

## Grothendieck’s problem for 3-manifold groups

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*For Fritz Grunewald*

**Abstract.** The following problem was posed by Grothendieck:

*Let  $u: H \rightarrow G$  be a homomorphism of finitely presented residually finite groups for which the extension  $\hat{u}: \hat{H} \rightarrow \hat{G}$  is an isomorphism. Is  $u$  an isomorphism?*

The problem was solved in the negative by Bridson and Grunewald who produced many examples of groups  $G$  and proper subgroups  $u: H \hookrightarrow G$  for which  $\hat{u}$  is an isomorphism, but  $u$  is not.

This paper addresses Grothendieck’s problem in the context of 3-manifold groups.

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### 1. Introduction

The profinite completion of a group  $\Gamma$  is the inverse limit of the directed system of finite quotients of  $\Gamma$ , and we shall denote this profinite group by  $\hat{\Gamma}$ . As is well-known, if  $\Gamma$  is residually finite then  $\Gamma$  injects into  $\hat{\Gamma}$ . In [19] the following problem was posed by Grothendieck (where it is pointed out that it is natural to assume the groups are residually finite):

*Let  $u: H \rightarrow G$  be a homomorphism of finitely presented residually finite groups for which the extension  $\hat{u}: \hat{H} \rightarrow \hat{G}$  is an isomorphism. Is  $u$  an isomorphism?*

This problem was solved in the negative by Bridson and Grunewald in [6] who produced many examples of groups  $G$  and proper subgroups  $u: H \hookrightarrow G$  for which  $\hat{u}$  is an isomorphism, but  $u$  is not. The method of proof of [6] was a far reaching generalization of an example of Platonov and Tavgen’ [35] that produced *finitely generated* examples that answered Grothendieck’s problem in the negative (see also [5]).

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Notice that if  $\hat{H} \rightarrow \hat{G}$  is an isomorphism, then the composite homomorphism  $H \hookrightarrow \hat{H} \rightarrow \hat{G}$  is an injection. Hence,  $H \rightarrow G$  must be an injection. Therefore, Grothendieck's Problem reduces to the case where  $H$  is a subgroup of  $G$ , and the homomorphism is inclusion. Henceforth, we will only consider the situation where  $u: H \rightarrow G$  is the inclusion homomorphism. We will on occasion suppress  $u$ , however as we remark upon later, it is important for us that we do consider the case where the isomorphism  $\hat{u}$  is induced by inclusion. All abstract groups are assumed infinite and finitely generated unless otherwise stated.

We introduce the following terminology. Let  $G$  be a group and  $H < G$ . We shall call  $(G, H)$  a *Grothendieck Pair* if  $u: H \rightarrow G$  provide *negative answers* to Grothendieck's problem; i.e.,  $\hat{u}$  is an isomorphism and  $u$  is not. If for all finitely generated subgroups  $H < G$ ,  $(G, H)$  is *never* a Grothendieck Pair then we will define  $G$  to be *Grothendieck Rigid*. Thus, if  $H \rightarrow G$  is a counterexample to Grothendieck's original question, then  $G$  is *not* Grothendieck Rigid.

In [19], Grothendieck explored general conditions on groups  $H$  and  $G$  that are not Grothendieck Pairs. This theme was taken up in [29] and [35], and discussed more recently in Bridson's talk at Grunewald's 60th Birthday Conference. For example, in [35] it is shown that if  $G$  is a discrete subgroup of  $\mathrm{SL}(2, \mathbb{Q}_p)$  or  $\mathrm{SL}(2, \mathbb{R})$  then  $G$  is Grothendieck Rigid. The proof of this is a simple consequence of the fact that virtually free groups and Fuchsian groups are LERF (see §2.4 below for more on this). On the other hand, it still seems to be an open question in general as to whether arithmetic lattices in semi-simple Lie groups having the Congruence Subgroup Property are Grothendieck Rigid, even for  $\mathrm{SL}(3, \mathbb{Z})$  (see [35]).

The aim of this paper is to explore Grothendieck's problem for (certain) 3-manifold groups. We will prove various results about Grothendieck Rigidity of such groups. As we discuss further below, Grothendieck's Problem seems naturally related to other problems concerning 3-manifold groups. Note that by Perelman's solution to the Geometrization Conjecture and a theorem of Hempel [22], the fundamental groups of all compact 3-manifolds are residually finite. Our first result generalizes part of Proposition 3 of [35].

**Theorem 1.1.** *Let  $M$  be a closed 3-manifold which admits a geometric structure. Then  $\pi_1(M)$  is Grothendieck Rigid.*

We can clearly assume that  $\pi_1(M)$  is infinite in our considerations, and the proof of Theorem 1.1 quickly reduces to the case of closed hyperbolic 3-manifolds; i.e., those that admit a complete finite volume metric of constant sectional curvature  $-1$  (since manifolds with the other geometric structures are known to have LERF fundamental groups). It is worth remarking that it is still a major open problem (although there is some evidence that it is true) as to whether the fundamental groups of hyperbolic 3-manifolds are LERF. Therefore our methods do not appeal to LERF.

We can also extend our results to cusped hyperbolic 3-manifolds.

**Theorem 1.2.** *Let  $M$  be a finite volume, cusped hyperbolic 3-manifold. Then  $\pi_1(M)$  is Grothendieck Rigid.*

A corollary of Theorems 1.1 and 1.2 that generalizes the corollary to Proposition 3 of [35] is (in contrast with the discussion above regarding arithmetic lattices having the Congruence Subgroup Property):

**Corollary 1.3.** *Let  $M$  be an arithmetic hyperbolic 3-manifold. Then  $\pi_1(M)$  is Grothendieck Rigid.*

The techniques in the proofs employ a combination of classical 3-manifold topology methods, as well as Thurston's hyperbolization theorem for atoroidal Haken manifolds (see [32] and [45]), applications of the character variety and more recent developments in the field. For example, we use the solutions to the Tameness Conjecture ([1], [10]), the Ending Lamination Conjecture ([7], [8]) and The Density Conjecture (see [12] for a discussion)

In addition to hyperbolic knots, we can also prove a Grothendieck Rigidity result for other knots in  $S^3$  (see Theorem 5.1).

We finish this Introduction with a discussion of Grothendieck's Problem in the context of other problems concerning 3-manifold groups.

Our use of the term "Grothendieck Rigidity" is in part motivated by other rigidity phenomena about 3-manifolds. In our setting we are trying to distinguish a 3-manifold from its covering spaces with finitely generated fundamental group by its profinite completion. Classically, one of the basic problems about compact, irreducible 3-manifolds with infinite fundamental group (see §3 for terminology) was "Topological Rigidity"; namely, *given  $M_1$  and  $M_2$  as above, assume that  $M_1$  and  $M_2$  are homotopy equivalent, are  $M_1$  and  $M_2$  homeomorphic?* This was solved for Haken manifolds by Waldhausen [47], for Seifert fiber spaces in [40] and for hyperbolic manifolds this was solved in a series of papers ([14], [15] and [16]).

A more precise question of particular interest to us can be posed:

*Suppose that  $M_1$  and  $M_2$  are geometric 3-manifolds with infinite fundamental group for which the profinite completions  $\widehat{\pi_1(M_1)}$  and  $\widehat{\pi_1(M_2)}$  are isomorphic. Are  $M_1$  and  $M_2$  homeomorphic?*

The hypothesis of the above question can be weakened somewhat, however, there are examples like the fundamental groups of the square knot and granny knot which are isomorphic but the complements of the knots are non-homeomorphic.

Using a basic, but remarkable, result about profinite completions (see §2 for more on this) this question has a more concrete reformulation:

*Suppose that  $M$  is a geometric 3-manifold with infinite fundamental group. Within the class of such 3-manifold groups, is  $M$  determined by the set*

$$\{G : G \text{ is a finite quotient of } \pi_1(M)\}?$$

This question has arisen, at least implicitly, in a totally different direction, in recent work of Calegari, Freedman and Walker, [9]. This paper addresses the properties of a “universal pairing” on a complex vector space that arises from looking at all compact oriented 3-manifolds that a fixed closed oriented surface bounds. To that end, in [9] a complexity function is defined on a compact oriented 3-manifold  $M$ , and part of this complexity function involves listing all finite quotients of  $\pi_1(M)$ . The question as to whether the finite quotients of the fundamental group of a compact orientable irreducible 3-manifold  $M$  determine  $M$  arises naturally here (see Remark 3.7 of [9]).

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## 2. Profinite preliminaries

Here we collect some basics about profinite groups, and discuss some connections of separability to Grothendieck’s problem. For details about profinite groups see [37].

**2.1.** Suppose that  $\Gamma$  is a residually finite (abstract) group. We recall a basic but important fact relating the subgroup structures of  $\Gamma$  and  $\hat{\Gamma}$  (see [37] Chapter 3.2).

**Proposition 2.1.** *With  $\Gamma$  as above, there is a one-to-one correspondence between the set  $\mathcal{X}$  of subgroups of  $\Gamma$  that are open in the profinite topology on  $\Gamma$ , and the set  $\mathcal{Y}$  of all open subgroups of  $\hat{\Gamma}$ . This is given by (where we assume  $\Gamma$  is embedded in  $\hat{\Gamma}$ ):*

- For  $H \in \mathcal{X}$ ,  $H \mapsto \bar{H}$ , where  $\bar{H}$  denotes the closure of  $H$  in  $\hat{\Gamma}$ .
- For  $Y \in \mathcal{Y}$ ,  $Y \mapsto Y \cap \Gamma$ .

Moreover,  $[\Gamma : H] = [\hat{\Gamma} : \bar{H}]$ .

Suppose now that  $\Gamma_1$  and  $\Gamma_2$  are finitely generated abstract groups such that  $\hat{\Gamma}_1$  and  $\hat{\Gamma}_2$  are isomorphic. Note that by “isomorphic” mean “isomorphic as groups”, since every isomorphism between the profinite completions is continuous (we do not assume that there is a homomorphism  $\Gamma_1 \rightarrow \Gamma_2$ ). It is easy to deduce from Proposition 2.1 that this implies that  $\Gamma_1$  and  $\Gamma_2$  have the same collection of finite quotient groups. In fact, the discussion on pp. 88–89 of [37] gives a stronger statement that we record for convenience.

**Theorem 2.2.** *Suppose that  $\Gamma_1$  and  $\Gamma_2$  are finitely generated abstract groups. For  $i = 1, 2$ , let  $\mathcal{C}_i = \{G : G \text{ is a finite quotient of } \Gamma_i\}$ . Then  $\hat{\Gamma}_1$  and  $\hat{\Gamma}_2$  are isomorphic if and only if  $\mathcal{C}_1 = \mathcal{C}_2$ .*

An easy corollary of this that we will appeal to repeatedly is recorded below. We will use the notation:  $b_1(\Gamma) = \text{rank}(\Gamma/[\Gamma, \Gamma]) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and if  $X$  is a manifold,  $b_1(X) = b_1(\pi_1(X))$ .

**Corollary 2.3.** *Suppose that  $\Gamma_1$  and  $\Gamma_2$  are finitely generated abstract groups for which  $\widehat{\Gamma}_1$  and  $\widehat{\Gamma}_2$  are isomorphic. Then  $b_1(\Gamma_1) = b_2(\Gamma_2)$ .*

*Proof.* If not, then one easily constructs a finite abelian quotient of one that cannot be a finite quotient of the other, and this contradicts Theorem 2.2. □

Now suppose that  $u: \Gamma_1 \rightarrow \Gamma_2$  is a homomorphism of finitely generated (residually finite abstract) groups. This determines a continuous homomorphism  $\hat{u}: \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$ . Proposition 2.1 can be applied to show that (see [37] Lemma 3.2.6 for (2), and also [35]):

- (1)  $\hat{u}$  is surjective if and only if  $u(\Gamma_1)$  is dense in the profinite topology on  $\Gamma_2$ .
- (2) Suppose that  $u$  is the inclusion homomorphism. Then  $\hat{u}$  is injective if and only if the profinite topology of  $\Gamma_2$  induces on  $\Gamma_1$  (or  $u(\Gamma_1)$ ) its profinite topology.

With this discussion, we prove:

**Lemma 2.4.** *Let  $\Gamma$  be a finitely generated (abstract) group, let  $\Delta < \Gamma$  be a subgroup of finite index, and  $H < \Gamma$ . If  $(\Gamma, H)$  is a Grothendieck Pair, then the inclusion map  $\Delta \cap H \hookrightarrow \Delta$  induces an isomorphism of profinite completions.*

*Proof.* Let  $\mathcal{T}$  denote the profinite topology on  $\Gamma$ . As above, we let  $u: H \rightarrow \Gamma$  denote the inclusion homomorphism with  $\hat{u}: \widehat{H} \hookrightarrow \widehat{\Gamma}$  an isomorphism. Let  $u'$  denote the restriction of  $u$  to  $H' = H \cap \Delta$ . Since  $\Delta$  has finite index in  $\Gamma$ ,  $\Delta$  is an open subgroup in  $\mathcal{T}$ . By (1) above,  $u(H)$  is dense in  $\mathcal{T}$ . It now follows from elementary point set topology that  $u'(H')$  is dense in the profinite topology on  $\Delta$ , since this is the subspace topology induced from  $\mathcal{T}$ .

To prove injectivity, note first that by (2) above, the profinite topology on  $H$  (or  $u(H)$ ) coincides with that induced by  $\mathcal{T}$ . Since  $H'$  has finite index in  $H$ , and  $\Delta$  has finite index in  $\Gamma$  it follows from the properties of the subspace topology that the profinite topology on  $u'(H')$  coincides with that induced by  $\Delta$ . Now (2) proves that  $\hat{u}'$  is injective. □

**Remark.** In our applications of Lemma 2.4 (see Corollary 4.5), the groups we are working with will have the property that  $\Delta \cap H$  is not isomorphic to  $\Delta$ , thereby showing that  $(\Delta, \Delta \cap H)$  is a Grothendieck Pair.

**2.2.** Let  $G$  be a group and  $H < G$ . Then  $G$  is called  $H$ -separable if given any  $g \in G \setminus H$ , there is a finite index subgroup  $N$  of  $G$  such that  $H < N$  but  $g \notin N$ . Equivalently,  $G$  is  $H$ -separable if  $H$  is closed in the profinite topology on  $G$ . The group  $G$  is called *LERF* or *subgroup separable* if  $G$  is  $H$ -separable for every

finitely generated subgroup  $H$  of  $G$ . LERF has attracted considerable interest recently through its connections with problems in 3-manifold topology (see for example [2], [20] and [28]).

A weaker property than  $H$ -separability is *engulfing*, which is also related to problems in 3-manifold topology (see [26]). In the notation above, suppose that  $H$  is a proper subgroup of  $G$ . We say that  $H$  is engulfed if  $H < K$  where  $K$  is a proper subgroup of  $G$  of finite index.

The following lemma illustrates the connection between the separability properties mentioned above, and Grothendieck’s problem (see also [35], pp. 90–91). Again, we are implicitly assuming that if  $H < \Gamma$ , then  $u : H \hookrightarrow \Gamma$  is the natural inclusion map.

**Lemma 2.5.** *Suppose  $\Gamma$  is a finitely generated abstract group and  $H < \Gamma$ . Assume that  $H$  is engulfed. Then  $(\Gamma, H)$  is not a Grothendieck Pair.*

*Proof.* We are assuming that  $u : H \hookrightarrow \Gamma$  induces the isomorphism  $\hat{u}$ . In particular,  $\hat{u}$  is onto, and so from (1) of §2.1,  $u(H)$  is dense in the profinite topology on  $\Gamma$ . On the other hand, if  $H$  is engulfed,  $H$  is contained in a proper subgroup of finite index, and so  $u(H)$  cannot be dense. Hence we have a contradiction.  $\square$

An immediate corollary of Lemma 2.5 is the following statement that was made in the Introduction.

**Corollary 2.6.** *Suppose that  $\Gamma$  is LERF, then  $\Gamma$  is Grothendieck Rigid.*

**2.3.** We conclude this section with a discussion of how character varieties of Grothendieck Pairs are related. We will restrict attention to  $SL(2, \mathbb{C})$  and  $PSL(2, \mathbb{C})$  character varieties. For a finitely generated (abstract) group  $G$ , we will denote by  $R(G)$  and  $X(G)$  the  $SL(2, \mathbb{C})$  representation and character varieties of  $G$ . If  $V$  is an algebraic set, we will let  $\dim(V)$  denote the maximal dimension of an irreducible component of  $V$ .

Suppose that  $H$  is a finitely generated subgroup of  $G$ . Then the inclusion mapping  $u$  induces maps

$$\tilde{u} : R(G) \rightarrow R(H) \quad \text{and} \quad u^* : X(G) \rightarrow X(H)$$

by

$$\tilde{u}(\rho) = \rho|_H \quad \text{and} \quad u^*(\chi_\rho) = \chi_{\rho|_H}.$$

It is easy to check that these maps are algebraic maps.

The following result is a variation of Proposition 4 of [35] in terms of character varieties.

**Proposition 2.7.** *Let  $(G_1, G_2)$  be a Grothendieck Pair. Let  $\mathcal{X}_i$  ( $i = 1, 2$ ) denote the collection of components of  $X(G_i)$  each of which contains the character of an irreducible representation. Then  $u^*$  determines a bijection from  $\mathcal{X}_1$  to  $\mathcal{X}_2$ . Furthermore,  $u^*$  is a birational isomorphism on each  $X \in \mathcal{X}_1$ .*

*Proof.* Firstly, we note that  $u^*$  maps  $\mathcal{X}_1$  to  $\mathcal{X}_2$ . The reason is this. Pick some irreducible representation  $\rho$  of  $G_1$ , and suppose its restriction to  $G_2$  were reducible. In our setting, this means it is conjugate to a group of upper triangular matrices and therefore soluble. Such a subgroup is verbal for the word  $[[a, b], [c, d]]$  and therefore separable in  $\rho(G_1)$  (see [25] for example). In particular, it can be engulfed in the image, so that  $G_2$  is engulfed in  $G_1$ , a contradiction.

We next claim that  $u^*$  is onto a Zariski dense subset of  $\mathcal{X}_2$ : Let  $X \in \mathcal{X}_2$ , and let  $\chi_\rho$  be the character of an irreducible representation  $\rho: G_2 \rightarrow \text{SL}(2, \mathbb{C})$ . Since  $(G_1, G_2)$  is Grothendieck Pair, it follows from [19] (see also [35]), that there exists a representation  $\rho_1: G_1 \rightarrow \text{SL}(2, \mathbb{C})$  such that  $\rho = \rho_1 \circ u$ ; i.e.,  $\rho_1$  restricted to  $G_2$  is  $\rho$ . Since  $X$  contains the character of an irreducible representation, the generic character of  $X$  is the character of an irreducible representation (see [13]) and the claim follows.

Finally, we show that  $u^*$  is injective. Suppose that  $\chi_\rho, \chi_\phi \in \mathcal{X}_1$  are characters of irreducible representations  $\rho$  and  $\phi$  (as above the generic character is the character of an irreducible representation) with  $u^*(\chi_\rho) = u^*(\chi_\phi)$ . Let  $\rho(G_1) = G$  and  $\phi(G_1) = G'$ .

Since  $u^*(\chi_\rho) = u^*(\chi_\phi)$ , and the restrictions to  $G_2$  determine the same *irreducible* character, we can conjugate  $G$  and  $G'$  so that they agree on the image,  $H$  say, of  $G_2$  (see [13] Proposition 1.5.2). Following the idea in [25], we can construct a homomorphism

$$\Phi: G_1 \rightarrow G \times G'$$

by setting

$$\Phi(g) = (\rho(g), \phi(g)).$$

By construction  $\Phi(G_2)$  maps to the diagonal subgroup  $H \times H$ , whereas  $\Phi(G_1)$  does not map diagonally, since the representations  $\rho$  and  $\phi$  are distinct. Since  $G$  and  $G'$  are finitely generated linear groups, they admit many non-trivial finite quotients, and so it follows that  $G_2$  can be engulfed. This contradicts Lemma 2.5. □

**Remark.** Proposition 2.7 can be formulated in exactly the same way for the  $\text{PSL}(2, \mathbb{C})$  character variety (see [24] for more on this character variety). We will use the notation  $Y(G)$  for the  $\text{PSL}(2, \mathbb{C})$  character variety.

### 3. 3-manifold preliminaries

We include some background from the topology of 3-manifolds that will be useful in the sequel.

**3.1.** Recall that a compact orientable 3-manifold  $M$  is called *irreducible* if every embedded 2-sphere bounds a 3-ball. A properly embedded orientable surface  $S \neq S^2, D^2$  in a compact orientable 3-manifold  $M$  is called *incompressible* if

$\ker(\pi_1(S) \rightarrow \pi_1(M)) = 1$ . The surface  $S$  is *essential* if it is in addition non-peripheral; that is to say  $\pi_1(S)$  is not conjugate into the fundamental group of a boundary component of  $M$ .  $M$  is called *atoroidal* if every  $\pi_1$ -injective map of a torus into  $M$  is homotopic into  $\partial M$ . A compact, orientable, irreducible 3-manifold  $M$  is called *Haken* if it contains an incompressible surface.

**Remark.** We are assuming that our incompressible surfaces are embedded, which is somewhat non-standard these days.

**3.2.** The following theorem of Jaco–Shalen and Johannson provides a canonical decomposition for compact orientable irreducible 3-manifolds.

**Theorem 3.1.** *Let  $M$  be a compact, orientable, irreducible 3-manifold. There exists a finite collection  $\mathcal{T}$  of disjoint incompressible tori such that each component of  $M \setminus \mathcal{T}$  is either a Seifert fibered space or is atoroidal. A minimal such collection  $T$  is unique up to isotopy.*

We will be interested only in the case when  $\partial M$  is a single incompressible torus and assume this in the following discussion. In this case, this minimal decomposition of  $M$  is called the JSJ decomposition of  $M$ . The collection  $\mathcal{T}$  in this case will be called the JSJ tori. It is easy to see that naturally attached to the JSJ decomposition is a dual graph with edges the JSJ tori and vertices the connected components of  $M \setminus \mathcal{T}$ . We will denote the submanifold determined by the vertex  $v$  by  $M_v$ . This in turn leads to a graph of group decomposition of  $\pi_1(M)$  that we also refer to as the JSJ decomposition for  $\pi_1(M)$ .

By Thurston’s hyperbolization theorem ([32] and [45]) the atoroidal pieces are typically hyperbolic (i.e., admit a complete finite volume hyperbolic structure), the only exceptions being  $S^1 \times S^1 \times [0, 1]$  and the twisted I-bundle over the Klein bottle (note that  $S^1 \times D^2$  does not arise in this situation). We will refer to a  $S^1 \times S^1 \times [0, 1]$  piece as a collar.

We now recall some of the terminology and results of [18] that are important for us. We will do this only for manifolds with a single torus boundary component and have no atoroidal piece that is a twisted I-bundle over the Klein bottle.

**Notation.** Let  $M$  be a compact orientable irreducible 3-manifolds with  $\partial M$  a single incompressible torus. Let  $\mathcal{T}$  be the collection of JSJ tori.

$\Sigma$  denotes the characteristic submanifold of  $M$ . For convenience we spell out the construction.

Suppose that  $T \in \mathcal{T}$ . If  $T$  lies in no Seifert piece add to  $\mathcal{T}$  a parallel copy of  $T$ . If  $T$  lies between two Seifert fibered pieces, remove the interior of a regular neighbourhood of  $T$  from  $M$ . Split  $M$  along the new  $\mathcal{T}$ , and let  $\Sigma'$  denote the Seifert fibered pieces obtained. If the boundary of  $M$  does not lie in a Seifert fibered piece add to  $\Sigma'$  the closure of a collared neighbourhood of  $\partial M$ . The collection of Seifert fibered pieces so obtained is  $\Sigma$ .

Let  $\beta M$  denote the piece of  $\Sigma$  that contains  $\partial M$ .

Let  $\gamma M$  denote the geometric piece adjacent to  $\partial M$ , together with a collar on  $\partial M$ .

**Remark.** Since  $M$  has one boundary component, if the unique piece of the JSJ decomposition that contains the boundary is hyperbolic then  $\beta M$  is a collar. In this case  $\gamma M$  can be identified with  $M_v$  where  $M_v$  is the vertex manifold containing the boundary of  $M$ .

Let  $M_1$  and  $M_2$  be compact, orientable, irreducible 3-manifolds with  $\partial M_i$  ( $i = 1, 2$ ) being a single incompressible torus. An essential map  $g: M_1 \rightarrow M_2$  (i.e.,  $g_*$  is injective on fundamental groups) is *loose* if it is homotopic to a map  $f: M_1 \rightarrow M_2$  for which  $f(M_1) \cap \gamma M_2 = \emptyset$ . Otherwise  $g$  is *tight*.

A subgroup  $H$  of  $\pi_1(M)$  is *loose* if for some component  $C$  of  $M \setminus \gamma M$ , there is a conjugate of  $H$  in  $i_*(\pi_1(C))$ , where  $i: C \rightarrow M$  is the inclusion map. Otherwise  $H$  is called *tight*.

The two notions of tight just described are consistent, since it is shown in [18], Proposition 6.5, that if  $M_1$  and  $M_2$  are as above and  $f: M_1 \rightarrow M_2$  is an essential map, then  $f$  is tight if and only if  $f_*(\pi_1(M_1))$  is a tight subgroup in  $\pi_1(M_2)$ .

We record the following proposition (which is a corollary of results in [18]).

**Proposition 3.2.** *Let  $M_1$  and  $M_2$  be Haken manifolds whose boundaries both consist of a single incompressible torus, and assume that  $\beta M_2$  is a collar. Assume that  $f: M_1 \rightarrow M_2$  is a tight essential map. Then  $\beta M_1$  is a collar and  $f|_{\beta M_1}: \beta M_1 \rightarrow \beta M_2$  is a covering map.*

*Proof.* Since  $f$  is a tight essential map, we can homotope  $f$  so that it satisfies the conclusions of Theorem 6.1 of [18]. Consider  $f(\beta M_1)$ . By properties of tight essential maps, this is a component  $Q$  of the JSJ decomposition of  $M_2$  (see the Deformation Theorem 1.2 and Theorem 6.1 of [18]). Indeed, tightness shows that  $Q$  must coincide with that piece of the JSJ decomposition of  $M_2$  containing  $\partial M_2$ , and this is a collar by assumption. It now follows that  $\beta M$  is a collar, and the map  $f$  is a covering map.  $\square$

**3.3.** We will be particularly interested in the discussion of the previous section when the manifold  $M_2$  is the exterior of a knot  $K \subset S^3$ ; that is the closure of the complement of a small open tubular neighbourhood of  $K$ . We will denote the exterior of a knot  $K$  by  $E(K)$ . Recall that a knot  $K$  is prime if it is not the connect sum of non-trivial knots.

The JSJ graph associated to  $E(K)$  is a rooted tree, where the root vertex  $v_0$  corresponds to the unique vertex manifold containing  $\partial E(K)$ . In the notation §3.2, this is simply the manifold  $\gamma M$ .

The Seifert fibered pieces that can arise in the JSJ decomposition of  $E(K)$  are also well understood (see [23], Lemma VI 3.4). These are the following:

- *torus knot exterior*;
- *a cable space*: i.e., a manifold obtained from  $D^2 \times S^1$  by removing an open regular neighbourhood in  $\text{Int}(D^2) \times S^1$  of an essential simple closed curve  $c$  which lies in a torus  $J \times S^1$ , where  $J$  is a simple closed curve in  $\text{Int}(D^2) \times S^1$  and  $c$  is non-contractible in  $J \times S^1$ ;
- *an ( $n$ -fold) composing space*: i.e., a compact 3-manifold homeomorphic to  $W \times S^1$ , where  $W$  is an  $n$ -times punctured disc.

In addition, since a Klein bottle does not embed in  $S^3$ , the only exceptional atoroidal piece is  $S^1 \times S^1 \times [0, 1]$ . If  $\gamma M$  is a cable space, then  $K$  is a cable knot, and when  $\gamma M$  is a composing space,  $K$  is a composite knot; that is a non-prime knot (see [23]).

A corollary of this discussion and that contained in §3.1 that will be useful to record is the following.

**Corollary 3.3.** *Suppose that  $K$  is a prime satellite knot that is not a cable knot. Then  $\beta E(K)$  is a collar and  $\gamma E(K)$  is the piece of the JSJ graph containing the root vertex.*

*Proof.* Only the statement about  $\beta E(K)$  needs any comment. Since  $K$  is a satellite knot it is not a torus knot. In addition, since  $K$  is a prime knot that is not a cable knot, it follows from above that  $\gamma E(K)$  is hyperbolic. The statement about  $\beta E(K)$  now follows.  $\square$

**3.4.** A standard property about compact 3-manifolds with non-empty boundary is the so-called “half-lives, half-dies” which is a consequence of Poincaré–Lefschetz duality.

**Theorem 3.4.** *Let  $M$  be a compact orientable 3-manifold with non-empty boundary. Then the rank of*

$$\ker(H_1(\partial M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}))$$

*is  $b_1(\partial M)/2$ .*

A corollary of this that will be useful for us is the following.

**Corollary 3.5.** *Let  $M$  be a compact orientable irreducible 3-manifold with a single torus boundary component,  $H < \pi_1(M)$  a finitely generated non-abelian subgroup and let  $X_H$  denote the cover of  $M$  corresponding to  $H$ . Assume that  $H_1(X_H; \mathbb{Q}) \cong \mathbb{Q}$ . Then  $X_H$  is homotopy equivalent to  $\Sigma \setminus K$ , a knot complement in a closed orientable 3-manifold  $\Sigma$ .*

*Proof.* Since  $H$  is finitely generated, [38] guarantees the existence of a compact core  $C_H$  for  $X_H$ ; that is a compact co-dimension zero submanifold  $C_H$  of  $X_H$  such that the inclusion mapping  $C_H \hookrightarrow X_H$  induces a homotopy equivalence. In particular,  $\pi_1(C_H) \cong H$ .

In addition since  $M$  is irreducible,  $C_H$  is irreducible and so we may assume that there are no 2-sphere boundary components in  $\partial C_H$ . We are assuming that  $b_1(X_H) = 1$ , and so  $C_H$  is a compact manifold with non-empty boundary and  $b_1(C_H) = 1$ . It follows from Theorem 3.4 that  $\partial C_H$  is therefore a torus.

Since  $H$  is non-abelian,  $C_H$  is not a solid torus, and by irreducibility  $\partial C_H$  must be incompressible. Thus  $C_H$  is a compact, orientable, 3-manifold with incompressible torus boundary. This proves the corollary.  $\square$

#### 4. Proofs

Before commencing with the proofs of Theorems 1.1 and 1.2, we make some preliminary comments and prove a proposition and a lemma which require some additional notation.

**4.1.** Let  $W$  be a compact orientable 3-manifold that is hyperbolizable – so that the interior of  $W$  admits at least one complete hyperbolic structure. If this is of finite volume, then this is the unique hyperbolic structure.

We will let  $Y(W)$  denote the character variety  $Y(\pi_1(W))$ . If  $\partial W$  is empty or consists of a disjoint union of  $n_T$  incompressible tori, then Thurston proved [44] (see also [33]) that  $Y(W)$  contains a so-called *canonical component* (denoted by  $X_0$ ) that contains the character of the faithful discrete representation. In the former case, a well-known consequence of Mostow–Weil Rigidity is that  $X_0$  is a single point. In the latter case, Thurston [44] proved that  $X_0$  has complex dimension  $n_T$ .

In the following discussion, we will assume that  $\partial W$  contains a non-toroidal component. In which case the interior of  $W$  admits many hyperbolic structures. Denote by  $\text{AH}(W)$  the subset of  $Y(W)$  consisting of all the characters of discrete faithful representations. In addition,  $\text{AH}_T(W)$  will denote the subset of characters of *minimally parabolic representations*, i.e., those  $\chi_\rho \in \text{AH}_T(W)$  satisfying the condition that, for  $g \in \pi_1(W)$ ,  $\rho(g)$  is parabolic if and only if  $g$  lies in a rank 2 abelian subgroup. We will also call the group  $\rho(\pi_1(W))$  minimally parabolic.

It will be convenient for our purposes to record some facts about  $\text{AH}(W)$ . A convenient reference for the first result stated below is [24], Theorem 8.44. Recall that a Kleinian group  $\Gamma$  (or the quotient  $\mathbb{H}^3/\Gamma$ ) is called *geometrically finite* if  $\Gamma$  admits a finite sided convex fundamental polyhedron. Otherwise,  $\Gamma$  (or  $\mathbb{H}^3/\Gamma$ ) is called geometrically infinite.

**Theorem 4.1.** *Let  $W$  be as above and  $\chi_\rho$  a minimally parabolic geometrically finite representation.*

*Then  $\chi_\rho$  is a smooth point of  $Y(W)$  and the complex dimension of a component of  $Y(W)$  containing  $\chi_\rho$  is  $-3\chi(\partial W)/2 + n_T = -3\chi(W) + n_T$ .*

Understanding the detailed structure of  $\text{AH}(W)$  (resp.  $\text{AH}_T(W)$ ) has been one of the main goals in the deformation theory of Kleinian groups. For example, combining work of Marden [31] and Sullivan [42] shows that the subset  $\text{GAH}_T \subset \text{AH}_T(W)$  consisting of characters of geometrically finite representations is the interior of  $\text{AH}(W)$ . The Density Conjecture asserts that  $\text{AH}(W)$  is the closure of  $\text{GAH}_T(W)$ . This has been recently established as a consequence of the solutions to the Tameness and Ending Lamination Conjectures ([1], [10], [7] and [8]), as well as many other developments (see the survey papers and [11] and [12] for more on this).

**Theorem 4.2** (Density Theorem). *With  $W$  as above, then  $\text{AH}(W)$  is the closure of its interior.*

The topology of  $\text{AH}_T(W)$  has also been investigated in some detail. It is known that  $\text{AH}_T(W)$  can be highly disconnected ([3], [4]) and it remains an open problem to determine if these components can be contained in more than one irreducible component of  $Y(W)$  (see [12] §10). However, for our purposes all we require is the following.

**Proposition 4.3.** *Let  $W$  be as above. Let  $V$  be an irreducible component of  $Y(W)$  such that  $V \cap \text{AH}(W) \neq \emptyset$ . Then  $\dim(V) = -3\chi(\partial W)/2 + n_T = -3\chi(W) + n_T$ .*

*Proof.* By Theorem 4.2 (the Density Theorem) and the discussion preceding it, every component of  $\text{AH}(W)$  contains a component of  $\text{GAH}_T(W)$ . Thus if  $V$  is a component of  $Y(W)$  such that  $V \cap \text{AH}(W) \neq \emptyset$ , we can assume that  $V$  contains a component  $U \subset \text{GAH}_T(W)$ . By Theorem 4.1,  $\text{GAH}_T(W)$  is contained in the set of smooth points of  $Y(W)$ . Thus the dimension of  $V$  is now given by Theorem 4.1.  $\square$

Two special cases that are worth pointing out are the cases when  $W = S \times I$  where  $S$  is either a closed orientable surface of genus  $\geq 2$  or an orientable punctured surface  $\neq$  disc or annulus. In the former case Goldman [17] established that  $Y(W)$  has two irreducible components of dimension  $6g - 6$ , and the latter case is that of a free group, and in this case,  $Y(W)$  is affine space of dimension  $3r - 3$  where  $r$  is the rank of the free group.

Before commencing with the proofs, we prove a general lemma that will be useful in the setting of hyperbolic 3-manifolds.

**Lemma 4.4.** *Let  $M = \mathbb{H}^3/\Gamma$  be a finite volume orientable hyperbolic 3-manifold, and  $H < \Gamma$  a finitely generated subgroup such that  $(\Gamma, H)$  is a Grothendieck Pair. Then  $H$  must be geometrically finite.*

*Proof.* Suppose that  $H$  were geometrically infinite. It follows from the solution to the Tameness Conjecture ([1], [10]) and work of Canary (see [11]) that  $H$  is isomorphic to the fundamental group of either a closed surface or a punctured surface. Indeed, in either case, the surface in question is the fibre in a fibration over the circle (or

a bundle over a mirrored interval in the non-orientable case) of some finite cover of  $M$ . However, in both cases the surface group is separable and so we can apply Lemma 2.5.

This completes the proof that  $H$  is geometrically finite.  $\square$

**4.2. Proof of Theorem 1.1.** Since  $\pi_1(M)$  can be assumed to be infinite,  $M$  admits a geometric structure modelled on  $\mathbb{H}^3$ , SOLV,  $\mathbb{E}^3$ ,  $S^2 \times \mathbb{R}$ ,  $\mathbb{H}^2 \times \mathbb{R}$ ,  $\widetilde{\text{PSL}}_2$ , or NIL (see [46]). The last five of these are Seifert fibered geometries and so [39] implies that  $\Gamma$  is LERF. Hence Corollary 2.6 applies. Manifolds in SOLV have virtually solvable fundamental groups, and these are also known to be LERF; since for example the fundamental group of any torus bundle with SOLV geometry is a subgroup of  $\text{SL}(3, \mathbb{Z})$  and Theorem 2 of [48] applies. Thus it remains to deal with the case of hyperbolic 3-manifolds.

It will be convenient to state the following corollary of Lemma 2.4 in the hyperbolic setting that we will make use of on several occasions below. Recall that a group is called *cohopfian* if it does not inject as a proper subgroup of itself.

**Corollary 4.5.** *Let  $M = \mathbb{H}^3/\Gamma$  be a finite volume hyperbolic 3-manifold,  $\Delta < \Gamma$  a subgroup of finite index and  $H < \Gamma$ . If  $(\Gamma, H)$  is a Grothendieck Pair, then  $(\Delta, \Delta \cap H)$  is a Grothendieck Pair.*

*Proof.* From Lemma 2.4 and the remark that follows it, we simply need to note that it is known that the fundamental group of a finite volume hyperbolic 3-manifold is cohopfian (see for example [36]).  $\square$

Returning to the proof of Theorem 1.1, let  $M = \mathbb{H}^3/\Gamma$  and  $u: H \hookrightarrow \Gamma$  determine a Grothendieck Pair. Note that by Corollary 4.5, it suffices to deal with the case that  $M$  is orientable, and we assume this henceforth.

Let  $X_H$  (resp.  $C_H$ ) denote the cover of  $M$  corresponding to  $H$  (resp. denote a compact core  $C_H$  for  $X_H$ ). Thus  $C_H$  is a compact manifold with non-empty boundary ( $H$  must have infinite index in  $\Gamma$ ). This boundary may or may not be incompressible.

By Lemma 4.4,  $H$  is geometrically finite, and since  $M$  is closed,  $\Gamma$  has no parabolic elements. Hence  $H$  is minimally parabolic. Proposition 2.7 implies that  $u^*(X_0)$  is a component of  $Y(H)$ . Furthermore, this component contains the character  $u^*(\chi_0)$  which by definition is the character of a minimally parabolic representation onto  $H$ . Since the core corresponding to  $H$  has non-empty boundary, Proposition 4.3 applies to give a contradiction.  $\square$

When  $M$  is not geometric, one can still prove some results.

**Theorem 4.6.** *Suppose that  $M$  is a closed 3-manifold that is an irreducible rational homology 3-sphere. Then  $\pi_1(M)$  is Grothendieck Rigid.*

*Proof.* Since  $M$  is a rational homology 3-sphere it is orientable. The theorem easily follows from Theorem 3.4 and the observation that any finitely generated infinite index subgroup  $H$  of  $\pi_1(M)$  has infinite abelianization since after capping off any 2-spheres, a compact core  $C_H$  of  $H$  must have non-empty boundary.  $\square$

**4.3. Proof of Theorem 1.2.** As above, we will assume that  $M$  is orientable, and also as above, we let  $\Gamma = \pi_1(M)$ , with  $H < \Gamma$  for which  $(\Gamma, H)$  is a Grothendieck Pair.

To illustrate some of the ideas in the proof, we first give the proof in a special case, that of a 1-cusped manifold.

*$M$  is 1-cusped.*

The case of  $b_1(M) = 1$  can be handled without resort to the character variety and we give this argument first. As in the proof of Theorem 1.1, we consider the compact core  $C_H$ . Note that by Corollary 2.3, the assumption that  $\hat{H} \cong \hat{\Gamma}$ , implies that  $b_1(C_H) = 1$ . Hence we deduce from Corollary 3.5 that  $C_H$  is homotopy equivalent to a compact 3-manifold with non-empty incompressible torus boundary. Furthermore  $C_H$  is irreducible since  $X_H$  is a quotient of  $\mathbb{H}^3$ . Hence  $C_H$  is Haken.

Since  $H < \Gamma$ , it follows that  $C_H$  satisfies the criteria required for Thurston's hyperbolization theorem for Haken manifolds (see [32] and [45]). That is to say,  $C_H \subset \mathbb{H}^3/H$  with  $\mathbb{H}^3/H$  having finite volume. This is a contradiction, since  $X_H$  is then a finite cover of  $M$ .

We now deal with the general 1-cusped case. Note that in this case  $X_0$  has dimension 1. With notation as above, some component of  $\partial C_H$  is not an incompressible torus, for otherwise, as in the previous paragraph,  $\mathbb{H}^3/H$  has finite volume which is a contradiction.

Thus, there is some component  $S$  of  $\partial C_H$  of genus at least two. As in the closed case, Proposition 2.7 implies that  $u^*(X_0)$  is a component of  $Y(H)$ . Furthermore, this component contains the character  $u^*(\chi_0)$  which by definition is the character of the discrete faithful representation of  $H$ . Now by Lemma 4.4,  $H$  is geometrically finite, and if it is also minimally parabolic then we can argue as in the closed case and apply Proposition 4.3. If  $H$  is not minimally parabolic, we simply note that nearby to  $H$  in the component containing  $u^*(\chi_0)$ , there are geometrically finite minimally parabolic representations and the dimension argument can still therefore be made.

*$M$  has  $m \geq 2$  cusps.*

As above, we let  $H < \Gamma$  with  $(\Gamma, H)$  a Grothendieck Pair. Again, as before, the compact core  $C_H$  is a compact manifold with non-empty boundary. As in the 1-cusped case,  $H$  is geometrically finite by Lemma 4.4.

Notice that any torus in  $\partial C_H$  must be incompressible, for if it is not, then the Loop Theorem gives a compression and hence an embedded 2-sphere. By irreducibility, this sphere bounds a ball in the manifold  $\mathbb{H}^3/H$  and this ball cannot contain the non-compact end which abuts the torus in question. It follows that  $\partial C_H$  is a solid torus, which is impossible.

It follows that if  $\partial C_H$  consists of  $m$  tori, then these are all incompressible and we can then argue exactly as in the case of one cusp.

Thus we can assume that some component of  $\partial C_H$  has genus at least two, and so  $\chi(C_H) < 0$ . We will assume that  $C_H$  has  $n_T < m$  (possibly zero) torus cusps. As in the argument for the Theorem 1.1 and the case of one cusp given above, the character  $\chi_0$  of the faithful discrete representation of  $\pi_1(M)$  determines a character  $u^*(\chi_0)$  of  $H$ , and furthermore  $u^*(X_0) = V$ , a component of  $Y(C_H)$  with  $\dim(X_0) = m = \dim(V)$ . Note that, as in the argument for the 1-cusped case if  $H$  is not minimally parabolic, nearby to  $u^*(\chi_0)$  in  $V$ , there are geometrically finite minimally parabolic representations and the dimension argument can still therefore be made. We therefore deduce from Proposition 4.3 that  $\dim(V) = -3\chi(C_H) + n_T$ , and so  $m = -3\chi(C_H) + n_T$ .

The idea now is to exhibit a finite index subgroup  $\Delta < \Gamma$  such that the dimension of the canonical component of the manifold  $\mathbb{H}^3/\Delta$  is strictly less than the dimension of the component containing the character associated to the faithful discrete representation given by  $H \cap \Delta$ . In particular,  $(\Delta, H \cap \Delta)$  is not a Grothendieck Pair which contradicts Corollary 4.5. This will complete the proof.

To that end, [27] provides an infinite collection of rational primes  $p > 3$  so that reduction homomorphisms surject both  $\Gamma$  and  $H$  onto the finite simple groups  $\text{PSL}(2, p)$ . Let  $N_p$  denote the kernel of the reduction epimorphism  $\Gamma \rightarrow \text{PSL}(2, p)$ , and  $M_p$  the finite cover of  $M$  determined by  $N_p$ . We let  $C_p$  denote the cover of the compact core  $C_H$  determined by  $H \cap N_p$ .

Now we can, by discarding a further finite number of  $p$ , assume that for each of the primes  $p$ , the image under these reduction homomorphisms of any of the  $m$  peripheral subgroups of  $\Gamma$ , and any of the  $n_T$  toroidal peripheral subgroups of  $H$  is non-trivial. In particular, using the structure of subgroups of  $\text{PSL}(2, p)$  generated by unipotent elements (see [43], Chapter 3, §6 for example), it follows that any of the aforementioned peripheral subgroups is mapped to a cyclic group of order  $p$ . The order of  $\text{PSL}(2, p)$  is  $p(p^2 - 1)/2$ , and so we deduce that  $M_p$  has  $m(p^2 - 1)/2$  cusps.

In addition,  $H$  surjects onto  $\text{PSL}(2, p)$ , and so by multiplicativity of Euler characteristic, we deduce that the Euler characteristic of  $C_p$  is  $\chi(C_H)p(p^2 - 1)/2$ . Furthermore, we also arranged that  $C_p$  has  $n_T(p^2 - 1)/2$  torus cusps. We now compare the dimensions of the canonical component of  $M_p$  with the dimension of the component  $V_p$  of  $X(C_p)$  containing the character associated to  $H \cap N_p$ .

From above,  $M_p$  has  $m(p^2 - 1)/2$  cusps, and therefore the dimension of  $X_0(M_p)$  is  $m(p^2 - 1)/2$ . By Proposition 4.3 the dimension of  $V_p$  is  $-3\chi(C_p) + n_T(p^2 - 1)/2$ . Now, using the earlier computation for  $\dim(V)$  we have:

$$-3\chi(C_p) = -3\chi(C_H)p(p^2 - 1)/2 = p(p^2 - 1)(m - n_T)/2.$$

Since  $(N_p, H \cap N_p)$  is a Grothendieck Pair, Proposition 2.7 allows us to equate

the dimensions of  $X_0(M_p)$  and  $V_p$ . This gives:

$$m(p^2 - 1)/2 = p(p^2 - 1)(m - n_T)/2 + n_T(p^2 - 1)/2.$$

This is easily rearranged to show that  $m = n_T$ , which is a contradiction.

**Remarks.** (1) We are very grateful to a referee who suggested the argument above in the proof of the case when  $M$  has at least 2 cusps.

(2) We are grateful to a second referee who made the following observation in the context of trying to prove Theorems 1.1 and 1.2 using only pro- $p$  completions.

If two groups have isomorphic profinite completions, then they have isomorphic pro- $p$  completions for any prime  $p$ . However, having isomorphic pro- $p$  completions is not enough to prove the main results of the paper.

For example, let  $L \subset S^3$  be a boundary link (i.e., the components of  $L$  bound disjoint Seifert surfaces) and let  $G = \pi_1(S^3 \setminus L)$ . There is a map from  $S^3 \setminus L$  to a wedge of circles which induces a surjective homomorphism from  $G$  to a non-abelian free group  $F$ . This has a section  $F \rightarrow G$ . By Stallings' theorem (see [41]), any homomorphism between groups  $G$  and  $K$  that induces an isomorphism on  $H_1(-; \mathbb{F}_p)$  and a surjection on  $H_2(-; \mathbb{F}_p)$  induces an isomorphism of pro- $p$  completions. Applying this to  $G$  and  $F$  we see that the homomorphism  $G \rightarrow F$  induces an isomorphism of pro- $p$  completions. Hence, the inclusion  $F \hookrightarrow G$  induces an isomorphism of pro- $p$  completions.

## 5. Knots in $S^3$

In this section we prove the following result. As in §3.3,  $E(K)$  will denote the exterior of  $K$ .

**Theorem 5.1.** *Let  $K \subset S^3$  be a knot in  $S^3$  which is either a torus knot or for which  $\beta E(K)$  is a collar. Then  $\pi_1(S^3 \setminus K)$  is Grothendieck Rigid.*

It is a consequence of Thurston's hyperbolization theorem for Haken manifolds (see [32] and [45]), that a knot  $K$  is either hyperbolic (so that  $S^3 \setminus K$  is hyperbolic), is a torus knot (in which case  $S^3 \setminus K$  is a Seifert fibered space) or is a satellite knot (in which case  $S^3 \setminus K$  contains an embedded incompressible torus that is not boundary parallel). It is known by [39] that the fundamental groups of Seifert fibered spaces are LERF, which proves Grothendieck Rigidity for torus knot groups. Thus, given Theorem 1.2, the main part of the proof of Theorem 5.1 deals with the case that  $\beta E(K)$  is a collar.

*Proof.* Following the discussion above, we will assume that  $\beta E(K)$  is a collar. In particular, the discussion in §3.3 (see Corollary 3.3) shows that  $\gamma E(K)$  is hyperbolic.

Let  $H$  be a finitely generated subgroup of  $\Gamma$  such that the inclusion of  $H$  into  $\Gamma$  induces an isomorphism of profinite completions. Let  $X_H$  (resp.  $C_H$ ) denote the cover of  $E(K)$  corresponding to  $H$  (resp. denote a compact core  $C_H$  for  $X_H$ ). Corollary 3.5 shows that  $C_H$  is homotopy equivalent to a compact 3-manifold with non-empty incompressible torus boundary.

Note that the assumption that  $\widehat{H} \cong \widehat{\Gamma}$  implies that  $H$  has infinite index in  $\Gamma$  by Lemma 2.5. In addition,  $H$  is not infinite cyclic since  $\Gamma$  is a non-abelian group. Also note that  $\pi_1(\partial C_H)$  is not peripheral; i.e., it is not conjugate into  $\pi_1(\partial E(K))$ . For if this were the case,  $C_H$  would be a finite sheeted covering of  $E(K)$  and so  $H$  would be a finite index subgroup of  $\Gamma$  which is a contradiction.

The proof will be completed by the following two lemmas and Lemma 2.5 (recall the terminology from §3.2).  $\square$

**Lemma 5.2.** *In the notation above,  $H$  is a loose subgroup of  $\Gamma$ .*

*Proof.* Suppose that  $H$  is not loose, and so by definition  $H$  is a tight subgroup of  $\Gamma$ . From §3.2, we can suppose that there is a tight essential map  $f: C_H \rightarrow E(K)$  realizing  $H = f_*(\pi_1(C_H))$ . Now Proposition 3.2 shows that  $\beta C_H$  is a collar, and  $f$  can be deformed to a covering map of collars. In particular, this covering map has finite degree. Hence  $H$  has finite index in  $\Gamma$  which is a contradiction.  $\square$

**Lemma 5.3.** *A loose subgroup of  $\Gamma$  can be engulfed in  $\Gamma$ .*

*Proof.* Suppose that  $H$  is a loose subgroup of  $\Gamma = \pi_1(S^3 \setminus K)$ , so we can assume that  $H < \pi_1(C)$  where  $C$  is a component of  $E(K) \setminus \gamma E(K)$ .

By definition,  $C$  does not contain  $\partial E(K)$ . This affords the following decomposition of the JSJ graph of  $E(K)$ ; namely it has the form of a rooted tree  $T = T_C \cup \{v_0\} \cup T_D$  where  $T_C$  and  $T_D$  are subtrees of the JSJ graph of  $E(K)$ , and meet precisely in  $\{v_0\}$ . This yields the free product with amalgamation decomposition  $\Gamma = G_C *_{\pi_1(P)} G_D$  where  $P$  is some JSJ torus lying in the boundary of  $\gamma M$ , and  $G_C$  and  $G_D$  the fundamental groups of the connected submanifolds associated to  $T_C$  and  $T_D$ , with  $H < G_C$ . Note that  $G_D \neq \pi_1(P)$ , so that this is a non-trivial amalgamated product decomposition.

Now  $\pi_1(P)$  is Abelian, so it is separable in  $\pi_1(E(K))$  (see [21] and [25]). A standard consequence of the separability of  $\pi_1(P)$  is that we can arrange finite groups  $A$ ,  $B$  and  $C$  and an epimorphism  $\phi: G_C *_{\pi_1(P)} G_D \rightarrow A *_Q B$  with  $\phi(G_C) = A$ ,  $\phi(G_D) = B$  and  $\phi(\pi_1(P)) = Q$  (see for example [30]). In particular,  $A *_Q B$  is a non-trivial free product with amalgamation of finite groups, and as such, is itself LERF ([39]). Thus if  $g \in B \setminus Q$ , we can find a finite index subgroup of  $A *_Q B$  that contains  $A$  but not  $g$ . Since  $A$  contains  $\phi(H)$ , it follows that  $\phi(H)$  is contained in a proper subgroup of finite index in  $A *_Q B$ . That is to say,  $\phi(H)$  is engulfed, and so  $H$  is engulfed.  $\square$

**Remarks.** (1) Even when  $K$  is a cable knot or composite knot, Lemma 5.3 still holds. Thus to prove Theorem 5.1 for cable knots and composite knots, one needs to rule out the case of a Grothendieck Pair  $(\pi_1(S^3 \setminus K), H)$  where  $H$  is a tight subgroup of  $\pi_1(S^3 \setminus K)$ .

(2) If  $(\pi_1(S^3 \setminus K), H)$  is a Grothendieck Pair, then as in the proof of Theorem 5.1,  $X_H$  is homotopy equivalent to a knot complement in some closed manifold. Furthermore, the Grothendieck Pair assumption implies that  $H_1(X_H; \mathbb{Z}) \cong \mathbb{Z}$ . It follows that  $X_H$  is homotopy equivalent to a knot complement in integral homology 3-sphere (see Lemma 3.0 of [18] for example). At present we do not know that  $X_H$  is homotopy equivalent to a knot complement in  $S^3$ .

(3) There are satellite knot groups that are known not to be LERF [34], for example connect sums of torus knots.

(4) Lemma 5.3 is in the spirit of a result of Wilton and Zaleskii [49], where it is proved that fundamental groups of the vertex manifolds in the JSJ decomposition are separable.

(5) Note that if a group is not cohopfian, it will contain a proper subgroup with isomorphic profinite completion. The point here is that the isomorphism is not induced by the canonical inclusion map. Now much is known about which 3-manifold groups are cohopfian (see [18] and [36] for example), and indeed it is shown in [18] that a non-trivial knot group is cohopfian if and only if  $K$  is not a torus knot, a cable knot or a non-trivial connect sum. In this latter case, the knot group injects in itself as a subgroup of infinite index. The proof of Theorem 5.1 relied on the techniques of [18], and the hypothesis of Theorem 5.1 in part indicates the fact that these are the knot groups that are not cohopfian. In particular, this discussion also illustrates why using  $\hat{u}$ , with  $u$  the inclusion homomorphism is crucial in our discussions.

## References

- [1] I. Agol, Tameness of hyperbolic 3-manifolds. Preprint 2004. [arXiv:math/0405568](https://arxiv.org/abs/math/0405568)
- [2] I. Agol, D. D. Long, and A. W. Reid, The Bianchi groups are separable on geometrically finite subgroups. *Ann. of Math. (2)* **153** (2001), 599–621. [Zbl 1067.20067](https://doi.org/10.2307/2374771) [MR 1836283](https://doi.org/10.2307/2374771)
- [3] J. W. Anderson and R. D. Canary, Algebraic limits of Kleinian groups which rearrange the pages of a book. *Invent. Math.* **126** (1996), 205–214. [Zbl 0874.57012](https://doi.org/10.1007/BF01205201) [MR 1411128](https://doi.org/10.2307/2374771)
- [4] J. W. Anderson, R. D. Canary, and D. McCullough, The topology of deformation spaces of Kleinian groups. *Ann. of Math. (2)* **152** (2000), 693–741. [Zbl 0976.57016](https://doi.org/10.2307/2374771) [MR 1815699](https://doi.org/10.2307/2374771)
- [5] H. Bass and A. Lubotzky, Nonarithmetic superrigid groups: counterexamples to Platonov’s conjecture. *Ann. of Math. (2)* **151** (2000), 1151–1173. [Zbl 0963.22005](https://doi.org/10.2307/2374771) [MR 1779566](https://doi.org/10.2307/2374771)
- [6] M. R. Bridson and F. J. Grunewald, Grothendieck’s problems concerning profinite completions and representations of groups. *Ann. of Math. (2)* **160** (2004), 359–373. [Zbl 1083.20023](https://doi.org/10.2307/2374771) [MR 2119723](https://doi.org/10.2307/2374771)

- [7] J. R. Brock, R. D. Canary, and Y. N. Minsky, The classification of Kleinian surface groups, II: the ending lamination conjecture. Preprint 2004. [arXiv:math/0412006](#)
- [8] J. R. Brock, R. D. Canary, and Y. N. Minsky, The classification of finitely generated Kleinian groups. In preparation.
- [9] D. Calegari, M. H. Freedman, and K. Walker, Positivity of the universal pairing in 3 dimensions. *J. Amer. Math. Soc.* **23** (2010), 107–188. [Zbl 1201.57024](#) [MR 2552250](#)
- [10] D. Calegari and D. Gabai, Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **19** (2006), 385–446. [Zbl 1090.57010](#) [MR 2188131](#)
- [11] R. D. Canary, Marden's tameness conjecture: history and applications. In *Geometry, analysis and topology of discrete groups*, Adv. Lect. Math. (ALM) 6, Internat. Press, Somerville, MA, 2008, 137–162. [Zbl 1149.57028](#) [MR 2464394](#)
- [12] R. D. Canary, Introductory bumponomics: the topology of deformation spaces of hyperbolic 3-manifolds. In *Teichmüller theory and moduli problem*, Ramanujan Math. Soc. Lect. Notes Ser. 10, Ramanujan Math. Soc., Mysore 2010, 131–150. [Zbl 1202.57018](#) [MR 2667553](#)
- [13] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)* **117** (1983), 109–146. [Zbl 0529.57005](#) [MR 683804](#)
- [14] D. Gabai, Homotopy hyperbolic 3-manifolds are virtually hyperbolic. *J. Amer. Math. Soc.* **7** (1994), 193–198. [Zbl 0801.57009](#) [MR 1205445](#)
- [15] D. Gabai, On the geometric and topological rigidity of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **10** (1997), 37–74. [Zbl 0870.57014](#) [MR 1354958](#)
- [16] D. Gabai, G. R. Meyerhoff, and N. Thurston, Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math. (2)* **157** (2003), 335–431. [Zbl 1052.57019](#) [MR 1973051](#)
- [17] W. M. Goldman, Topological components of spaces of representations. *Invent. Math.* **93** (1988), 557–607. [Zbl 0655.57019](#) [MR 952283](#)
- [18] F. González-Acuña and W. C. Whitten, Imbeddings of three-manifold groups. *Mem. Amer. Math. Soc.* **99** (1992), no. 474. [Zbl 0756.57002](#) [MR 1117167](#)
- [19] A. Grothendieck, Représentations linéaires et compactification profinie des groupes discrets. *Manuscripta Math.* **2** (1970), 375–396. [Zbl 0239.20065](#) [MR 0262386](#)
- [20] F. Haglund and D. T. Wise, Special cube complexes. *Geom. Funct. Anal.* **17** (2008), 1551–1620. [Zbl 1155.53025](#) [MR 2377497](#)
- [21] E. Hamilton, Abelian subgroup separability of Haken 3-manifolds and closed hyperbolic  $n$ -orbifolds. *Proc. London Math. Soc. (3)* **83** (2001), 626–646. [Zbl 1025.57007](#) [MR 1851085](#)
- [22] J. Hempel, Residual finiteness for 3-manifolds. In *Combinatorial group theory and topology*, Ann. of Math. Stud. 111, Princeton University Press, Princeton, NJ, 1987, 379–396. [Zbl 0772.57002](#) [MR 0895623](#)
- [23] W. H. Jaco and P. B. Shalen, Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.* **21** (1979), no. 220. [Zbl 0415.57005](#) [MR 539411](#)
- [24] M. Kapovich, *Hyperbolic manifolds and discrete groups*. Progr. Math. 183, Birkhäuser, Boston, MA, 2001. [Zbl 0958.57001](#) [MR 1792613](#)

- [25] D. D. Long, Immersions and embeddings of totally geodesic surfaces. *Bull. London Math. Soc.* **19** (1987), 481–484. [Zbl 0596.57011](#) [MR 898729](#)
- [26] D. D. Long, Engulfing and subgroup separability for hyperbolic groups. *Trans. Amer. Math. Soc.* **308** (1988), 849–859. [Zbl 0664.57001](#) [MR 951631](#)
- [27] D. D. Long and A. W. Reid, Simple quotients of hyperbolic 3-manifold groups. *Proc. Amer. Math. Soc.* **126** (1998), 877–880. [Zbl 0891.57012](#) [MR 1459136](#)
- [28] D. Long and A. W. Reid, *Surface subgroups and subgroup separability in 3-manifold topology*. Publicações Matemáticas do IMPA, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro 2005. [Zbl 1074.57010](#) [MR 2164951](#)
- [29] A. Lubotzky, Tannaka duality for discrete groups. *Amer. J. Math.* **102** (1980), 663–689. [Zbl 0449.20051](#) [MR 584465](#)
- [30] A. Lubotzky, Free quotients and the first Betti number of some hyperbolic manifolds. *Transform. Groups* **1** (1996), 71–82. [Zbl 0876.22015](#) [MR 1390750](#)
- [31] A. Marden, The geometry of finitely generated Kleinian groups. *Ann. of Math. (2)* **99** (1974), 383–462. [Zbl 0282.30014](#) [MR 0349992](#)
- [32] J. W. Morgan and H. Bass (Eds.), *The Smith conjecture*, Pure Appl. Math. 112, Academic Press, Orlando, FL, 1984. [Zbl 0599.57001](#) [MR 0758459](#)
- [33] W. D. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds. *Topology* **24** (1985), 307–332. [Zbl 0589.57015](#) [MR 815482](#)
- [34] G. A. Niblo and D. T. Wise, Subgroup separability, knot groups and graph manifolds. *Proc. Amer. Math. Soc.* **129** (2001), 685–693. [Zbl 0967.20018](#) [MR 1707529](#)
- [35] V. P. Platonov and O. I. Tavgen', Grothendieck's problem on profinite completions and representations of groups. *K-Theory* **4** (1990), 89–101. [Zbl 0722.20020](#) [MR 1076526](#)
- [36] L. Potyagailo and S. Van, On the co-Hopfity of 3-manifold groups. *Algebra i Analiz* **11** (1999), no. 5, 194–220; English transl. *St. Petersburg Math. J.* **11** (2000), no. 5, 861–881. [Zbl 0977.57017](#) [MR 1734354](#)
- [37] L. Ribes and P. Zalesskii, *Profinite groups*. *Ergeb. Math. Grenzgeb. (3)* **40**, Springer-Verlag, Berlin 2000. [Zbl 0949.20017](#) [MR 1775104](#)
- [38] G. P. Scott, Compact submanifolds of 3-manifolds. *J. London Math. Soc. (2)* **7** (1973), 246–250. [Zbl 0266.57001](#) [MR 0326737](#)
- [39] P. Scott, Subgroups of surface groups are almost geometric. *J. London Math. Soc. (2)* **17** (1978), 555–565; Correction *ibid.* **32** (1985), 217–220. [Zbl 0412.57006](#) [MR 0494062](#); [Zbl 0581.57005](#) [MR 0811778](#)
- [40] P. Scott, There are no fake Seifert fibre spaces with infinite  $\pi_1$ . *Ann. of Math. (2)* **117** (1983), 35–70. [Zbl 0516.57006](#) [MR 683801](#)
- [41] J. Stallings, Homology and central series of groups. *J. Algebra* **2** (1965), 170–181. [Zbl 0135.05201](#) [MR 0175956](#)
- [42] D. Sullivan, Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.* **155** (1985), 243–260. [Zbl 0606.30044](#) [MR 806415](#)
- [43] M. Suzuki, *Group theory I*. Grundlehren Math. Wiss. 247, Springer-Verlag, Berlin 1982. [Zbl 0472.20001](#) [MR 0648772](#)

- [44] W. P. Thurston, The geometry and topology of 3-manifolds. Electronic edition of the 1980 lecture notes distributed by Princeton University. <http://library.msri.org/books/gt3m/>
- [45] W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 357–381. [Zbl 0496.57005](#) [MR 648524](#)
- [46] W. P. Thurston, *Three-dimensional geometry and topology*. Vol. 1, Princeton Math. Ser. 35, Princeton University Press, Princeton, NJ, 1997. [Zbl 0873.57001](#) [MR 1435975](#)
- [47] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)* **87** (1968), 56–88. [Zbl 0157.30603](#) [MR 0224099](#)
- [48] B. A. F. Wehrfritz, Two examples of soluble groups that are not conjugacy separable. *J. London Math. Soc. (2)* **7** (1973), 312–316. [Zbl 0272.20048](#) [MR 0338176](#)
- [49] H. Wilton and P. Zalesskii, Profinite properties of graph manifolds. *Geom. Dedicata* **147** (2010), 29–45. [Zbl 1204.57019](#) [MR 2660565](#)

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