z}$ that P_a is split by the segment $[\mathbf{0}, (-n, 0)]$ is nonempty and has a nonnegative maximal element.*

Proof. Obviously, $P_0 = P$, so the set in question contains 0 and its maximal element, if it exists, is nonnegative. To prove the lemma, it remains to show that the set is bounded from above, i.e. that the segment $[\mathbf{0}, (-n, 0)]$ does not split the polygon P_a for large a.

As P does not contain the point $\mathbf{0} \in n\mathbb{Z}^2$, there exists a linear form $\ell(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ such that

$$\ell(\mathbf{x}) > 0, \qquad \mathbf{x} \in P. \tag{5.1}$$

Choosing points $\check{\mathbf{x}} \in P \cap [\mathbf{0}, (0, n)]$ and $\hat{\mathbf{x}} \in P \cap [\mathbf{0}, (-n, 0)]$, so that $\check{\mathbf{x}} = (0, \check{x}_2)$, $\check{x}_2 > 0$, and $\hat{\mathbf{x}} = (\hat{x}_1, 0)$, $\hat{x}_1 < 0$ and computing $\ell(\check{\mathbf{x}})$ and $\ell(\hat{\mathbf{x}})$, we see that in view of (5.1),

$$\alpha_1 < 0, \qquad \alpha_2 > 0. \tag{5.2}$$

Fix an integer a such that

$$a \geq -\frac{\alpha_1}{\alpha_2}. \tag{5.3}$$

Consider the linear form

$$\tilde{\ell}(\tilde{x}_1, \tilde{x}_2) = (\alpha_1 + a\alpha_2)\tilde{x}_1 + \alpha_2\tilde{x}_2.$$

It is easy to check that $\ell(\mathbf{x}) = \tilde{\ell}(\tilde{\mathbf{x}})$ whenever $\mathbf{x} = A_a^{-1}\tilde{\mathbf{x}}$. In particular, if $\tilde{\mathbf{x}} \in P_a$, we have $\mathbf{x} = A_{a_0}^{-1}\tilde{\mathbf{x}} \in P$, and according to (5.1), we obtain

$$\tilde{\ell}(\tilde{\mathbf{x}}) > 0, \qquad \tilde{\mathbf{x}} \in P_a. \tag{5.4}$$

On the other hand, if $\tilde{\mathbf{x}} = (\tilde{x}_1, 0) \in [0, (-n, 0)]$, then $\tilde{x}_1 \leq 0$, and

$$\tilde{\ell}(\tilde{\mathbf{x}}) = (\alpha_1 + a\alpha_2)\tilde{x}_1 < 0, \qquad \tilde{\mathbf{x}} \in [\mathbf{0}, (-n, 0)] \tag{5.5}$$

according to the choice of a. Comparing (5.4) and (5.5), we see that P_a has no common points with the segment $[\mathbf{0}, (-n, 0)]$. This is true for any a satisfying (5.3), so the set in question is bounded from above, as claimed. □

Let $a_0 \geq 0$ be the greatest integer such that P_{a_0} is split by the segment $[\mathbf{0}, (-n, 0)]$. We say that the polygon $\hat{P} = P_{a_0}$ is the *lift* of P and that A_{a_0} is the *lift transformation* of P.

Lemma 5.4. *Let that $\hat{P}$ be the lift of P; then the segment $[\mathbf{0}, (0, n)]$ splits $\hat{P}$ and the segment $[\mathbf{0}, (-n, -n)]$ does not. If the segment $[(0, n), (n, 2n)]$ does not split P, it does not split $\hat{P}$ either.*

Proof. The operator A_{a_0} leaves the points of the segment $[\mathbf{0}, (0, n)]$ fixed. The segment splits P, so it splits $A_{a_0}P = \hat{P}$ as well.

By the definition of a_0, the segment $[\mathbf{0}, (-n, 0)]$ does not split the polygon $A_{a_0+1}P = A_1\hat{P}$. Consequently, the segment $[\mathbf{0}, (-n, -n)] = A_1^{-1}[\mathbf{0}, (-n, 0)]$ does not split $\hat{P}$, as claimed.

Suppose that the segment $[(0, n), (n, 2n)]$ does not split P. The intersection $P \cap \{0 \leq x_1 \leq n\}$ lies in the half-plane $x_2 \leq x_1 + n$. It suffices to check that the intersection $\hat{P} \cap \{0 \leq x_1 \leq n\}$ lies in the same half-plane. Indeed, let $(\hat{x}_1, \hat{x}_2) \in \hat{P}$ and $0 \leq \hat{x}_1 \leq n$; then $\hat{x}_1 = x_1$ and $\hat{x}_2 = -a_0x + x_2$ for some $(x_1, x_2) \in P \cap \{0 \leq x_1 \leq n\}$, so $\hat{x}_2 \leq x_2 \leq x_1 + n = \hat{x}_1 + n$, as claimed. □

5.3. Type VI polygons.

Lemma 5.5. *Suppose that P is a type VI_n polygon; then there exists an affine automorphism ψ of $n\mathbb{Z}^2$ such that $\psi(P)$ is a polygon of one of the types I_n, II_n, III_n, or VII_n.*

Proof. The polygon P is split by the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$, so its lift $\hat{P}$ is well-defined. The polygon $\hat{P}$ is split by the segment $[\mathbf{0}, (-n, 0)]$ by the definition of the lift and by the segment $[\mathbf{0}, (0, n)]$ by Lemma 5.4. By the same lemma, the segment $[\mathbf{0}, (-n, -n)]$ does not split $\hat{P}$. The lines $x_1 = \pm n$ are invariant under the lift transformation, so they do not split $\hat{P}$ either. Besides, $\hat{P}$ has points in the slab

$$-n \le x_1 \le n, \tag{5.6}$$

(e.g. on the segment $[\mathbf{0}, (0, n)]$), so we conclude that it is contained in this slab.

If the line $x_2 = n$ does not split $\hat{P}$, the latter is a type VII_n polygon. Otherwise, $\hat{P}$ is split by one of the segments $[(-n, n), (0, n)]$ and $[(0, n), (n, n)]$, since their union is exaclty the set of common points of the line $x_2 = n$ and the slab (5.6).

Suppose that the segment $[(-n, n), (0, n)]$ splits $\hat{P}$. Let T be the translation by the vector $(n, 0) \in n\mathbb{Z}^2$. The polygon $T\hat{P}$ is split by the segments

$$\begin{aligned} [(0, n), (n, n)] &= T[(-n, n), (0, n)], \\ [(n, 0), \mathbf{0}] &= T[\mathbf{0}, (-n, 0)], \\ [(n, 0), (n, n)] &= T[\mathbf{0}, (0, n)] \end{aligned}$$

and is not split by the line $x_1 = 0$ being the image of $x_1 = -n$ under T. In other words, $T\hat{P}$ is a type III_n polygon, and we are done.

It remains to consider the case of the segment $[(0, n), (n, n)]$ splitting $\hat{P}$. Let φ be an affine automorphism of $n\mathbb{Z}^2$ defined by

$$\varphi(x_1, x_2) = (-x_1, n - x_2),$$

which is the symmetry with respect to $(0, n/2)$, and set $P' = \varphi(\hat{P})$. The polygon P' lies in the slab (5.6), which is invariant under φ; also P' is split by the segments

$$\begin{aligned} [(-n, 0), \mathbf{0}] &= \varphi\big([(n, n), (0, n)]\big), \\ [\mathbf{0}, (0, n)] &= \varphi\big([(0, n), \mathbf{0}]\big) \end{aligned}$$

and is not split by the segment

$$[(0, n), (n, 2n)] = \varphi\big([\mathbf{0}, (-n, -n)]\big).$$

Thus, the lift $\hat{P}'$ is well-defined. By the definition of the lift and by Lemma 5.4, the polygon $\hat{P}'$ is split by the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$ and is not split by the segments $[\mathbf{0}, (-n, -n)]$ and $[(0, n), (n, 2n)]$; moreover, $\hat{P}'$ lies in the slab (5.6). Consequently, the line $x_1 = -n$ does not split $\hat{P}'$. If the line $x_2 = n$ does not split it either, it is a type VII_n polygon, and we are done. Otherwise, as before, we infer that either $[(-n,n),(0,n)]$ splits $\hat{P}'$, and we conclude by noticing that $T\hat{P}'$ is a type III_n polygon, or $[(0,n),(n,n)]$ splits $\hat{P}'$, which we assume in what follows.

The intersection of the line $x_1 - x_2 = -n$ and the slab (5.6) is the union of the segments $[(-n,0),(0,n)]$ and $[(0,n),(n,2n)]$. The latter segment does not split $\hat{P}'$, the line $x_1 - x_2 = -n$ splits $\hat{P}'$ if and only if the segment $[(-n,0),(0,n)]$ does so. Likewise, the line $x_1 - x_2 = 0$ splits $\hat{P}'$ if and only if the segment $[\mathbf{0},(n,n)]$ does so, for $\hat{P}'$ is not split by $[(-n,-n),\mathbf{0}]$. Thus, we have four logical possibilities.

Case 1. The lines $x_1 - x_2 = -n$ and $x_1 - x_2 = 0$ do not split $\hat{P}'$.

Case 2. The segments $[(-n,0),(0,n)]$ and $[\mathbf{0},(n,n)]$ split $\hat{P}'$.

Case 3. The segment $[(-n,0),(0,n)]$ splits $\hat{P}'$ and the line $x_1 - x_2 = 0$ does not.

Case 4. The segment $[\mathbf{0},(n,n)]$ splits $\hat{P}'$ and the line $x_1 - x_2 = -n$ does not.

In Case 1 define the affine automorphism of $n\mathbb{Z}^2$ by

$$\psi_1(x_1,x_2) = (-x_1 + x_2, x_2)$$

and consider the polygon $\psi_1(\hat{P}')$. It is not split by the lines $x_1 = 0$ and $x_1 = n$, being the images of $x_1 - x_2 = 0$ and $x_1 - x_2 = -n$, respectively, and $\psi_1(\hat{P}')$ has points inside the slab $0 \le x_1 \le n$, e.g. on the segment

$$[(n,0),(0,n)] = \psi_1([\mathbf{0},(0,n)]).$$

Consequently, $\psi_1(\hat{P}')$ is a type I_n polygon.

In Cases 2 and 3 we use the same automorphism ψ_1. It is not hard to check that in Case 2, $\psi_1(\hat{P}')$ is a type II_n polygon and in Case 3, it is a type III_n polygon.

In Case 4, define the automorphism of $n\mathbb{Z}^2$ by

$$\psi_2(x_1,x_2) = (x_1 - x_2 + n, x_2).$$

It is easily seen that $\psi_2(\hat{P}')$ is a type III_n polygon. □

5.4. Type VII polygons. It follows from the definition that any type VII_n polygon is split by the segments $[\mathbf{0},(-n,0)]$ and $[\mathbf{0},(0,n)]$, so its lift is well-defined.

Lemma 5.6. *Let P be a type VII_n polygon and $\hat{P}$ be its lift; then $\mathscr{S}(\hat{P}) \ge \mathscr{S}(P)$ and $\mathscr{S}(\hat{P}) = \mathscr{S}(P)$ if and only if $P = \hat{P}$.*

Proof. Let A_{a_0} be the lift transformation. If $a_0 = 0$, we have $P = \hat{P}$, so $\mathscr{S}(\hat{P}) = \mathscr{S}(P)$. It remains to show that

$$a_0 \ge 1 \tag{5.7}$$

implies

$$\mathscr{S}(\hat{P}) > \mathscr{S}(P). \tag{5.8}$$

As the segments $[\mathbf{0},(-n,0)]$ and $[\mathbf{0},(0,n)]$ split both P and $\hat{P}$, by Proposition 3.11 the frame $(\mathbf{0};-\mathbf{e}_1,\mathbf{e}_2)$ splits the slopes $Q_1(P)$ and $Q_1(\hat{P})$, whence

$$\mathscr{S}_+(P) \le -1, \qquad \mathscr{S}_+(\hat{P}) \le -1.$$

Thus,

$$\mathscr{S}(P) = \min\{x_2 : (x_1,x_2) \in P,\ x_1 \le -1\},$$
$$\mathscr{S}(\hat{P}) = \min\{x_2 : (x_1,x_2) \in \hat{P},\ x_1 \le -1\}.$$

Using these representations and (5.7), we get

$$\begin{aligned}\mathscr{S}(\hat{P}) &= \min\{\hat{x}_2 : (\hat{x}_1,\hat{x}_2) \in \hat{P},\ \hat{x}_1 \le -1\}\\ &= \min\{-a_0x_1 + x_2 : (x_1,x_2) \in P,\ x_1 \le -1\}\\ &\ge \min\{x_2 + 1 : (x_1,x_2) \in P,\ x_1 \le -1\} = \mathscr{S}(P) + 1,\end{aligned}$$

so (5.8) is proved. □

Lemma 5.7. *Let P be a type VII_n polygon and $\hat{P}$ be its lift. Then either $\hat{P}$ is a type VII_n polygon, or the translation of $\hat{P}$ by the vector $(n,0)$ is a type III_n polygon.*

Proof. Let T be the translation by the vector $(n,0)$. Note that $\hat{P}$ and $T\hat{P}$ are obtained by applying affine automorphisms of $n\mathbb{Z}^2$ to P, so they are free of points of this lattice.

The polygon $\hat{P}$ is split by the segments $[\mathbf{0},(0,n)]$ (by Lemma 5.4) and $[\mathbf{0},(-n,0)]$ (by the definition of lift), but not by the line $x_1 = -n$ (because by the definition of a type VII_n polygon this line does not split P and it is invariant under the lift transformation). Assume for a moment that the segment $[(0,n),(-n,n)]$ splits $\hat{P}$. Then the segments

$$[\mathbf{0},(n,0)] = T[(-n,0),\mathbf{0}],$$
$$[(n,0),(n,n)] = T[\mathbf{0},(0,n)],$$
$$[(0,n),(n,n)] = T[(-n,n),(0,n)].$$

split TP, while the line $x_1 = 0$, being the image of $x_1 = -n$ under T, does not. Consequently, TP is a type III_n polygon.

It remains to show that if the segment $[(0,n),(-n,n)]$ does not split $\hat{P}$, the latter is a type VII_n polygon. We know already that the segments $[\mathbf{0},(-n,0)]$ and $[\mathbf{0},(0,n)]$ split $\hat{P}$, while the line $x_1 = -n$ does not, so we only need to show that the line $x_2 = n$ does not split $\hat{P}$ either, or, equivalently, that $\hat{P}$ lies in the half-plane $x_2 \leq n$. Indeed, as $\hat{P}$ lies in the half-plane $x_1 \geq -n$ and the segment $[(-n,n),(0,n)]$ does not split $\hat{P}$, it is clear that

$$\max\{x_2 : (x_1,x_2) \in \hat{P}, x_1 \leq 0\} \leq n. \tag{5.9}$$

On the other hand,

$$\begin{aligned} &\max\{x_2 : (x_1,x_2) \in \hat{P}, x_1 \geq 0\} \\ &\quad = \max\{-a_0x_1' + x_2' : (x_1',x_2') \in P, x_1 \leq 0\} \\ &\quad \leq \max\{x_2' : (x_1',x_2') \in P, x_1 \leq 0\} \leq n. \end{aligned} \tag{5.10}$$

Bounds (5.9) and (5.10) imply that $\hat{P}$ lies in the half-plane $x_2 \leq n$, as claimed. □

Lemma 5.8. *For any type VII_n polygon there exists an affine automorphism of $n\mathbb{Z}^2$ mapping it onto a type III_n or a type V_n polygon.*

Proof. Take a type VII_n polygon P_0, and let $\hat{P}_0$ be its lift. If the translation by the vector $(n,0)$ maps $\hat{P}_0$ onto a type III_n polygon, we are done. Otherwise, by Lemma 5.7, $\hat{P}_0$ is a type VII_n polygon. Let P_0' be the reflection of $\hat{P}_0$ about the line $x_1 + x_2 = 0$. It is easy to check that it is again a type VII_n polygon. Let $\hat{P}_0'$ be its lift. As before, either the translation of $\hat{P}_0'$ by $(n,0)$ is a type III_n polygon and we are done, or $\hat{P}_0'$ is a type VII_n polygon, in which case we define the type VII_n polygon P_1 to be the reflection of $\hat{P}_0'$ about the line $x_1 + x_2 = 0$.

Iterating this procedure, we either find an affine automorphism of $n\mathbb{Z}^2$ mapping P_0 onto a type III_n polygon, or construct the sequences of type VII_n polygons $\{P_k\}$, $\{\hat{P}_k\}$, $\{P_k'\}$, and $\{\hat{P}_k'\}$. In the latter case consider the sequence of integers $\{\mathscr{S}(P_k)\}_{k=0}^{\infty}$. As P_k are type VII_n polygons, it is easily seen that the members of this sequence are negative (this follows e.g. from the fact that by Proposition 3.11 the frame $(\mathbf{0};-\mathbf{e}_1,-\mathbf{e}_2)$ splits any type VII_n polygon). Observe that the sequence increases. Indeed, it is easy to check that

$$\mathscr{S}(P_{k+1}) = -\mathscr{E}(\hat{P}_k') = -\mathscr{E}(P_k') = \mathscr{S}(\hat{P}_k);$$

furthermore, by Lemma 5.6 we have

$$\mathscr{S}(\hat{P}_k) \geq \mathscr{S}(P_k).$$

Thus, we see that the sequence $\{\mathcal{S}(P_k)\}$ increases; moreover, we have $\mathcal{S}(P_{k+1}) = \mathcal{S}(P_k)$ if and only if $\mathcal{S}(\hat{P}_k) = \mathcal{S}(P_k)$, which by Lemma 5.6 is equivalent to $\hat{P}_k = P_k$.

The sequence of integers $\{\mathcal{S}(P_k)\}$ increases and is bounded from above, so it stabilizes. We show in the same way that the sequence $\{\mathcal{S}(P_k')\}$ stabilizes, too. Consequently, there exists k_0 such that $P_{k_0} = \hat{P}_{k_0}$ and $P_{k_0}' = \hat{P}_{k_0}'$. Then also $P_{k_0} = P_{k_0+1}$. Set $\hat{P} = P_{k_0}$.

We claim that $\hat{P}$ lies in the triangle Δ being the solution set of the system

$$\begin{cases} x_1 \geq -n, \\ x_2 \leq n, \\ x_1 - x_2 \leq 0. \end{cases}$$

Since $\hat{P}$ is a type VII_n polygon, it lies in the angle

$$\begin{cases} x_1 \geq -n, \\ x_2 \leq n. \end{cases}$$

The intersection of the line $x_1 - x_2 = 0$ with this angle is the segment $[(-n,-n), (n,n)]$, so we only need to show that neither of the segments $I_1 = [(-n,-n), \mathbf{0}]$ and $I_2 = [\mathbf{0}, (n,n)]$ splits $\hat{P}$. It the case of the former this is true by Lemma 5.4, as $\hat{P}$ is the lift of P_{k_0}. Likewise, I_1 does not split $\hat{P}_{k_0}'$, so I_2, being the reflection of I_1 about the line $x_1 + x_2 = 0$, does not split $P_{k_0+1} = \hat{P}$, as claimed.

By construction, $\hat{P} = BP$, where B is a unimodular transformation. The affine automorphism of $n\mathbb{Z}^2$ defined by

$$\psi(x_1, x_2) = (x_1 + n, -x_2 + n)$$

maps Δ onto the triangle defined by (2.1). Consequently, the polygon $\psi(BP)$ is of type V_n, i.e. $\varphi = \psi B$ is the required automorphism. $\square$

6. Type II polygons

In this section we prove Theorem 1.2 for type II polygons.

Lemma 6.1. *Suppose that $n \geq 3$ is an integer and P is a type II_n polygon; then*

(i) $\big((n,0); -\mathbf{e}_1, \mathbf{e}_2\big)$ *splits* Q_1*;*

(ii) $\big((n,n); -\mathbf{e}_1, -\mathbf{e}_2\big)$ *splits* Q_2*;*

(iii) $\big((0,n); \mathbf{e}_1, -\mathbf{e}_2\big)$ *splits* Q_3*;*

(iv) $(\mathbf{0}; \mathbf{e}_1, \mathbf{e}_2)$ *splits* Q_4*;*

Proof. Statements (i)–(iv) immediately follow from the definition of a type II_n polygon and Proposition 3.11. □

Proof of Theorem 1.2 *for type II polygons.* Assume that P is a type II_n N-gon. We begin by translating the geometrical constraints on P into inequalities.

Evoking Corollary 3.8 for the maximal slopes of P and correspondent frames indicated in Lemma 6.1, we obtain:

$$\begin{aligned} 2N_1 &\le -\mathcal{S}_+ + \mathcal{E}_- + n, \\ 2N_2 &\le -\mathcal{N}_+ - \mathcal{E}_+ + 2n, \\ 2N_3 &\le \mathcal{N}_- - \mathcal{W}_+ + n, \\ 2N_4 &\le \mathcal{S}_- + \mathcal{W}_-. \end{aligned}$$

Further, by Proposition 3.12 we have

$$\begin{aligned} \mathcal{S}_+ - \mathcal{S}_- &\ge M_1, \\ \mathcal{E}_+ - \mathcal{E}_- &\ge M_2, \\ \mathcal{N}_+ - \mathcal{N}_- &\ge M_3, \\ \mathcal{W}_+ - \mathcal{W}_- &\ge M_4. \end{aligned}$$

Using the above inequalities, we obtain:

$$\begin{aligned} 2N &= \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\le (-\mathcal{S}_+ + \mathcal{E}_- + n) + (-\mathcal{N}_+ - \mathcal{E}_+ + 2n) + (\mathcal{N}_- - \mathcal{W}_+ + n) \\ &\quad + (\mathcal{S}_- + \mathcal{W}_-) + 2M_1 + 2M_2 + 2M_3 + 2M_4 \\ &= 4n + \big(M_1 - (\mathcal{S}_+ - \mathcal{S}_-)\big) + \big(M_2 - (\mathcal{E}_+ - \mathcal{E}_-)\big) + \big(M_3 - (\mathcal{N}_+ - \mathcal{N}_-)\big) \\ &\quad + \big(M_4 - (\mathcal{W}_+ - \mathcal{W}_-)\big) + (M_1 + M_2 + M_3 + M_4) \\ &\le 4n + 4, \end{aligned}$$

so $N \le 2n + 2$, as claimed. □

7. Type III polygons

In this section we prove Theorem 1.2 for type III polygons.

Lemma 7.1. *Given an integer $n \ge 3$ and a type III_n polygon P, the following assertions hold:*

(i) *The frame* $\big((n,0);\mathbf{e}_2,-\mathbf{e}_1\big)$ *splits* Q_1.

(ii) *The frame* $\big((n,n);-\mathbf{e}_2,-\mathbf{e}_1\big)$ *splits* Q_2.

(iii) *We have*

$$\mathscr{W} \geq 0. \tag{7.1}$$

Proof. Assertions (i) and (ii) follow from the definition of a type III_n polygon and Proposition 3.11. Assertion (iii) is obvious. □

Proof of Theorem 1.2 *for type II polygons.* Let P be a type III_n N-gon. We begin by translating the geometrical constraints on P into inequalities.

The frame $\big((n,0);\mathbf{e}_2,-\mathbf{e}_1\big)$ splits Q_1, so by Theorem 3.7 there exist integers s_1 and t_1 such that

$$2N_1 \leq \mathscr{E}_- - \mathscr{S}_+ + n - t_1 + s_1, \tag{7.2}$$

$$-\mathscr{S}_+ + n - s_1 \geq 0, \tag{7.3}$$

$$-\mathscr{S} < t_1 s_1 - \frac{s_1^2 - s_1}{2} + (-\mathscr{S}_+ + n - s_1)(t_1 + 1), \tag{7.4}$$

$$0 \leq s_1 \leq t_1. \tag{7.5}$$

Likewise, $\big((n,n);-\mathbf{e}_2,-\mathbf{e}_1\big)$ splits Q_2, so there exist integers s_2 and t_2 such that

$$2N_2 \leq -\mathscr{E}_+ - \mathscr{N}_+ + 2n - t_2 + s_2, \tag{7.6}$$

$$-\mathscr{N}_+ + n - s_2 \geq 0, \tag{7.7}$$

$$\mathscr{N} - n < t_2 s_2 - \frac{s_2^2 - s_2}{2} + (-\mathscr{N}_+ + n - s_2)(t_2 + 1), \tag{7.8}$$

$$0 \leq s_2 \leq t_2. \tag{7.9}$$

As Q_3 is a slope with respect to $(\mathbf{e}_1,-\mathbf{e}_2)$, by Proposition 3.2 there exists $s_3 \in \mathbb{Z}$ such that

$$2N_3 \leq \mathscr{N}_- - \mathscr{W} + s_3, \tag{7.10}$$

$$\mathscr{N} - \mathscr{W}_+ \geq \frac{1}{2} s_3(s_3 + 1), \tag{7.11}$$

$$0 \leq s_3 \leq N_3. \tag{7.12}$$

Likewise, applying Proposition 3.2 to Q_4 and $(\mathbf{e}_1,\mathbf{e}_2)$, we conclude that there exists $s_4 \in \mathbb{Z}$ such that

$$2N_4 \le \mathcal{S}_- - \mathcal{W} + s_4, \tag{7.13}$$

$$\mathcal{W}_- - \mathcal{S} \ge \frac{1}{2} s_4(s_4 + 1), \tag{7.14}$$

$$0 \le s_4 \le N_4. \tag{7.15}$$

Further, by Proposition 3.12,

$$\mathcal{S}_+ - \mathcal{S}_- \ge M_1, \tag{7.16}$$

$$\mathcal{E}_+ - \mathcal{E}_- \ge M_2, \tag{7.17}$$

$$\mathcal{N}_+ - \mathcal{N}_- \ge M_3, \tag{7.18}$$

$$\mathcal{W}_+ - \mathcal{W}_- \ge M_4. \tag{7.19}$$

Arguing by contradiction, we assume that

$$2N \ge 4n + 6. \tag{7.20}$$

Summing (7.2), (7.6), (7.10), and (7.13) and subsequently using (7.16)–(7.19), we obtain:

$$\begin{aligned} 2N &= \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\le 3n + s_1 + s_2 + s_3 + s_4 - t_1 - t_2 \\ &\quad - (\mathcal{S}_+ - \mathcal{S}_-) - (\mathcal{E}_+ - \mathcal{E}_-) - (\mathcal{N}_+ - \mathcal{N}_-) \\ &\quad - 2\mathcal{W} + 2M_1 + 2M_2 + 2M_3 + 2M_4 \\ &\le 3n + s_1 + s_2 + s_3 + s_4 - t_1 - t_2 + M_1 + M_2 + M_3 + 2M_4. \end{aligned}$$

Comparing this with (7.20), we deduce

$$n - s_1 - s_2 - s_3 - s_4 + t_1 + t_2 - M_1 - M_2 - M_3 - 2M_4 + 6 \le 0. \tag{7.21}$$

Now we use (7.4) and (7.8) to estimate $\mathcal{N} - \mathcal{S}$ from above:

$$\begin{aligned} \mathcal{N} - \mathcal{S} < n + t_1 s_1 - \frac{s_1^2 - s_1}{2} + t_2 s_2 - \frac{s_2^2 - s_2}{2} + (-\mathcal{S}_+ + n - s_1)(t_1 + 1) \\ + (-\mathcal{N}_+ + n - s_2)(t_2 + 1). \end{aligned} \tag{7.22}$$

Let us estimate $\mathcal{S}_+$ and $\mathcal{N}_+$. Using (7.16), (7.13), (7.15), and (7.1), we obtain

$$\mathcal{S}_+ \ge \mathcal{S}_- + M_1 \ge 2N_4 + \mathcal{W} - s_4 + M_1 \ge s_4 + \mathcal{W} + M_1 \ge s_4 + M_1,$$

whence

$$-\mathscr{S}_+ + n - s_1 \leq n - s_1 - s_4 - M_1. \tag{7.23}$$

Incidentally, note that the left-hand side is nonnegative by virtue of (7.3), so

$$n - s_1 - s_4 - M_1 \geq 0. \tag{7.24}$$

Likewise, from (7.18), (7.11), (7.12), and (7.1) we derive

$$-\mathscr{N}_+ + n - s_2 \leq n - s_2 - s_3 - M_3, \tag{7.25}$$

which together with (7.7) implies

$$n - s_2 - s_3 - M_3 \geq 0. \tag{7.26}$$

As $t_1 + 1 > 0$ and $t_2 + 1 > 0$, we can use (7.23) and (7.25) to obtain from (7.22)

$$\begin{aligned}\mathscr{N} - \mathscr{S} < n + t_1 s_1 - \frac{s_1^2 - s_1}{2} + t_2 s_2 - \frac{s_2^2 - s_2}{2} + (n - s_1 - s_4 - M_1)(t_1 + 1) \\ + (n - s_2 - s_3 - M_3)(t_2 + 1).\end{aligned} \tag{7.27}$$

Now we estimate $\mathscr{N} - \mathscr{S}$ from below by summing (7.11), (7.14), and (7.19):

$$\mathscr{N} - \mathscr{S} \geq M_4 + \frac{1}{2} s_3(s_3 + 1) + \frac{1}{2} s_4(s_4 + 1). \tag{7.28}$$

Consider the second term on the right-hand side. Inequality (7.21) gives

$$s_3 - 1 \geq (n - s_1 - s_4 + t_1 - M_1) + (t_2 - s_2) + (5 - M_2 - M_3 - 2M_4).$$

The second term on the right-hand side is nonnegative by virtue of (7.9) and the third one is also nonnegative (even positive). Consequently, we have

$$s_3 - 1 \geq n - s_1 - s_4 + t_1 - M_1. \tag{7.29}$$

By virtue of (7.24) we have $n - s_1 - s_4 + t_1 - M_1 \geq t_1 \geq 0$, so using (7.29), we get

$$\begin{aligned}\frac{1}{2} s_3(s_3 + 1) &= s_3 + \frac{1}{2} s_3(s_3 - 1) \\ &\geq s_3 + \frac{1}{2}(n - s_1 - s_4 + t_1 - M_1 + 1)(n - s_1 - s_4 + t_1 - M_1).\end{aligned}$$

Set

$$A = n - s_1 - s_4 - M_1, \qquad B = t_1 + 1$$

(A and B are integers) and continue as follows:

$$\begin{aligned}\frac{1}{2}s_3(s_3+1) &\geq s_3 + \frac{1}{2}(A+B)(A+B-1) \\ &= s_3 + \frac{1}{2}(A^2 - A) + \frac{1}{2}(B^2 - B) + AB \geq s_3 + \frac{1}{2}(B^2 - B) + AB.\end{aligned}$$

For the terms on the right-hand side we have

$$\begin{aligned}\frac{1}{2}(B^2 - B) = \frac{1}{2}(t_1^2 + t_1) &= t_1 s_1 - \frac{s_1^2 - s_1}{2} + \frac{1}{2}(t_1 - s_1)(t_1 - s_1 + 1) \\ &\geq t_1 s_1 - \frac{s_1^2 - s_1}{2}\end{aligned}$$

(since $t_1 - s_1 \geq 0$ according to (7.5)), and

$$AB = (n - s_1 - s_4 - M_1)(t_1 + 1),$$

and we finally obtain

$$\frac{1}{2}s_3(s_3+1) \geq s_3 + t_1 s_1 - \frac{s_1^2 - s_1}{2} + (n - s_1 - s_4 - M_1)(t_1 + 1). \qquad (7.30)$$

One can estimate the third term on the right-hand side of (7.28) in the same way by making use of (7.5), (7.26), and (7.9). Eventually,

$$\frac{1}{2}s_4(s_4+1) \geq s_4 + t_2 s_2 - \frac{s_2^2 - s_2}{2} + (n - s_2 - s_3 - M_3)(t_2 + 1). \qquad (7.31)$$

Now, using (7.30) and (7.31), we derive from (7.28) the following bound:

$$\begin{aligned}\mathcal{N} - \mathcal{S} \geq M_4 + s_3 + s_4 + t_1 s_1 - \frac{s_1^2 - s_1}{2} + t_2 s_2 - \frac{s_2^2 - s_2}{2} \\ + (n - s_1 - s_4 - M_1)(t_1 + 1) + (n - s_2 - s_3 - M_3)(t_2 + 1). \qquad (7.32)\end{aligned}$$

Comparing (7.27) with (7.32), we obtain

$$-n + s_3 + s_4 + M_4 < 0.$$

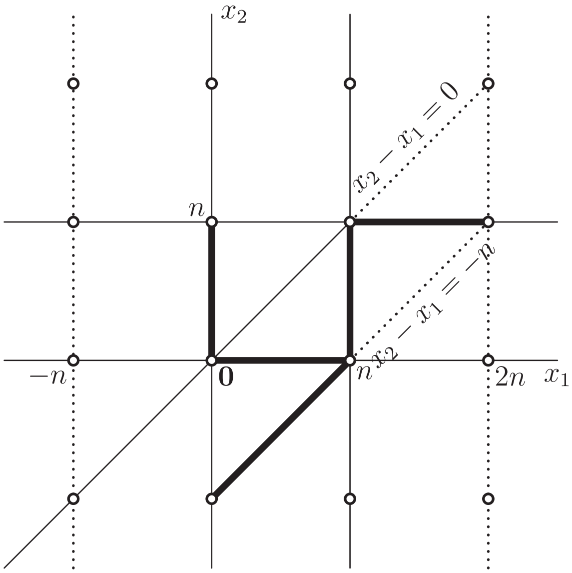

Figure 5. The segments splitting the polygon in the hypothesis of Lemma 8.2 are thick, and P does not intersect dotted lines. The inequalities (8.3)–(8.6) are obvious.

Summing this inequality with (7.21), we get

$$(t_1 - s_1) + (t_2 - s_2) + (6 - M_1 - M_2 - M_3 - M_4) < 0.$$

However, the summands on the left-hand side are nonnegative. Indeed, in the case of the first and the second ones it follows from (7.5) and (7.9), respectively, and the third one is ≥ 2. Contradiction. □

8. Type IV polygons

In this section we prove Theorem 1.2 for type IV polygons.

Lemma 8.1. *Suppose that the line $x_1 - x_2 = n$ splits a type IV_n polygon; then so does one of the segments $[(0, -n), (n, 0)]$ and $[(n, 0), (2n, n)]$.*

Proof. All the points of the line $x_1 - x_2 = n$ belonging to the slab $-n + 1 \leq x_1 \leq 2n - 1$ lie on the segments $[(-n, -2n), (0, -n)]$, $[(0, -n), (n, 0)]$, and $[(n, 0), (2n, n)]$, and as the polygon is free of $n\mathbb{Z}^2$-points, exactly one of the segments splits it. However, it cannot be the first one, because then P would contain the origin. □

Lemma 8.2. *Suppose that P is a type IV_n polygon and the segment $[(0,-n),(n,0)]$ splits it. Then the following assertions hold:*

(i) *The intersection of P with the half-plane $x_1 \geq n$ lies in the slab*

$$-n < x_2 - x_1 < 0. \tag{8.1}$$

(ii) *The frame $\big((n,0);\mathbf{e}_2,-\mathbf{e}_1\big)$ splits the slope Q_1 and forms small angle with it.*

(iii) *The frame $\big((n,n);\mathbf{e}_1,-\mathbf{e}_2\big)$ splits the slope Q_3 and forms small angle with it.*

(iv) *The frame $(\mathbf{0};\mathbf{e}_1,\mathbf{e}_2)$ splits the slope Q_4.*

(v) *The slope Q_1 has a vertex $\mathbf{v}=(v_1,v_2)$ satisfying*

$$v_2 - v_1 \leq -n-1. \tag{8.2}$$

(vi) *The following inequalities hold:*

$$n < \mathcal{N}_+ \leq \mathcal{E} \leq 2n-1, \tag{8.3}$$

$$n < \mathcal{N} \leq \mathcal{N}_+ - 1, \tag{8.4}$$

$$-n < \mathcal{W} < 0, \tag{8.5}$$

$$0 < \mathcal{W}_+ < n. \tag{8.6}$$

Proof. To prove (i), it suffices to observe that P cannot have common points with the segments $[(n,0),(2n,n)]$ and $[(n,n),(2n,2n)]$. Indeed, if P had common points with the former segment, by convexity it would containt the point $(n,0)$; if it had common points with the latter, it would contain the point (n,n). Thus, the part of P contained in the half-plane $x_1 \geq n$ must lie between the lines $x_2 - x_1 = -n$ and $x_2 - x_1 = 0$.

Let us prove (v). Clearly, the functional $x_2 - x_1$ attains its maximum on P on a vertex $\mathbf{v} \in Q_1$. As the line $x_2 - x_1 = -n$ splits P, this minimum is less than n, and (v) follows.

The fact that the frames split correspondent slopes in assertions (ii)–(iv) follows from Proposition 3.11. To prove that $\big((n,0);\mathbf{e}_2,\mathbf{e}_1\big)$ forms small angle with Q_1, we apply Proposition 3.6 taking the vertex from assertion (v) as $\mathbf{y}$. To prove that $\big((n,n);\mathbf{e}_1,-\mathbf{e}_2\big)$ splits Q_3, we use the same theorem with $\mathbf{y} = (\mathcal{W},\mathcal{W}_+)$.

The inequalities in (vi) are fairly intuitive, see Figure 8. □

Proof of Theorem 1.2 *for type IV polygons.* First, assume that the segment $[(0,-n),(n,0)]$ splits P, so that we can apply Lemma 8.2.

The frame $\big((n,0);\mathbf{e}_2,-\mathbf{e}_1\big)$ forms small angle with the slope Q_1, so by Corollary 3.8 we have

$$2N_1 \le \mathscr{E}_- - \mathscr{S}_+ + n - \left\lceil \frac{\mathscr{E}-n}{2} \right\rceil + 1. \tag{8.7}$$

Likewise, as $\big((n,n);\mathbf{e}_1,-\mathbf{e}_2\big)$ forms small angle with Q_3, we obtain

$$2N_3 \le \mathscr{N}_- - \mathscr{W}_+ - \left\lceil \frac{\mathscr{N}-n}{2} \right\rceil + 1. \tag{8.8}$$

Applying Proposition 3.2 to the basis $(-\mathbf{e}_1,-\mathbf{e}_2)$ and the slope Q_2, we see that there exists an integer s_2 such that

$$2N_2 \le \mathscr{E} - \mathscr{N}_+ + s_2, \tag{8.9}$$

$$\mathscr{N} - \mathscr{E}_+ \ge \frac{s_2^2+s_2}{2}, \tag{8.10}$$

$$0 \le s_2 \le N_2. \tag{8.11}$$

As the frame $(\mathbf{0};\mathbf{e}_1,\mathbf{e}_2)$ splits Q_4, by Corollary 3.8 we have

$$2N_4 \le \mathscr{S}_- + \mathscr{W}_-. \tag{8.12}$$

Finally, by Proposition 3.12 we have

$$\mathscr{S}_+ - \mathscr{S}_- \ge M_1, \tag{8.13}$$

$$\mathscr{E}_+ - \mathscr{E}_- \ge M_2, \tag{8.14}$$

$$\mathscr{N}_+ - \mathscr{N}_- \ge M_3, \tag{8.15}$$

$$\mathscr{W}_+ - \mathscr{W}_- \ge M_4. \tag{8.16}$$

We estimate $2N$ by means of (8.7)–(8.9), and (8.12):

$$\begin{aligned} 2N &= \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\le n + 2 - \left\lceil \frac{\mathscr{E}-n}{2} \right\rceil + \mathscr{E} - \left\lceil \frac{\mathscr{N}-n}{2} \right\rceil + s_2 + \mathscr{E}_- + 2M_2 - (\mathscr{S}_+ - \mathscr{S}_-) \\ &\quad - (\mathscr{N}_+ - \mathscr{N}_-) - (\mathscr{W}_+ - \mathscr{W}_-) + 2M_1 + 2M_3 + 2M_4. \end{aligned}$$

Dropping the ceilings, using (8.3) and (8.13)–(8.16) and subsequently estimating $M_k \leq 1$, we obtain

$$2N \leq 3n + \frac{9}{2} + \left(-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2\right). \tag{8.17}$$

Let us estimate the term in parentheses on the right-hand side. From (8.10) and (8.14) we get

$$\mathcal{N} \geq \mathcal{E}_+ + \frac{s_2^2 + s_2}{2}, \qquad \mathcal{E}_- \leq \mathcal{E}_+ - M_2,$$

whence

$$-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2 \leq \frac{\mathcal{E}_+}{2} - \frac{s_2^2 - 3s_2}{4} + M_2. \tag{8.18}$$

It follows from assertion (iv) of Lemma 8.2 that the vertex $(\mathcal{E}_+, \mathcal{E})$ of P lies in the half-plane $x_1 \geq n$, so using assertion (i) and (8.3), we get $\mathcal{E}_+ \leq \mathcal{E} - 1 \leq 2n - 2$. Moreover, $M_2 \leq 1$ and $s_2^2 - 3s_2 \geq -2$, since s_2 is an integer, so from (8.18) we obtain

$$-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2 \leq n + \frac{1}{2}.$$

Combining this with (8.17), we get

$$2N \leq 4n + 5.$$

Dividing both sides by 2 and taking the floor, we obtain $N \leq 2n + 2$, as claimed.

To conclude the proof, we show that if the segment $[(0, -n), (n, 0)]$ does not split P, there exists an affine automorphism of $\mathbb{Z}^2$ mapping P on a type II_n or a type III_n polygon.

Define the automorphism φ by

$$\varphi(x_1, x_2) = (-x_1 + x_2 + n, x_2).$$

By definition, the segments $[\mathbf{0}, (n, 0)]$ and $[(n, n), (2n, n)]$ split the polygon P, and so does $[\mathbf{0}, (n, n)]$ by virtue of Lemma 8.1. Consequently, the images of those segments under φ – i.e., the segments $[(n, 0), \mathbf{0}]$, $[(n, n), (0, n)]$, $[(n, n), (n, 2n)]$ – split $\varphi(P)$. If P is also split by $[(n, 0), (2n, n)]$, then $\varphi(P)$ is split by $[\mathbf{0}, (0, n)]$, and consequently, $\varphi(P)$ is a type II_n polygon. Otherwise, the line $x_1 - x_2 = n$ does not split P, so the line $x_1 = 0$ does not split $\varphi(P)$ either, and the latter is a type III_n polygon. □

9. Type V polygons

In this section we prove Theorem 1.2 for type V polygons.

We will denote by Δ_n the triangle with the vertices $\mathbf{0}$, $(2n, 0)$, and $(0, 2n)$ defined by (2.1).

Lemma 9.1. *Suppose that P is an N-gon of type V_n and that the frame $\big((n,n); -\mathbf{e}_2, -\mathbf{e}_1\big)$ splits the slope Q_2 and forms small angle with it. Suppose that either*

$$\mathscr{S}_+ \leq n \tag{9.1}$$

or

$$\mathscr{S}_+ \geq n+1 \quad \text{and} \quad \mathscr{W}_+ \leq n. \tag{9.2}$$

Then

$$N \leq 2n+2. \tag{9.3}$$

Proof. As Q_1 is a slope with respect to the basis $(\mathbf{e}_2, -\mathbf{e}_1)$, by Proposition 3.2 there exists an integer s_1 such that

$$2N_1 \leq \mathscr{E}_- - \mathscr{S} + s_1, \tag{9.4}$$

$$\mathscr{E} - \mathscr{S}_+ \geq \frac{1}{2} s_1(s_1+1), \tag{9.5}$$

$$0 \leq s_1 \leq N_1. \tag{9.6}$$

The same proposition applied to Q_3 and $(\mathbf{e}_1, -\mathbf{e}_2)$ ensures the existence of an integer s_3 such that

$$2N_3 \leq \mathscr{N}_- - \mathscr{W} + s_3, \tag{9.7}$$

$$\mathscr{N} - \mathscr{W}_+ \geq \frac{1}{2} s_3(s_3+1), \tag{9.8}$$

$$0 \leq s_3 \leq N_3. \tag{9.9}$$

As Q_4 is a slope with respect to the bases $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_2, \mathbf{e}_1)$, by the same proposition there exist integers s and s' such that

$$2N_4 \le \mathscr{S}_- - \mathscr{W} + s, \tag{9.10}$$

$$\mathscr{W}_- - \mathscr{S} \ge \frac{1}{2}s(s+1), \tag{9.11}$$

$$0 \le s \le N_4, \tag{9.12}$$

$$2N_4 \le \mathscr{W}_- - \mathscr{S} + s', \tag{9.13}$$

$$\mathscr{S}_- - \mathscr{W} \ge \frac{1}{2}s'(s'+1), \tag{9.14}$$

$$0 \le s' \le N_4. \tag{9.15}$$

The frame $\big((n,n); -\mathbf{e}_2, -\mathbf{e}_1\big)$ forms small angle with Q_2, so by Corollary 3.8

$$2N_2 \le 2n - \mathscr{N}_+ - \mathscr{E}_+ - \left\lceil \frac{\mathscr{E} - n}{2} \right\rceil + 1. \tag{9.16}$$

By Proposition 3.12,

$$\mathscr{S}_+ - \mathscr{S}_- \ge M_1, \tag{9.17}$$

$$\mathscr{E}_+ - \mathscr{E}_- \ge M_2, \tag{9.18}$$

$$\mathscr{N}_+ - \mathscr{N}_- \ge M_3, \tag{9.19}$$

$$\mathscr{W}_+ - \mathscr{W}_- \ge M_4. \tag{9.20}$$

Moreover, as the points of P satisfy (2.1), we have

$$\mathscr{W} \ge 0, \tag{9.21}$$

$$\mathscr{S} \ge 0. \tag{9.22}$$

Assume that (9.3) does not hold. Then

$$2N \ge 4n + 6. \tag{9.23}$$

First, assume that (9.1) holds.

Let us estimate $2N$ from above. First, estimate the sum $2N_1 + 2N_2 + 2M_2$. Using (9.4), (9.16), (9.18), and (9.22), we have

$$2N_1 + 2N_2 + 2M_2 \le 2n + s_1 - \mathscr{N}_+ + M_2 - \left\lceil \frac{\mathscr{E} - n}{2} \right\rceil + 1. \tag{9.24}$$

Estimating the ceiling by means of (9.5), we obtain:

$$\left\lceil \frac{\mathscr{E} - n}{2} \right\rceil \ge -n + \mathscr{S}_+ + s_1 + \left\lceil \frac{n - \mathscr{S}_+}{2} + \frac{1}{4}(s_1^2 - 3s_1) \right\rceil. \tag{9.25}$$

It follows from (9.1) that $(n - \mathcal{S}_+)/2 \geq 0$, and because s_1 is an integer, we have $1/4(s_1^2 - 3s_1) \geq -1/2$, so we get

$$\left\lceil \frac{n - \mathcal{S}_+}{2} + \frac{1}{4}(s_1^2 - 3s_1) \right\rceil \geq \left\lceil -\frac{1}{2} \right\rceil = 0.$$

Combining this with (9.25), we get

$$\left\lceil \frac{\mathcal{E} - n}{2} \right\rceil \geq -n + \mathcal{S}_+ + s_1,$$

and further combining this with (9.24) and the inequality $M_2 \leq 1$, we obtain

$$2N_1 + 2N_2 + 2M_2 = 3n - \mathcal{N}_+ - \mathcal{S}_+ + M_2 + 1 \leq 3n - \mathcal{N}_+ - \mathcal{S}_+ + 2.$$

By means of the last bound and (9.7), (9.10), (9.17), (9.19), and (9.21), we obtain

$$\begin{aligned} 2N &= (2N_1 + 2N_2 + 2M_2) + 2N_3 + 2N_4 + 2M_1 + 2M_3 + 2M_4 \\ &\leq (3n - \mathcal{N}_+ - \mathcal{S}_+ + 2) + (\mathcal{N}_- - \mathcal{W} + s_3) \\ &\quad + (\mathcal{S}_- - \mathcal{W} + s) + 2M_1 + 2M_3 + 2M_4 \\ &\leq 3n + 2 + s_3 + s + M_1 + M_3 + 2M_4 \leq 3n + 3 + s_3 + s + M_3 + 2M_4. \end{aligned}$$

Comparing this bound with (9.23), we get

$$3n + 3 + s_3 + s + M_3 + 2M_4 \geq 4n + 6,$$

whence

$$n \leq s_3 + s + M_3 + 2M_4 - 3. \tag{9.26}$$

As $M \subset \Delta_n$, we have

$$\mathcal{N}_+ + \mathcal{N} \leq 2n.$$

Let us estimate the terms on the left-hand side. Using (9.21), (9.7), (9.9), and (9.19), we get

$$\mathcal{N}_+ = \mathcal{W} + (\mathcal{N}_- - \mathcal{W}) + (\mathcal{N}_+ - \mathcal{N}_-) \geq 2N_3 - s_3 + M_3 \geq s_3 + M_3.$$

Using (9.22), (9.11), (9.20), and (9.8), we obtain

$$\begin{aligned}\mathcal{N} &= \mathcal{S} + (\mathcal{W}_- - \mathcal{S}) + (\mathcal{W}_+ - \mathcal{W}_-) + (\mathcal{N} - \mathcal{W}_+)\\ &\geq \frac{1}{2}s(s+1) + M_4 + \frac{1}{2}s_3(s_3+1).\end{aligned}$$

Thus,

$$(s_3 + M_3) + \left(\frac{1}{2}s(s+1) + M_4 + \frac{1}{2}s_3(s_3+1)\right) \leq 2n,$$

or, equivalently,

$$\frac{1}{2}(s_3^2 + 3s_3) + \frac{1}{2}(s^2 + s) + M_3 + M_4 \leq 2n. \tag{9.27}$$

Now we use (9.26) to estimate n on the right-hand side of (9.27):

$$\frac{1}{2}(s_3^2 + 3s_3) + \frac{1}{2}(s^2 + s) + M_3 + M_4 \leq 2(s_3 + s + M_3 + 2M_4 - 3).$$

Hence

$$\frac{1}{2}(s_3^2 - s_3) + \frac{1}{2}(s^2 - 3s) \leq M_3 + 3M_4 - 6 \leq -2,$$

so

$$s_3^2 - s_3 + s^2 - 3s \leq -4.$$

Completing the squares, we obtain a contradiction:

$$\left(s_3 - \frac{1}{2}\right)^2 + \left(s - \frac{3}{2}\right)^2 \leq -\frac{3}{2},$$

Thus, we have proved (9.3) provided that (9.1) holds.

Now assume that (9.2) holds.

Let us estimate $2N$ from above starting with the sum $2N_1 + 2N_2 + 2M_1 + 2M_2$. Using (9.4), (9.16), (9.18), and (9.5), we obtain

$$\begin{aligned}&2N_1 + 2N_2 + 2M_1 + 2M_2\\ &\quad\leq (\mathcal{E}_- - \mathcal{S} + s_1) + \left(2n - \mathcal{N}_+ - \mathcal{E}_+ - \left\lceil\frac{\mathcal{E} - n}{2}\right\rceil + 1\right) + 2M_1 + 2M_2\\ &\quad\leq 2n - \mathcal{N}_+ - \mathcal{S} + 2M_1 + M_2 + 1 - \left\lceil\frac{\mathcal{S}_+ - n}{2} + \frac{s_1^2 - 3s_1}{4}\right\rceil.\end{aligned}$$

Estimating $(s_1^2 - 3s_1)/4 \geq -1/2$, we get

$$\begin{aligned}
&2N_1 + 2N_2 + 2M_1 + 2M_2 \\
&\quad \leq 2n - \mathscr{N}_+ - \mathscr{S} + 2M_1 + M_2 + 1 - \left\lceil \frac{\mathscr{S}_+ - n - 1}{2} \right\rceil \\
&\quad = 2n - \mathscr{N}_+ - \mathscr{S} + 2M_1 + M_2 + 1 - \left\lfloor \frac{\mathscr{S}_+ - n}{2} \right\rfloor.
\end{aligned}$$

Write the bound in the form

$$\begin{aligned}
2N_1 + 2N_2 + 2M_1 + 2M_2 \leq 2n - \mathscr{N}_+ + M_1 + M_2 + 1 \\
- \left(\left\lfloor \frac{\mathscr{S}_+ - n}{2} \right\rfloor + \mathscr{S} - M_1 \right).
\end{aligned} \tag{9.28}$$

Let us show that

$$\left\lfloor \frac{\mathscr{S}_+ - n}{2} \right\rfloor + \mathscr{S} - M_1 \geq 0. \tag{9.29}$$

Assume that $M_1 = 1$. According to (9.2), we have either $\mathscr{S}_+ \geq n + 2$ or $\mathscr{S}_+ = n + 1$. In the former case we use (9.22) and obtain (9.29). In the latter case by (9.17) we have $\mathscr{S}_- \leq \mathscr{S}_+ - M_1 = n$, so the edge $[(\mathscr{S}_-, \mathscr{S}), (\mathscr{S}_+, \mathscr{S})]$ of P contains the point $(n, \mathscr{S})$. Thus, we cannot have $\mathscr{S} = 0$, since P is free of points of $n\mathbb{Z}^2$. Thus, we must have $\mathscr{S} \geq 1$, and (9.29) follows.

If $M_1 = 0$, inequality (9.29) follows from (9.2) and (9.22).

Thus, we have proved (9.29) for all possible cases. Combining it with (9.28), we obtain

$$2N_1 + 2N_2 + 2M_1 + 2M_2 \leq 2n - \mathscr{N}_+ + M_1 + M_2 + 1.$$

Now estimate $2N$ using the last inequality and (9.7), (9.13), (9.19), (9.20), (9.21), and (9.22):

$$\begin{aligned}
2N &= (2N_1 + 2N_2 + 2M_1 + 2M_2) + 2N_3 + 2N_4 + 2M_3 + 2M_4 \\
&\leq (2n - \mathscr{N}_+ + M_1 + M_2 + 1) + (\mathscr{N}_- - \mathscr{W} + s_3) \\
&\quad + (\mathscr{W}_- - \mathscr{S} + s') + 2M_3 + 2M_4 \\
&\leq 2n + 1 + s_3 + s' + \big(M_3 - (\mathscr{N}_+ - \mathscr{N}_-)\big) \\
&\quad + (\mathscr{W}_- + M_4) + M_1 + M_2 + M_3 + M_4 \\
&\leq 2n + 1 + s_3 + s' + \mathscr{W}_+ + M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

Comparing this bound with (9.23) we obtain

$$\mathscr{W}_+ \geq -s_3 - s' + 2n + 5 - M_1 - M_2 - M_3 - M_4. \tag{9.30}$$

Together with (9.2) this implies

$$n \leq s_3 + s' - 5 + M_1 + M_2 + M_3 + M_4. \tag{9.31}$$

The triangle Δ_n lies in the half-plane $x_1 \leq 2n$, so we have

$$\mathscr{S}_+ \leq 2n. \tag{9.32}$$

We can estimate the left-hand side by means of (9.21), (9.14), and (9.17) as follows:

$$\mathscr{S}_+ = \mathscr{W} + (\mathscr{S}_- - \mathscr{W}) + (\mathscr{S}_+ - \mathscr{S}_-) \geq \frac{1}{2}s'(s'+1) + M_1.$$

Using this bound and (9.31), we obtain from (9.32):

$$\frac{1}{2}s'(s'+1) + M_1 \leq 2s_3 + 2s_4 - 10 + 2M_1 + 2M_2 + 2M_3 + 2M_4,$$

whence

$$\begin{aligned} 2s_3 &\geq \frac{1}{2}(s'^2 - 3s') + 10 - M_1 - 2M_2 - 2M_3 - 2M_4 \\ &\geq \frac{1}{2}(s'^2 - 3s') + 3 = \frac{1}{2}(s'^2 - 3s' + 6), \end{aligned}$$

and finally

$$s_3 \geq \frac{1}{4}(s'^2 - 3s' + 6). \tag{9.33}$$

The vertices of P solve (2.1), so

$$\mathscr{N}_+ + \mathscr{N} \leq 2n. \tag{9.34}$$

Using (9.21), (9.7), (9.9), and (9.19), we deduce

$$\mathscr{N}_+ = \mathscr{W} + (\mathscr{N}_- - \mathscr{W}) + (\mathscr{N}_+ - \mathscr{N}_-) \geq 2N_3 - s_3 + M_3 \geq s_3 + M_3,$$

while by virtue of (9.30) and (9.8) we obtain

$$\begin{aligned}\mathcal{N} &= \mathcal{W}_+ + (\mathcal{N} - \mathcal{W}_+)\\ &\geq (-s_3 - s' + 2n + 5 - M_1 - M_2 - M_3 - M_4) + \frac{1}{2}s_3(s_3 + 1).\end{aligned}$$

Combining (9.34) with two last bounds, we get

$$s' \geq \frac{1}{2}s_3(s_3 + 1) + 5 - M_1 - M_2 - M_4 \geq \frac{1}{2}s_3(s_3 + 1) + 2 = \frac{1}{2}(s_3^2 + s_3 + 4),$$

and finally

$$s' \geq \frac{1}{2}(s_3^2 + s_3 + 4). \tag{9.35}$$

It is not hard to check that the inequalities (9.33) and (9.35) are incompatible. This contradiction proves (9.3) in case (9.2) holds. □

Definition 9.2. We call an integral polygon *minimal* if it does not contain other integral polygon with the same number of vertices.

We note two simple properties of minimal polygons.

Proposition 9.3. *Any edge of a minimal polygon contains precisely two interger points – its endpoints.*

Proof. If $\mathbf{v}_1, \dots, \mathbf{v}_N$ are the vertices of an integral N-gon and its edge $[\mathbf{v}_1, \mathbf{v}_2]$ contains an integral point $\mathbf{v}$ different from $\mathbf{v}_1$ and $\mathbf{v}_2$, then it is easily seen that the convex hull of the points $\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an integral N-gon contained in the original one and different from it. This means that the original polygon is not minimal. □

Proposition 9.4. *Affine automorphisms of the integral lattice map minimal polygons onto minimal polygons.*

This proposition is obvious.

Proof of Theorem 1.2 *for type V polygons.* Let P be a type V_n polygon having N vertices. We must prove that

$$N \leq 2n + 2. \tag{9.36}$$

We can certainly assume that P is minimal, for if not, we replace P by a minimal polygon contained in P (of course, this minimal polygon is of type V_n as well).

First, assume that P satisfies either

$$\mathcal{S}_+ \leq n \tag{9.37}$$

or

$$\mathcal{S}_+ \geq n+1, \qquad \mathcal{W}_+ \leq n. \tag{9.38}$$

If P lies in a slab of the form

$$0 \leq x_1 \leq n, \qquad n \leq x_1 \leq 2n, \qquad 0 \leq x_2 \leq n, \qquad n \leq x_2 \leq 2n,$$

it is a type I_n polygon, and the bound (9.37) follows from Theorem 1.2 for type I polygons (Remark 2.8). Otherwise, P is split by the segments $[(n,0),(n,n)]$ and $[(0,n),(n,n)]$, which are the intersections of the lines $x_1 = n$ and $x_2 = n$ with Δ_n. Therefore, by Proposition 3.11 the frame $\big((n,n); -\mathbf{e}_2, -\mathbf{e}_1\big)$ splits the slope Q_2. If this frame forms small angle with the slope, Lemma 9.1 provides (9.36). If not, it follows from Proposition 3.6 that the frame $\big((n,n); -\mathbf{e}_1, -\mathbf{e}_2\big)$ forms small angle with Q_2. Let P' be the reflection of P about the line $x_1 = x_2$. It is not hard to check that $\big((n,n); -\mathbf{e}_2, -\mathbf{e}_1\big)$ forms small angle with $Q_2(P')$; moreover, P' is a minimal type V_n polygon, and since

$$\mathcal{S}_+(P') = \mathcal{W}_+(P), \qquad \mathcal{W}_+(P') = \mathcal{S}_+(P),$$

we see that P' satisfies (9.37) or (9.38). Applying the established part of the lemma to P', we obtain (9.36).

Now suppose that P satisfies neither (9.37), nor (9.38). Thus, in particular,

$$\mathcal{S}_+(P) \geq n+1.$$

Consider the affine automorphism of $n\mathbb{Z}^2$ given by

$$\varphi(x_1,x_2) = (-x_1 - x_2 + 2n, x_2).$$

By Proposition 9.4, the polygon $\varphi(P)$ is minimal. Moreover, it lies in the triangle Δ_n, since $\Delta_n = \varphi(\Delta_n)$. Obviously, we have

$$\mathcal{S}\big(\varphi(P)\big) = \mathcal{S}(P).$$

A straightforward computation gives

$$\mathscr{S}_-\big(\varphi(P)\big) = -\mathscr{S}_+(P) - \mathscr{S}(P) + 2n.$$

As the points of P satisfy system (2.1), we have $\mathscr{S}(P) \geq 0$, so

$$\mathscr{S}_-\big(\varphi(P)\big) \leq -\mathscr{S}_+(P) + 2n \leq n - 1.$$

By Proposition 9.3,

$$\mathscr{S}_+\big(\varphi(P)\big) \leq \mathscr{S}_-\big(\varphi(P)\big) + 1,$$

so we have

$$\mathscr{S}_+\big(\varphi(P)\big) \leq n.$$

Thus, the polygon $\varphi(P)$ satisfies (9.37), and applying the established part of the lemma to $\varphi(P)$, we obtain (9.36). □

10. Proof of the main theorem

Proof of Theorem 1.2. Suppose that P is a convex integral N-gon free of points of $n\mathbb{Z}^2$. By Theorem 2.6 in can be mapped onto an N-gon of one of the types I_n–V_n by a suitable affine transformation. The bound $N \leq 2n + 2$ is obvious for type I_n polygons (see Remark 2.8) and in Sections 6–9 this bound is established for polygons of types II_n–V_n. The theorem is proved. □

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