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Accuracy of a coupled mixed and Galerkin finite element approximation for poroelasticity

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Abstract. In this paper, we consider a combined mixed finite element and continuous Galerkin finite element formulation for a coupled flow and geomechanics model. We use the lowest order Raviart–Thomas finite elements for the spatial approximation of the flow variables and continuous piecewise linear finite elements for the deformation variable. This numerical approach appears to be a common choice in the existing reservoir engineering simulators. We focus on deriving error estimates in a discrete-in-time setting. Previous *a priori* error estimates described in the literature e.g. [2], [19], which are optimal, show first order convergence in space with respect to the L^2-norm for the pressure and for the average fluid velocity and also first order convergence in space with respect to the H^1-norm for the displacement. Here we prove one extra order of convergence for the displacement approximation with respect to the L^2-norm. We also demonstrate that, by including a post-processing step in the scheme, the order of convergence of the approximation of pressure can be improved. Even though this result is critical for deriving the L^2-norm error estimates for the approximation of the deformation variable, surprisingly the corresponding gain of one convergence order holds independently of including or not the post-processing step in the method.

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1. Introduction

Poroelasticity theory is used to model the interaction of fluid flow and the mechanical response in fluid-saturated porous media. The deformation of the medium influences the flow of fluid and vice versa. The development of the coupled geomechanics and flow models emerged in the context soil mechanics, in particular with the seminal work of Karl von Terzaghi [25] and the theory proposed by Maurice Anthony Biot [4], [5], known as Biot Theory.

Poroelasticity models are widely used in geomechanics and reservoir engineering, and they have relevance in diverse other fields as, for example, biomechanics and environmental engineering. Due to the high importance of the applications, there is an ever-growing demand for reliable models and numerical tools. Applications range from reservoir simulation [9], [20], [23], [24], modelling carbon sequestration [12], estimating the mechanical behaviour of fluid-saturated living bone tissue [11], among others as highlighted in [18].

As a prototype of the geomechanical coupling between the single-phase flow of pore fluids and the deformation of the solid skeleton, we consider in this paper the linear poroelastic Biot Theory. The flow (pressures and fluxes) and deformations (displacements) in the poroelastic medium are modeled based on the Darcy's law and on the momentum and mass conservation principles. The momentum equation is similar to the linear elasticity equation, with a fluid pressure term acting as a force.

We summarize the governing equations below. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, denote the domain of interest. The coupled balance equations are written as follow: find $(\mathbf{u}, p)$ such that

$$
\begin{aligned}
-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla^2\mathbf{u} + \alpha\nabla p &= \mathbf{f} \text{ in } \Omega\times(0,T] \\
\frac{\partial}{\partial t}(c_o p + \alpha\nabla\cdot\mathbf{u}) - \frac{1}{\mu_f}\nabla\cdot K(\nabla p - \rho_f\mathbf{g}) &= s_f \text{ in } \Omega\times(0,T] \\
p &= p_D \text{ on } \Gamma_p\times(0,T] \\
-\frac{1}{\mu_f}K(\nabla p - \rho_f\mathbf{g})\cdot\boldsymbol{\eta} &= q \text{ on } \Gamma_f\times(0,T] \\
\mathbf{u} &= \mathbf{u}_D \text{ on } \Gamma_0\times(0,T] \\
\tilde{\boldsymbol{\sigma}}\boldsymbol{\eta} &= \mathbf{r}_N \text{ on } \Gamma_N\times(0,T] \\
p(0) &= p^0 \text{ in } \Omega,
\end{aligned}
\tag{1}
$$

where $\partial\Omega = \Gamma_p \cup \Gamma_f$ and $\partial\Omega = \Gamma_0 \cup \Gamma_N$, with $\operatorname{meas}(\Gamma_0) > 0$. The symbol $\boldsymbol{\eta}$ represents the outward normal vector on $\partial\Omega$. The primary variables are the pressure p and the deformation $\mathbf{u}$. The physical parameters of the model are: λ, μ, the Lamé constants, c_o, the constrained specific storage coefficient, α, the Biot–Willis constant, μ_f, the fluid viscosity, ρ_f, the fluid mass density and $\mathbf{g}$, the body force per unit of mass. The effective stress $\boldsymbol{\sigma}$, is the standard stress tensor from elasticity,

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda\,\mathrm{tr}\big(\varepsilon(\mathbf{u})\big)I,$$

where

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\big(\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^t\big),$$

I is the identity matrix in $\mathbb{R}^d \times \mathbb{R}^d$, and the total stress, $\tilde{\boldsymbol{\sigma}}$, is given by

$$\tilde{\boldsymbol{\sigma}}(\mathbf{u}, p) = \boldsymbol{\sigma}(\mathbf{u}) - \alpha p I.$$

K denotes the symmetric permeability tensor. We require the existence of the inverse of the operator K and we assume that K^{-1} is uniformly bounded and positive definite, that is, there exists a positive constant ζ such that, for all $\mathbf{s} \in \left(L^2(\Omega)\right)^d$,

$$\left(K^{-1}(\mathbf{x}, t)\mathbf{s}, \mathbf{s}\right) \geq \zeta \|\mathbf{s}\|_{L^2(\Omega)}, \qquad \forall \mathbf{x} \in \Omega,\ t \in [0, T]. \tag{2}$$

We assume the storage coefficient to be strictly positive and uniformly bounded,

$$0 < \gamma_c \leq c_o(\mathbf{x}) \leq L_c, \qquad \forall \mathbf{x} \in \Omega, \tag{3}$$

and the Biot–Willis constant with a range of values $0 < \alpha \leq 1$.

In practice, if the initial condition p^0 is unknown, it can be found by considering $\nabla p(0) = \rho_f \mathbf{g}$. And then the first equation of (1) is used to obtain $\mathbf{u}(0)$.

The complete system (1) can be solved either simultaneously, in a fully coupled approach, or sequentially, in a loosely coupled scheme. The analysis of the fully coupled numerical method, combining a mixed method and a continuous or discontinuous Galerkin method, was considered e.g. in [2], [13], [19]. The iteratively coupled methods were considered e.g. in [14], [16], [28].

In this paper we focus on the fully coupled method which combines lowest order Raviart–Thomas mixed finite elements for the Darcy flow and Galerkin piecewise linear finite elements for elasticity. We analyze the effect on convergence of considering a post-processing step in the scheme and we prove second order of convergence in space for the pressure approximation. Moreover we derive L^2-error estimates for the approximation of the deformation and we also obtain second order of convergence in space. Both results, which are here proved for the fully coupled approach, are also useful to analyse the iteratively coupled schemes which converge to fully coupled schemes [28].

2. The coupled variational formulation

In order to introduce the mixed formulation [17], [21], we consider the variable for the flux $\mathbf{z} = -\frac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g})$.

The function space for pressure is $L^2(\Omega)$. The space used for the flux variable is

$$\mathbf{H}(\mathrm{div}) := \left\{\mathbf{s} \in \left(L^2(\Omega)\right)^d : \nabla \cdot \mathbf{s} \in L^2(\Omega)\right\}$$

and we define its subset

$$\mathbf{S}_0 := \{\mathbf{s} \in \mathbf{H}(\text{div}) : \mathbf{s} \cdot \boldsymbol{\eta}|_{\Gamma_f} = 0\}.$$

The function space for the deformation is

$$\mathbf{V}_0 := \left\{\mathbf{v} \in \left(H^1(\Omega)\right)^d : \mathbf{v}|_{\Gamma_0} = 0\right\}.$$

Associated to this space, we define the bilinear form $a_{\mathbf{u}}(.\,,.)$ by

$$a_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x},$$

or equivalently

$$a_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) = \int_\Omega \big(2\mu\big(\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\big) + \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\big)\, d\mathbf{x}.$$

The bilinear form is continuous and coercive in $\mathbf{V}_0 \times \mathbf{V}_0$ ([7]); therefore, for some positive real numbers C_{cont} and C_{coer} holds

$$\begin{aligned} a_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) &\leq C_{cont}\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_0, \\ a_{\mathbf{u}}(\mathbf{v}, \mathbf{v}) &\geq C_{coer}\|\mathbf{v}\|^2_{H^1(\Omega)}, \qquad \forall \mathbf{v} \in \mathbf{V}_0. \end{aligned} \tag{4}$$

The space $\mathbf{V}_0$ is endowed with the norm $\|.\|_{a_{\mathbf{u}}}$, where $\|\mathbf{v}\|^2_{a_{\mathbf{u}}} := a_{\mathbf{u}}(\mathbf{v}, \mathbf{v})$.

We define the linear functionals

$$\begin{aligned} \ell_1(\mathbf{v}) &:= \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N \cdot \mathbf{v}, \qquad \mathbf{v} \in \mathbf{V}_0, \\ \ell_2(w) &:= \int_\Omega s_f w, \qquad w \in L^2(\Omega), \\ \ell_3(\mathbf{s}) &:= -\int_{\Gamma_p} p_D \mathbf{s} \cdot \boldsymbol{\eta} + \int_\Omega \rho_f \mathbf{g} \cdot \mathbf{s}, \qquad \mathbf{s} \in \mathbf{S}_0. \end{aligned}$$

Since the boundary conditions are allowed to be inhomogeneous, we need to select, for each $t \in [0, T]$, a function $\mathbf{u}_d(.\,, t) \in \left(H^1(\Omega)\right)^d$ such that $\mathbf{u}_d(.\,, t)|_{\Gamma_0} = \mathbf{u}_D(.\,, t)$ and a function $\mathbf{z}_d(.\,, t) \in \mathbf{H}(\text{div})$ such that $\mathbf{z}_d(.\,, t)|_{\Gamma_f} \cdot \boldsymbol{\eta} = q(.\,, t)$.

The variational problem becomes: find $\mathbf{u} \in \mathbf{u}_d + H^1([0, T]; \mathbf{V}_0)$, $p \in H^1\big([0, T]; L^2(\Omega)\big)$ and $\mathbf{z} \in \mathbf{z}_d + L^2([0, T]; \mathbf{S}_0)$ such that

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) - \alpha(\nabla\cdot\mathbf{v}, p) = \ell_1(\mathbf{v}), \tag{5}$$

$$\left(c_o \frac{\partial p}{\partial t}, w\right) + \alpha\left(\frac{\partial}{\partial t}\nabla\cdot\mathbf{u}, w\right) + (\nabla\cdot\mathbf{z}, w) = \ell_2(w), \tag{6}$$

$$\mu_f(K^{-1}\mathbf{z},\mathbf{s}) - (p, \nabla\cdot\mathbf{s}) = \ell_3(\mathbf{s}) \tag{7}$$

holds for all $(\mathbf{v}, w, \mathbf{s}) \in \mathbf{V}_0 \times L^2(\Omega) \times \mathbf{S}_0$ and $t \in [0, T]$.

We also make the following smoothness assumptions, in order the above variational formulation makes sense:

$$\begin{aligned}
&\mathbf{f} \in C^1\big([0,T]; \big(H^{-1}(\Omega)\big)^d\big),\\
&s_f \in C\big([0,T]; L^2(\Omega)\big),\\
&p_D \in C\big([0,T]; L^2(\Gamma_p)\big),\\
&q \in C([0,T]; TrS), \qquad TrS = \{\mathbf{s}\cdot\boldsymbol{\eta}_{|_{\Gamma_f}} : \mathbf{s} \in \mathbf{H}(\mathrm{div})\},\\
&\mathbf{u}_D \in C^1\big([0,T]; \big(H^{1/2}(\Gamma_0)\big)^d\big),\\
&\mathbf{r}_N \in C^1\big([0,T]; \big(H^{-1/2}(\Gamma_N)^d\big)\big),\\
&\mathbf{g} \in C\big([0,T]; \big(L^2(\Omega)\big)^d\big),\\
&\mathbf{u}^0 \in \big(H^1(\Omega)\big)^d,\\
&p^0 \in L^2(\Omega).
\end{aligned}$$

For the study of the error, we assume that the weak solution of the problem (5)–(7) is sufficiently regular: $\mathbf{u} \in W^{2,\infty}\big([0,T]; \big(H^2(\Omega)\big)^d\big)$, $p \in W^{2,\infty}\big([0,T]; H^1(\Omega)\big)$, and $\mathbf{z} \in L^2\big([0,T]; \big(H^1(\Omega)\big)^d\big)$.

In order to approximate the variational problem (5)–(7) with a finite element scheme we need to provide some definitions.

Let $\mathscr{E}_h$ and $\mathscr{E}_H$ be two nondegenerate partitions of the polyhedral domain Ω, with maximal element diameter h and H, respectively. The elements of $\mathscr{E}_h$ and $\mathscr{E}_H$ are triangles, if $d = 2$, and tetrahedra, if $d = 3$.

Let $(W_h, \mathbf{S}_h) \subset L^2(\Omega) \times \mathbf{H}(\mathrm{div})$ denote a standard mixed finite element space on $\mathscr{E}_h$, called lowest order Raviart–Thomas approximating space (RT0) (e.g. [8], [21]) and

$$\mathbf{S}_{h,0} := \{\mathbf{s} \in \mathbf{S}_h : \mathbf{s}\cdot\boldsymbol{\eta}_{|_{\Gamma_f}} = 0\}.$$

We consider the linear operators $\Pi_h : \mathbf{H}(\mathrm{div}) \to \mathbf{S}_h$ and $I_h : L^2(\Omega) \to W_h$ which satisfy the following properties:

$$\begin{aligned}
&\big(\nabla\cdot(\mathbf{s}-\Pi_h\mathbf{s}),w\big)=0, \qquad \forall\mathbf{s}\in\mathbf{H}(\mathrm{div}),\ w\in W_h,\\
&\|\mathbf{s}-\Pi_h\mathbf{s}\|_{L^2(\Omega)}\le C_\Pi h\|\mathbf{s}\|_{H^1(\Omega)}, \qquad \forall\mathbf{s}\in\big(H^1(\Omega)\big)^d,\\
&\nabla\cdot\Pi_h=I_h\nabla\cdot,\\
&(\nabla\cdot\mathbf{s}_h,p-I_hp)=0, \qquad \forall\mathbf{s}_h\in\mathbf{S}_h,\ p\in L^2(\Omega),\\
&\|p-I_hp\|_{L^2(\Omega)}\le C_I h\|p\|_{H^1(\Omega)}, \qquad \forall p\in H^1(\Omega).
\end{aligned}$$

Let $\mathbf{V}_H\subset\big(H^1(\Omega)\big)^d$ be the space of continuous piecewise polynomials of degree 1 defined on $\mathscr{E}_H$ and

$$\mathbf{V}_{H,0}:=\{\mathbf{v}\in\mathbf{V}_H:\mathbf{v}|_{\Gamma_0}=0\}.$$

The elliptic projector $E_H:\big(H^1(\Omega)\big)^d\to\mathbf{V}_H$ is defined by

$$a_{\mathbf{u}}(\mathbf{u}-E_H\mathbf{u},\mathbf{v}_H)=0, \qquad \forall\mathbf{u}\in\big(H^1(\Omega)\big)^d,\ \mathbf{v}_H\in\mathbf{V}_H, \tag{8}$$

and satisfies (see [7])

$$\|\mathbf{u}-E_H\mathbf{u}\|_{a_{\mathbf{u}}}\le C_E H\|\mathbf{u}\|_{H^2(\Omega)}, \qquad \forall\mathbf{u}\in\big(H^2(\Omega)\big)^d. \tag{9}$$

The fully discrete method is derived by discretizing the time derivatives. Here we considered the backward Euler method. We define $\Delta t=T/N$, where N denotes the number of time steps and $t^n=n\Delta t$. In what follows, we will use the notation $g^n=g(.,t^n)$.

Let $\mathbf{u}_{dH}(\mathbf{x},t)=E_H\mathbf{u}_d(\mathbf{x},t)$ and $\mathbf{z}_{dh}(\mathbf{x},t)=\Pi_h\mathbf{z}_d(\mathbf{x},t)$. The complete numerical formulation becomes: find $\mathbf{u}_H^n\in\mathbf{u}_{dH}^n+\mathbf{V}_{H,0}$, $p_h^n\in W_h$, $\mathbf{z}_h^n\in\mathbf{z}_{dh}^n+\mathbf{S}_{h,0}$ such that

$$a_{\mathbf{u}}(\mathbf{u}_H^n,\mathbf{v})-\alpha(p_h^n,\nabla\cdot\mathbf{v})=\ell_1^n(\mathbf{v}), \tag{10}$$

$$\left(c_o\frac{p_h^n-p_h^{n-1}}{\Delta t},w\right)+\alpha\left(\nabla\cdot\frac{\mathbf{u}_H^n-\mathbf{u}_H^{n-1}}{\Delta t},w\right)+(\nabla\cdot\mathbf{z}_h^n,w)=\ell_2^n(w), \tag{11}$$

$$\mu_f\big((K^n)^{-1}\mathbf{z}_h^n,\mathbf{s}\big)-(p_h^n,\nabla\cdot\mathbf{s})=\ell_3^n(\mathbf{s}), \tag{12}$$

for all $(\mathbf{v},w,\mathbf{s})\in(\mathbf{V}_{H,0},W_h,\mathbf{S}_{h,0})$.

Additionally, we consider the initial conditions $\mathbf{u}_H^0\in\mathbf{u}_{dH}^0+\mathbf{V}_{H,0}$, $p_h^0\in W_h$, such that

$$a_{\mathbf{u}}(\mathbf{u}_H^0, \mathbf{v}) = a_{\mathbf{u}}(\mathbf{u}^0, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}_H,$$
$$(p_h^0, w) = (p^0, w), \qquad \forall w \in W_h.$$

The fully coupled scheme involves calculating $\mathbf{u}_H^n$, p_h^n and $\mathbf{z}_h^n$ simultaneously.

The convergence result in the next theorem can be found in [2], [19].

Theorem 2.1. *Let* $(\mathbf{u}, p, \mathbf{z})$ *be the solution of* (5)–(7) *and* $(\mathbf{u}_H, p_h, \mathbf{z}_h)$ *be the solution of* (10)–(12). *Then, for* Δt *small enough, there exists* $C > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^\infty(H^1)} + \|p - p_h\|_{L^\infty(L^2)} + \|\mathbf{z} - \mathbf{z}_h\|_{L^2(L^2)} \leq C(H + h) + \mathcal{O}(\Delta t), \tag{13}$$

where C *depends on the model parameters and on the solution of the continuous model* (5)–(7) *but is not dependent on* H, h *and* Δt.

3. Post-processing step for pressure

The objective of this section is to obtain a higher order approximation for pressure. To improve accuracy, a post-processing step can be included in the numerical scheme, following the idea proposed by Arbogast and Wheeler in [1].

To simplify the notation, we use $\|.\|_0$ and $\|.\|_1$, respectively, for the $L^2(\Omega)$ and $H^1(\Omega)$ norms. When we consider a subset $R \subset \Omega$, the $L^2(R)$ inner product is denoted by $(.\,,.)_R$ and the corresponding norm by $\|.\|_{0,R}$.

We start by defining the space $\tilde{W}_h$ consisting of functions that are discontinuous and piecewise linear over the grid $\mathcal{E}_h$. We locally post-process the pressure by finding $\tilde{p}_h \in \tilde{W}_h$ such that on each element of $R \in \mathcal{E}_h$,

$$\big(c_o(\tilde{p}_h^n - p_h^n), w\big)_R = 0 \qquad \forall w \in W_h, \tag{14}$$
$$(K^n \nabla \tilde{p}_h^n + \mathbf{z}_h^n, \nabla w)_R = 0 \qquad \forall w \in \tilde{W}_h. \tag{15}$$

We will demonstrate that this post-processing technique improves the approximation p_h^n so that the L^2-error between $\tilde{p}_h^n$ and p^n is of second order in space.

In the error analysis we will compare the post-processed finite element solution to an elliptic projection of the solution of (5)–(7). We define the projection $(P_h, \mathbf{Z}_h) \in W_h \times \mathbf{S}_h$ of $(p, \mathbf{z})$ [1], [27], by

$$\big(c_o(P_h - p), w\big) + \big(\nabla \cdot (\mathbf{Z}_h - \mathbf{z}), w\big) = 0 \qquad \forall w \in W_h, \tag{16}$$
$$\mu_f\big(K^{-1}(\mathbf{Z}_h - \mathbf{z}), \mathbf{s}\big) = (P_h - p, \nabla \cdot \mathbf{s}) \qquad \forall \mathbf{s} \in \mathbf{S}_h, \tag{17}$$

and on each element $R \in \mathcal{E}_h$ we define $\tilde{P}_h \in \tilde{W}_h$ by

$$\left(c_o(\tilde{P}_h - P_h), w\right)_R = 0 \qquad \forall w \in W_h, \tag{18}$$

$$(K\nabla\tilde{P}_h + \mathbf{Z}_h, \nabla w)_R = 0 \qquad \forall w \in \tilde{W}_h. \tag{19}$$

For convenience, we now introduce some additional notation, in particular for auxiliary and projection errors:

$$\xi^n = p_h^n - P_h^n \in W_h, \qquad \tilde{\xi}^n = \tilde{p}_h^n - \tilde{P}_h^n \in \tilde{W}_h, \qquad \boldsymbol{\zeta}^n = \mathbf{z}_h^n - \mathbf{Z}_h^n \in \mathbf{S}_h,$$

and

$$\eta^n = P_h^n - p^n, \qquad \tilde{\eta}^n = \tilde{P}_h^n - p^n.$$

Lemma 3.1. *The following inequalities hold*

$$\|\sqrt{c_o}\xi^n\|_0 \le \|\sqrt{c_o}\tilde{\xi}^n\|_0, \tag{20}$$

$$\left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n\right) \le Q\|(K^n)^{-1/2}\boldsymbol{\zeta}^n\|_0^2 h^2, \tag{21}$$

where the constant Q depends on the positive upper and lower bounds of c_o and K.

Proof. For any element $R \in \mathscr{E}_h$, by (14), (15), (18) and (19) we deduce that

$$\left(c_o(\tilde{\xi}^n - \xi^n), 1\right)_R = 0 \tag{22}$$

and

$$(K^n\nabla\tilde{\xi}^n + \boldsymbol{\zeta}^n, \nabla w)_R = 0, \qquad w \in \tilde{W}_h. \tag{23}$$

Since ξ^n is constant on R, from (22) we have that $\left(c_o(\xi^n - \tilde{\xi}^n), \xi^n\right)_R = 0$. Then

$$(c_o\xi^n, \xi^n)_R = (c_o\tilde{\xi}^n, \xi^n)_R,$$

and (20) follows.

Using approximation properties ([22], Theorem 2.6), for a good choice of the constant C_ξ we get

$$\begin{aligned}\left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n - \xi^n\right)_R &= \left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n\right)_R = \left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n - C_\xi\right)_R \\ &\le C_L\|\sqrt{c_o}(\tilde{\xi}^n - \xi^n)\|_{0,R}\|\nabla\tilde{\xi}^n\|_{0,R}h\end{aligned}$$

where C_L depends on the upper bound of c_o. Then

$$\|\sqrt{c_o}(\tilde{\xi}^n - \xi^n)\|_{0,R} \le C_L\|\nabla\tilde{\xi}^n\|_{0,R}h.$$

Taking $w = \tilde{\xi}^n$ in (23) results

$$\|(K^n)^{1/2}\nabla\tilde{\xi}^n\|_{0,R} \le \|(K^n)^{-1/2}\boldsymbol{\zeta}^n\|_{0,R},$$

and we obtain (21). □

The detailed arguments for proving the next lemma can be found in [1], Theorem 2.

Lemma 3.2. *Assume sufficient regularity of data and of the solution of* (5)–(7). *For each $t \in (0, T]$ and for h sufficiently small, holds*

$$\|\eta\|_0 = \|P_h - p\|_0 \le C_1\|\mathbf{z}\|_1 h, \tag{24}$$

$$\|\tilde{\eta}\|_0 = \|\tilde{P}_h - p\|_0 \le C_2(\|\mathbf{z}\|_1 + \|\nabla\cdot\mathbf{z}\|_1)h^2, \tag{25}$$

$$\|(\tilde{\eta})_t\|_0 = \|(\tilde{P}_h - p)_t\|_0 \le C_3\big(\|\mathbf{z}\|_1 + \|\nabla\cdot\mathbf{z}\|_1 + \|(\mathbf{z})_t\|_1 + \|\nabla\cdot\mathbf{z}_t\|_1\big)h^2, \tag{26}$$

where C_1, C_2 and C_3 are independent of t, p, h and Δt.

The next result will be central in the convergence analysis.

Lemma 3.3. *Let E_H be defined by* (8). *The following estimate holds*

$$\|\nabla\cdot E_H\mathbf{u} - \nabla\cdot\mathbf{u}_H\|_0 \le \frac{\alpha}{\lambda}\|p - p_h\|_0. \tag{27}$$

Proof. For any element $R \in \mathscr{E}_H$ we have

$$\begin{aligned}\lambda\|\nabla\cdot(E_H\mathbf{u} - \mathbf{u}_H)\|_{0,R}^2 &\le a_{\mathbf{u}}(E_H\mathbf{u} - \mathbf{u}_H, E_H\mathbf{u} - \mathbf{u}_H)\\ &= \alpha\big(p - p_h, \nabla\cdot(E_H\mathbf{u} - \mathbf{u}_H)\big)_R\\ &= \alpha\big(p - \tilde{p}_h, \nabla\cdot(E_H\mathbf{u} - \mathbf{u}_H)\big)_R\\ &\le \alpha\|p - \tilde{p}_h\|_{0,R}\|\nabla\cdot(E_H\mathbf{u} - \mathbf{u}_H)\|_{0,R}.\end{aligned}$$ □

The convergence result for the post-processed numerical solution for pressure is given in the next theorem.

Theorem 3.4. *Consider that $\mathscr{E}_h$ and $\mathscr{E}_H$ coincide or $\mathscr{E}_h$ to be a refinement of $\mathscr{E}_H$. Assume sufficient regularity of the solution of* (5)–(7) *and that the initialization error satisfy*

$$\|\tilde{p}_h^0 - p^0\|_0 \le C_0 h^2,$$

for some constant C_0 *depending on* p_0. *Let* Δt *satisfy* $\Delta t \frac{\mu_f}{2} > Qh^2$ *and* $\Delta t = c' H^2$ *for some positive constant* c'. *If* $\frac{\alpha^2}{\lambda} < \frac{1}{4}\gamma_c$, *then for* h *and* H *sufficiently small,*

$$\max_n \| p^n - \tilde{p}_h^n \|_0 \le C_p (h^2 + H^2), \tag{28}$$

where C_p *depends on* p *but not on* h, H *or* Δt.

Proof. For convenience, we use the notation

$$\ell_4(\mathbf{u}, w) = \alpha\left(\frac{\partial}{\partial t}\nabla \cdot \mathbf{u}, w\right),$$
$$\bar{\ell}_4(\mathbf{u}_H^n, w) = \alpha\left(\nabla \cdot \frac{\mathbf{u}_H^n - \mathbf{u}_H^{n-1}}{\Delta t}, w\right).$$

Combining (6), (11) and (14), we obtain, for all $w \in W_h$,

$$\big(c_o(\tilde{p}_h^n - p^n), w\big) - \big(c_o(\tilde{p}_h^{n-1} - p^{n-1}), w\big) + \Delta t(\nabla \cdot \mathbf{z}_h^n, w) - \int_{t^{n-1}}^{t^n} (\nabla \cdot \mathbf{z}, w)\, dt$$
$$= \Delta t\big(\ell_2(w) - \bar{\ell}_4(\mathbf{u}_H^n, w)\big) - \int_{t^{n-1}}^{t^n} \big(\ell_2(w) - \ell_4(\mathbf{u}, w)\big)\, dt.$$

Then

$$\big(c_o(\tilde{\xi}^n + \tilde{\eta}^n), w\big) - \big(c_o(\tilde{\xi}^{n-1} + \tilde{\eta}^{n-1}), w\big) + \Delta t(\nabla \cdot \boldsymbol{\zeta}^n, w)$$
$$+ \Delta t(\nabla \cdot \mathbf{Z}_h^n, w) - \int_{t^{n-1}}^{t^n} (\nabla \cdot \mathbf{z}, w)\, dt$$
$$= \Delta t\big(\ell_2(w) - \bar{\ell}_4(\mathbf{u}_H^n, w)\big) - \int_{t^{n-1}}^{t^n} \big(\ell_2(w) - \ell_4(\mathbf{u}, w)\big)\, dt,$$

and by (16) we get

$$\big(c_o(\tilde{\xi}^n + \tilde{\eta}^n), w\big) - \big(c_o(\tilde{\xi}^{n-1} + \tilde{\eta}^{n-1}), w\big) + \Delta t(\nabla \cdot \boldsymbol{\zeta}^n, w)$$
$$+ \int_{t^{n-1}}^{t^n} \big(\ell_2(w) - \ell_4(\mathbf{u}, w)\big)\, dt$$
$$= \Delta t\big(\ell_2(w) - \bar{\ell}_4(\mathbf{u}_H^n, w)\big) + \int_{t^{n-1}}^{t^n} (\nabla \cdot \mathbf{z}, w)\, dt$$
$$- \Delta t(\nabla \cdot \mathbf{z}^n, w) + \Delta t(c_o \eta^n, w). \tag{29}$$

Combining (7), (12) and (17), we have

$$\begin{aligned}
&\mu_f(K^{-1}\boldsymbol{\zeta}^n, \mathbf{s}) \\
&\quad = (p_h^n, \nabla\cdot\mathbf{s}) + (\ell_3, \mathbf{s}) - \mu_f\big(K^{-1}(\mathbf{Z}_h^n - \mathbf{z}^n), \mathbf{s}\big) - \mu_f\big((K^n)^{-1}\mathbf{z}^n, \mathbf{s}\big) \\
&\quad = (p_h^n, \nabla\cdot\mathbf{s}) + (\ell_3, \mathbf{s}) - (P_h^n, \nabla\cdot\mathbf{s}) - \mu_f\big((K^n)^{-1}\mathbf{z}^n, \mathbf{s}\big) + (p^n, \nabla\cdot\mathbf{s}) \\
&\quad = (\xi^n, \nabla\cdot\mathbf{s}). \qquad (30)
\end{aligned}$$

Taking in (29) and (30) $w = \xi^n$ and $\mathbf{s} = \boldsymbol{\zeta}^n$, respectively, we obtain

$$\begin{aligned}
&\big(c_o(\tilde{\xi}^n + \tilde{\eta}^n), \xi^n\big) - \big(c_o(\tilde{\xi}^{n-1} + \tilde{\eta}^{n-1}), \xi^n\big) + \Delta t(\nabla\cdot\boldsymbol{\zeta}^n, \xi^n) \\
&\qquad + \int_{t^{n-1}}^{t^n} \big(\ell_2(\xi^n) - \ell_4(\mathbf{u}, \xi^n)\big)\, dt \\
&\quad = \Delta t\big(\ell_2(\xi^n) - \bar{\ell}_4(\mathbf{u}_H^n, \xi^n)\big) + \int_{t^{n-1}}^{t^n} (\nabla\cdot\mathbf{z}, \xi^n)\, dt \\
&\qquad - \Delta t(\nabla\cdot\mathbf{z}^n, \xi^n) + \Delta t(c_o\eta^n, \xi^n)
\end{aligned}$$

and

$$\mu_f(K^{-1}\boldsymbol{\zeta}^n, \boldsymbol{\zeta}^n) = (\xi^n, \nabla\cdot\boldsymbol{\zeta}^n). \qquad (31)$$

Using (18) we get

$$\begin{aligned}
&(c_o\tilde{\xi}^n, \xi^n) - (c_o\tilde{\xi}^{n-1}, \xi^n) + \Delta t\mu_f(K^{-1}\boldsymbol{\zeta}^n, \boldsymbol{\zeta}^n) \\
&\quad = \int_{t^{n-1}}^{t^n} (\nabla\cdot\mathbf{z} - \nabla\cdot\mathbf{z}^n, \xi^n)\, dt + \int_{t^{n-1}}^{t^n} \ell_4(\mathbf{u}, \xi^n)\, dt - \Delta t\bar{\ell}_4(\mathbf{u}_H^n, \xi^n) \\
&\qquad - (1 - \Delta t)(c_o\tilde{\eta}^n, \xi^n) + (c_o\tilde{\eta}^{n-1}, \xi^n). \qquad (32)
\end{aligned}$$

Since

$$(c_o\tilde{\xi}^{n-1}, \xi^n) \le \frac{1}{2}(c_o\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1}) + \frac{1}{2}(c_o\xi^n, \xi^n)$$

then

$$(c_o\tilde{\xi}^n, \xi^n) - (c_o\tilde{\xi}^{n-1}, \xi^n) \ge (c_o\tilde{\xi}^n, \xi^n) - \frac{1}{2}(c_o\tilde{\xi}^{n-1}, \tilde{\xi}^{n-1}) - \frac{1}{2}(c_o\xi^n, \xi^n).$$

From (14) and (18) we obtain $(c_o\tilde{\xi}^n, \xi^n) = (c_o\xi^n, \xi^n)$ and consequently,

$$(c_o\tilde{\xi}^n,\xi^n)-(c_o\tilde{\xi}^{n-1},\xi^n)\geq\frac{1}{2}(c_o\tilde{\xi}^n,\xi^n)-\frac{1}{2}(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1})$$
$$=\frac{1}{2}(c_o\tilde{\xi}^n,\tilde{\xi}^n)-\frac{1}{2}(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1})$$
$$-\frac{1}{2}\big(c_o(\tilde{\xi}^n-\xi^n),\tilde{\xi}^n\big). \qquad (33)$$

We will now analyze the right-hand side of (32). Bramble-Hilbert Lemma (e.g. [10]) implies that

$$\left\|\int_{t^{n-1}}^{t^n}\nabla\cdot\mathbf{z}-\nabla\cdot\mathbf{z}^n\,dt\right\|_0\leq C_4(\Delta t)^{3/2}\left\|\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z})_t\,dt\right\|_0,$$

where C_4 is independent of t and Δt. Hence,

$$\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z}-\nabla\cdot\mathbf{z}^n,\xi^n)\,dt\leq\frac{1}{2}\|\xi^n\|_0^2\Delta t+\frac{C_4^2}{2}\int_{t^{n-1}}^{t^n}\|(\nabla\cdot\mathbf{z})_t\|_0^2\,dt(\Delta t)^2. \qquad (34)$$

Summing and subtracting $(E_H\mathbf{u})^n$, where E_H is the elliptic projector defined by (9), we have that

$$\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u},\xi^n)\,dt-\Delta t\bar{\ell}_4(\mathbf{u}_H^n,\xi^n)=\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u}-E_H\mathbf{u},\xi^n)\,dt$$
$$+\Delta t\bar{\ell}_4\big((E_H\mathbf{u})^n-\mathbf{u}_H^n,\xi^n\big). \qquad (35)$$

Let us now consider the first term of the right-hand side of (35). For any $\varepsilon>0$, holds

$$\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u}-E_H\mathbf{u},\xi^n)\,dt\leq C_E\Delta tH\|\mathbf{u}_t\|_{L^\infty(H^2)}\|\xi^n\|_0$$
$$\leq\frac{C_E^2}{4\varepsilon}\Delta tH^4\|\mathbf{u}_t\|_{L^\infty(H^2)}^2+\varepsilon\Delta tH^{-2}\|\xi^n\|_0^2.$$

Since we have assumed that $\Delta t=c'H^2$, we obtain

$$\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u}-E_H\mathbf{u},\xi^n)\,dt\leq\frac{C_E^2}{4\varepsilon}\Delta tH^4\|\mathbf{u}_t\|_{L^\infty(H^2)}^2+\varepsilon c'\|\xi^n\|_0^2. \qquad (36)$$

For the other term, we use Lemma 3.3 and (25) to obtain the estimate

$$
\begin{aligned}
\Delta t\bar{\ell}_4(E_H\mathbf{u}-\mathbf{u}_H^n,\xi^n) &\le \frac{\alpha^2}{\lambda}(\|p^n-\tilde{p}_h^n\|_0+\|p^{n-1}-\tilde{p}_h^{n-1}\|_0)\|\xi^n\|_0\\
&\le \frac{\alpha^2}{\lambda}(\|\tilde{\eta}^n\|_0+\|\tilde{\eta}^{n-1}\|_0+\|\tilde{\xi}^n\|_0+\|\tilde{\xi}^{n-1}\|_0)\|\xi^n\|_0\\
&\le C_2(\|\mathbf{z}\|_{L^\infty(H^1)}+\|\nabla\cdot\mathbf{z}\|_{L^\infty(H^1)})h^2\|\tilde{\xi}^n\|_0\\
&\quad+\frac{3}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^n\|_0^2+\frac{1}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^{n-1}\|_0^2. \qquad (37)
\end{aligned}
$$

It remains to analyze the last two terms of the right-hand side of (32). Using Lemma 3.2 we deduce that

$$
\begin{aligned}
-(c_o\tilde{\eta}^n,\xi^n)+(c_o\tilde{\eta}^{n-1},\xi^n) &= -\int_{t^{n-1}}^{t^n}(c_o\tilde{\eta}_t,\xi^n)\,dt\\
&\le \frac{L_c^2}{2}\int_{t^{n-1}}^{t^n}\|\tilde{\eta}_t\|_0^2\,dt+\frac{1}{2}\|\xi^n\|_0^2\Delta t\\
&\le \frac{(L_cC_3)^2}{2}h^4\int_{t^{n-1}}^{t^n}\big(\|\mathbf{z}\|_1+\|\nabla\cdot\mathbf{z}\|_1+\|(\mathbf{z})_t\|_1\\
&\quad+\|\nabla\cdot\mathbf{z}_t\|_1\big)^2\,dt+\frac{1}{2}\|\xi^n\|_0^2\Delta t, \qquad (38)
\end{aligned}
$$

and

$$
\Delta t(c_o\tilde{\eta}^n,\xi^n)\le\frac{(L_cC_2)^2}{2}h^4\Delta t(\|\mathbf{z}^n\|_1+\|\nabla\cdot\mathbf{z}^n\|_1)^2+\frac{1}{2}\|\xi^n\|_0^2\Delta t. \qquad (39)
$$

Combining (32) with (33) and using (21), (34), (36), (37), (38) and (39) we obtain

$$
\begin{aligned}
&\frac{1}{2}\big((c_o\tilde{\xi}^n,\tilde{\xi}^n)-(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1})\big)+\big((K^n)^{-1}\boldsymbol{\zeta}^n,\boldsymbol{\zeta}^n\big)\left(\Delta t\mu_f-\frac{1}{2}Qh^2\right)\\
&\quad\le\left(\frac{3}{2}\|\tilde{\xi}^n\|_0^2+\frac{C_4^2}{2}\int_{t^{n-1}}^{t^n}\|(\nabla\cdot\mathbf{z})_t\|_0^2\,dt\Delta t\right)\Delta t+\frac{C_E^2}{4\varepsilon}\Delta tH^4\|\mathbf{u}_t\|^2_{L^\infty((t^{n-1},t^n),H^2)}\\
&\qquad+\varepsilon c'\|\xi^n\|_0^2+\frac{3}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^n\|_0^2+\frac{1}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^{n-1}\|_0^2\\
&\qquad+\frac{(L_cC_3)^2}{2}h^4\int_{t^{n-1}}^{t^n}\big(\|\mathbf{z}\|_1+\|\nabla\cdot\mathbf{z}\|_1+\|(\mathbf{z})_t\|_1+\|\nabla\cdot\mathbf{z}_t\|_1\big)^2\,dt\\
&\qquad+\frac{(L_cC_2)^2}{2}h^4\Delta t(\|\mathbf{z}^n\|_1+\|\nabla\cdot\mathbf{z}^n\|_1)^2. \qquad (40)
\end{aligned}
$$

Provided that Δt is sufficiently small, a summation on n, and an application of Gronwall's inequality yield to

$$\max_n \|\tilde{\xi}^n\|_0^2 \leq C\Big(\|\tilde{\xi}^0\|_0^2 + h^4 \int_0^T \|\mathbf{z}\|_1^2 + \|\nabla\cdot\mathbf{z}\|_1^2 + \|(\mathbf{z})_t\|_1^2 + \|\nabla\cdot\mathbf{z}_t\|_1^2\, dt$$
$$+ h^4(\|\mathbf{z}\|^2_{L^\infty(H^1)} + \|\nabla\cdot\mathbf{z}\|^2_{L^\infty(H^1)}) + H^4\|\mathbf{u}_t\|^2_{L^\infty(H^2)}\Big), \qquad (41)$$

for some constant C independent of h, H and Δt. □

Remark 3.5. It is interesting to observe that equation (10) remains unaltered if we replace $\tilde{p}_h^n$ by p_h^n, under the assumption that $\mathscr{E}_h$ and $\mathscr{E}_H$ coincide or that $\mathscr{E}_h$ is a refinement of $\mathscr{E}_H$. In fact, for any test function $\mathbf{v} \in \mathbf{V}_{H,0}$ we have that $\nabla\cdot\mathbf{v}$ is constant in every element $R \in \mathscr{E}_h$ and consequently $(p_h^n, \nabla\cdot\mathbf{v})_R = (\tilde{p}_h^n, \nabla\cdot\mathbf{v})_R$.

4. The L^2 estimates for deformation

The objective of this section is to derive the convergence order for the displacement approximation error with respect to the L^2-norm. The estimate we will be obtained using duality techniques.

Let $\mathbf{e}_H^n = \mathbf{u}^n - \mathbf{u}_H^n$. We will restrict our study to the case $\mathbf{e}_H \in \mathbf{V}_0$, which is satisfied for example when the Dirichlet condition for $\mathbf{u}$ on Γ_0 is homogeneous. For the general case of inhomogeneous Dirichlet data for $\mathbf{u}$ on Γ_0, the analysis required is more involving. We refer the paper [3] for some insight in this question, even though therein the study is restricted to the Laplace equation.

Consider the dual problem: find $\boldsymbol{\phi} \in \mathbf{V}_0$, such that

$$a_{\mathbf{u}}(\boldsymbol{\phi}, \mathbf{v}) = (\mathbf{e}_H^n, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}_0. \qquad (42)$$

For the derivation of L^2 error estimates we assume that the problem (42) is H^2-regular, that is, $\boldsymbol{\phi} \in H^2(\Omega)$ and

$$\|\boldsymbol{\phi}\|_{H^2(\Omega)} \leq C_{reg}\|\mathbf{e}_H^n\|_0, \qquad (43)$$

where C_{reg} is a positive constant which depends on the domain Ω. A sufficient condition for the H^2 regularity estimate (43) to hold is that the domain Ω is a convex polygonal domain in $\mathbb{R}^2$ and that (42) is a pure displacement problem ($\Gamma_0 = \partial\Omega$) [6]. Other conditions which guarantee (43) to be true are discussed for instance in [15] and [26].

In the next theorem we present the L^2-estimates.

Theorem 4.1. *Under the foregoing assumptions of this section and the same conditions as in Theorem* 3.4, *the following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^\infty(L^2)} \leq C(H^2 + h^2), \tag{44}$$

where C is independent of h, H and Δt.

Proof. Let $I_H\boldsymbol{\phi} \in \mathbf{V}_{H,0}$ be the nodal interpolation of $\boldsymbol{\phi}$. It is well known that

$$\|\boldsymbol{\phi} - I_H\boldsymbol{\phi}\|_1 \leq C_{interp} H \|\boldsymbol{\phi}\|_{H^2(\Omega)}. \tag{45}$$

Since $\mathbf{e}_H \in \mathbf{V}_0$ then,

$$\begin{aligned}\|\mathbf{e}_H^n\|_0^2 &= a_{\mathbf{u}}(\boldsymbol{\phi}, \mathbf{e}_H^n) = a_{\mathbf{u}}(\mathbf{e}_H^n, \boldsymbol{\phi} - I_H\boldsymbol{\phi}) + a_{\mathbf{u}}(\mathbf{e}_H^n, I_H\boldsymbol{\phi}) \\ &= a_{\mathbf{u}}(\mathbf{e}_H^n, \boldsymbol{\phi} - I_H\boldsymbol{\phi}) + \alpha(\nabla \cdot I_H\boldsymbol{\phi}, p^n - p_h^n) \\ &= a_{\mathbf{u}}(\mathbf{e}_H^n, \boldsymbol{\phi} - I_H\boldsymbol{\phi}) + \alpha\big(\nabla \cdot (I_H\boldsymbol{\phi} - \boldsymbol{\phi}), p^n - p_h^n\big) + \alpha(\nabla \cdot \boldsymbol{\phi}, p^n - p_h^n).\end{aligned}$$

Now, the trick is to sum and subtract the post-processed approximation for pressure. We get

$$\|\mathbf{e}_H^n\|_0^2 = a_{\mathbf{u}}(\mathbf{e}_H^n, \boldsymbol{\phi} - I_H\boldsymbol{\phi}) + \alpha\big(\nabla \cdot (I_H\boldsymbol{\phi} - \boldsymbol{\phi}), p^n - \tilde{p}_h^n\big) + \alpha(\nabla \cdot \boldsymbol{\phi}, p^n - \tilde{p}_h^n).$$

Using (4), (43) and (45), we obtain

$$\begin{aligned}\|\mathbf{e}_H^n\|_0^2 \leq\ & C_{cont} C_{interp} C_{reg} H \|\mathbf{e}_H^n\|_1 \|\mathbf{e}_H^n\|_0 + \alpha C_{cont} C_{interp} C_{reg} H \|\mathbf{e}_H^n\|_0 \|p^n - \tilde{p}_h^n\|_0 \\ & + \alpha C_{reg} \|\mathbf{e}_H^n\|_0 \|p^n - \tilde{p}_h^n\|_0,\end{aligned} \tag{46}$$

and consequently,

$$\|\mathbf{e}_H^n\|_0 \leq C(H\|\mathbf{e}_H^n\|_1 + H\|p^n - \tilde{p}_h^n\|_0 + \|p^n - \tilde{p}_h^n\|_0). \tag{47}$$

□

5. Conclusion

In this paper we have analyzed the convergence of a fully discrete numerical approximation for a coupled flow and geomechanics model. The numerical scheme combines lowest order mixed finite elements and Galerkin piecewise linear finite elements. We proposed a post-processing procedure to increase the order of convergence of the numerical approximation of pressure. Moreover, we were able to gain one order of convergence for the numerical approximation of displacement, estimating the error in the L^2-norm when compared to the error in the H^1-norm.

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