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Slowly non-dissipative equations with oscillating growth

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Abstract. The goal of this paper is to construct explicitly the global attractors of semilinear parabolic equations when the reaction term has an oscillating growth. In this case, solution can also grow-up, and hence the attractor is unbounded and induces a flow at infinity. In particular, we construct heteroclinic connections between bounded and/or unbounded hyperbolic equilibria when the reaction term is asymptotically linear.

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1. Main results

Consider the scalar parabolic differential equation

$$u_t = u_{xx} + b(x,u)u + f(x,u,u_x) \tag{1.1}$$

with initial data $u(0,x) = u_0(x)$ and $x \in (0,\pi)$ has Neumann boundary. Suppose that $b, f \in C^2$ are bounded, b is strictly positive, that is $0 < \varepsilon \leq b(x,u) \leq B$ for some $\varepsilon > 0$ and all $(x,u) \in [0,\pi] \times \mathbb{R}$.

The equation (1.1) defines a semiflow denoted by $(t, u_0) \mapsto u(t)$ in a Banach space $X^\alpha := H^{2\alpha}([0,\pi])$, where $\alpha \in (0,1)$ denotes the fractional power. We suppose that $\alpha > 3/4$ so that solutions are at least $C^1([0,\pi])$. The appropriate functional setting is described in Section 2.1.

We are interested in the asymptotic behaviour of solutions of (1.1) when grow-up can occur, namely, when solutions grow unboundedly as $t \to \infty$. A sufficient condition for grow-up to occur is $b > 0$, as we will prove later in Lemma 2.1. This class of asymptotics is also known as *slowly non-dissipative*. In this setting, there does not exist a global attractor in the usual sense, namely a maximal compact invariant set that attracts all bounded sets. Omitting the compactness condition, there is an *unbounded global attractor* $\mathscr{A} \subseteq X^\alpha$, defined as the minimal invariant non-empty set in X^α attracting all bounded sets, firstly introduced by Chepyzhov and Goritskii [3].

The choice of slowly non-dissipative settings in (1.1) is motivated by the following. For the analogous equation $u_t = u_{xx} + g(u)$ it is known that the larger the limit below, in case it exists,

$$\lim_{|u|\to\infty} \frac{g(u)}{u} = \tilde{b},$$

the more non-dissipative the semiflow becomes. More precisely, one can prove that if $k^2 < \tilde{b} < (k+1)^2$ then on the k-dimensional eigenspace E_k, spanned by the k first eigenvalues, the energy functional goes to $-\infty$ as the norm of u increases, see for instance [15]. In addition, we know that negative values of $\tilde{b}$ yields dissipative flows, while if $\tilde{b} = \infty$ superlinear nonlinearities arise and, therefore, solutions with finite time blow-up. Then, a crucial assumption to ensure slowly non-dissipativity in this setting is that $0 < \tilde{b} < \infty$. In our original case (1.1), this translates into

$$0 < \lim_{|u|\to\infty} b(x,u) + \frac{f(u)}{u} < \infty$$

which is equivalent to suppose that f grows sublinearly, and the regular function b to be positive and bounded in u, even though it might oscillate without a limit at infinity. Although we believe that the same results proved in this paper are true for sublinear functions f, we restricted ourselves to the bounded ones.

The goal of this paper is to decompose $\mathscr{A}$ into smaller invariant sets, describe them and show how they are related.

This geometric description of the attractor $\mathscr{A}$ in the semilinear dissipative case was carried out by Brunovský and Fiedler [2] for $f(u)$, by Fiedler and Rocha [4] for $f(x,u,u_x)$, for periodic boundary conditions by Fiedler, Rocha and Wolfrum [5], and for quasilinear equations by Lappicy [9]. Such attractors are known as *Sturm attractors*. When solutions can grow-up, the semilinear case was previously studied by Hell [7] in order to give an understanding of the structure at infinity, Ben-Gal [1] for $f(u)$, Pimentel and Rocha [14] for $f(x,u,u_x)$. The case of periodic boundary condition was treated separately in Pimentel [13]. Such attractors are known as *unbounded Sturm attractors*.

Despite non-dissipativity, there still exist a Lyapunov function constructed by Matano [11]. Hence, the following dichotomy hold: either solutions converge to a bounded equilibrium as $t\to\infty$, or it is a grow-up solution.

In the latter case, Hell [7] viewed such grow-up solutions as heteroclinic orbits to infinity. In order to describe the dynamics of unbounded solutions, Hell added an infinite dimensional sphere $\mathscr{S}$ at infinity with an appropriate flow to it, which is compatible with the dynamics outside arbitrarily large sets. From now on, we abuse notation and extend the flow to infinity by $X^\alpha \cup \mathscr{S}$ in order to add such dynamics at the sphere $\mathscr{S}$, and hence the global attractor $\mathscr{A} \subseteq X^\alpha \cup \mathscr{S}$ consists of such unbounded dynamics at infinity as well. This extension is made by rescaling the solution u of (1.1), which becomes unbounded, through the compactified rescaled variable χ which has a converging limit. In our case, the limit of grow-up solutions lie in $\mathscr{S}$ and are isolated, so that such limiting objects are called equilibria at infinity, denoted $\pm\Phi_j$. Moreover, there is a Lyapunov function at infinity, and hence its associated flow is gradient. This decomposes the attractor as the theorem below, and its rigorous description is carried in Section 2.2 for the general case, and in Section 2.3 in a particular setting.

Since the flow at the sphere at infinity is generally nonlinear and complicated, we suppose that the reaction term b converges uniformly to a bounded function $b^\infty(x, u)$ in this compactification, and hence there exists a well defined limiting flow at $\mathscr{S}$. Mathematically, for any $\delta > 0$ sufficiently small, there exists an $R > 0$ such that

$$|b(x, u) - b^\infty(x, \chi)| \leq \delta \tag{1.2}$$

for all $x \in [0, \pi]$ and any $u \in X^\alpha \backslash B_R(0)$, where $B_R(0)$ is the ball of radius R in X^α; and $\chi \in L^2$ is such that $\|\chi\| = 1$.

Below we present the first main theorem that decomposes the attractor. In particular, we show that the dynamics at infinity $\mathscr{S}$ is gradient, and hence only consists of equilibria and connections between them.

Theorem 1.1 (Decomposition of the unbounded Sturm attractor). *Consider $b, f \in C^2$ with b, f bounded, and $b(x, u) \geq \varepsilon > 0$ satisfying* (1.2). *Suppose that all bounded equilibria are hyperbolic. Then, the unbounded attractor $\mathscr{A}$ of* (1.1) *can be decomposed as*

$$\mathscr{A} = \mathscr{E} \cup \mathscr{H}$$

where the set of equilibria $\mathscr{E}$ consists of elements which are bounded $\mathscr{E}^b = \{e_j\}_{j=1}^N$ and unbounded $\mathscr{E}^\infty$; and the set of heteroclinics $\mathscr{H}$ contains bounded connections $\mathscr{H}^b$, grow-up solutions $\mathscr{H}^{up}$ from bounded to unbounded elements, and unbounded connections $\mathscr{H}^\infty$ between unbounded equilibria. Mathematically,

$$\mathscr{A} = \mathscr{E}^b \cup \mathscr{E}^\infty \cup \mathscr{H}^b \cup \mathscr{H}^{up} \cup \mathscr{H}^\infty.$$

The bounded equilibria and their bounded heteroclinic connections can be computed similarly as [9]. In the upcoming Theorem 1.2, we describe how the set of equilibria $\mathscr{E}$ is connected to itself, namely we will give necessary and sufficient conditions so that a heteroclinic orbit exist. Before that, we need to introduce a new hypothesis on b^∞ and some particular notions.

As mentioned before, the flow at the sphere at infinity $\mathscr{S}$ is generally nonlinear and we do not know the complications of the dynamics at infinity, even though we proved in the last theorem that $\mathscr{S}$ has gradient structure. Therefore, we restrict the possibilities in the case b is asymptotically linear with limiting growth $b^\infty \in \mathbb{R}_+$ outside arbitrarily large sets, so that we can compute the flow at $\mathscr{S}$. Mathematically, for any $\delta > 0$ sufficiently small, there exists an $R > 0$ such that

$$|b(x,u) - b^\infty| \leq \delta \tag{1.3}$$

for all $x \in [0,\pi]$ and any $u \in X^\alpha \backslash B_R(0)$, where $B_R(0)$ is the ball of radius R in X^α.

Alternatively, one can rewrite the assumption (1.2) as

$$\lim_{\|u\| \to \infty} |b(x,u) - b^\infty| = 0 \tag{1.4}$$

for all $x \in [0,\pi]$.

An example that satisfies such condition and monotonically grows to a constant b^∞ is $b(u) = b^\infty \arctan(\|u\| + 1)$ with $b^\infty \in \mathbb{R}_+$, since $\lim_{\|u\| \to \infty} b(u) = b^\infty$. Another example is a small oscillating function close to linear with smaller amplitude as u grows, namely $b(u) = b^\infty + \sin^2(\|u\|)/[\|u\| + 1]$ with $b^\infty > 0$. The case when $b(x,u)$ converges to two different constants b_1^∞ and b_2^∞ in different directions could also possibly be treated in a similar fashion as the uniform convergence to b^∞.

Denote by the *zero number* $0 \leq z(u_*) \leq \infty$ the number of strict sign changes of a function $u_* : [0,\pi] \to \mathbb{R} \cup \{\pm\infty\}$, rigorously defined as

$$z(u_*) := \sup_k \left\{ \begin{array}{c} \text{There is a partition } \{x_j\}_{j=1}^k \text{ of } [0,\pi] \\ \text{such that } u_*(x_j)u_*(x_{j+1}) < 0 \text{ for all } j \end{array} \right\} \tag{1.5}$$

and $z(u_*) = -1$ if $u \equiv 0$. Note we allow discontinuous and unbounded functions u_*. Nevertheless, the importance of the zero number lies in the sign changes, even though the function might have jumps or attain value ∞ or $-\infty$.

Recall that the *Morse index* $i(u_*)$ of an equilibrium $u_* \in \mathscr{E}$ is given by the number of positive eigenvalues of the linearized operator of the right hand side of (1.1) at such equilibrium, that is, the dimension of the unstable manifold of said equilibrium. Also, an equilibrium u_* is *hyperbolic* if such linearized operator has no eigenvalue being zero.

We say that two different equilibria $u_- \in \mathscr{E}^b$ and $u_+ \in \mathscr{E} = \mathscr{E}^b \cup \mathscr{E}^\infty$ of (1.1) are *adjacent* if there does not exist an equilibrium $u_* \in \mathscr{E}^b$ of (1.1) such that $u_*(0)$

lies between $u_-(0)$ and $u_+(0)$, and

$$z(u_- - u_*) = z(u_- - u_+) = z(u_+ - u_*).$$

This notion was firstly described by Wolfrum [16].

Both the zero number and Morse index can be computed from a permutation of the equilibria, as it was done in [6] and [4] for the semilinear dissipative case. For the unbounded structure, a permutation can be computed as Pimentel and Rocha [14]. Such permutation is called the *Sturm Permutation*.

Next, we present the connections in case of an asymptotic linear diffusion, yielding a linear structure at infinity of Chafee–Infante type.

Theorem 1.2 (Connections within the unbounded Sturm attractor). *Consider $b, f \in C^2$ with b, f bounded, and $b(x,u) \geq \varepsilon > 0$ satisfying* (1.3). *Suppose that all bounded equilibria are hyperbolic. Then, there are finitely many equilibria at infinity given by $\mathscr{E}^\infty = \{\pm\Phi_j\}_{j=0}^{N^\infty}$ where $N^\infty = \lfloor\sqrt{b^\infty}\rfloor$, and the following holds:*

(1) *There is a heteroclinic $u(t) \in \mathscr{H}^b$ between two equilibria $e_j, e_k \in \mathscr{E}^b$ so that*

$$e_j \xleftarrow{t\to-\infty} u(t) \xrightarrow{t\to\infty} e_k$$

if, and only if, e_j and e_k are adjacent and $i(e_j) > i(e_k)$.

(2) *There is a heteroclinic $u(t) \in \mathscr{H}^{up}$ between equilibria $e_j \in \mathscr{E}^b$ and $\Phi_k \in \mathscr{E}^\infty$ so*

$$e_j \xleftarrow{t\to-\infty} u(t) \xrightarrow{t\to\infty} \pm\Phi_k$$

if, and only if, e_j and Φ_k are adjacent.

(3) *There is an heteroclinic $\Phi(t) \in \mathscr{H}^\infty$ between two equilibria $\Phi_j, \Phi_k \in \mathscr{E}^\infty$ so*

$$\Phi_j \xleftarrow{t\to-\infty} \Phi(t) \xrightarrow{t\to\infty} \Phi_k$$

if, and only if, $j > k$.

The remaining is organized as follows. In Section 2.1 we provide necessary background theories and introduce the proper notation. In Section 2.2 we find the flow at infinity and prove it is gradient, and hence the attractor is composed of equilibria $\mathscr{E}^\infty$ and heteroclinics $\mathscr{H}^\infty$. In Section 2.3, under the assumption (1.3) and describe the sets $\mathscr{H}^{up}$ and $\mathscr{H}^\infty$.

2. Proof of main results

2.1. Background. In the abstract setting, we consider the Hilbert space $X = L^2([0,\pi])$ with norm $\|\cdot\|$ and the sectorial operator $A = \partial_x^2$ with domain $D(A) =$

$H^2([0,\pi])$. For $\alpha \in (0,1)$, we consider the fractional power spaces $X^\alpha := D(A^\alpha)$ with graph norm $\|u\|_\alpha := \|A^\alpha u\|$ in case $u \in X^\alpha$, which interpolates between X and $D(A)$. We take $\alpha > 3/4$ so that $X^\alpha([0,\pi]) \subseteq C^1([0,\pi])$.

Therefore, the solution $u(t)$ generates a semiflow in the underlying space X^α for $t \geq 0$. Since f is bounded, the dynamical system $u(t)$ generated by (1.1) is globally defined. Indeed, $\|u(t)\|_\alpha$ is bounded for each $t \in (0,\infty)$ and $\alpha \in (3/4, 1)$, where $u(t)$ has initial data in $u_0 \in X^\alpha$. If $T(u_0)$ denotes the maximal time of existence of the solution through u_0, then it follows from [8] that $T(u_0) = +\infty$ for all $u_0 \in X^\alpha$. This implies that blow-up in finite time does not occur.

Consider the orthonormal basis $\{\varphi_j(x)\}_{j\in\mathbb{N}_0}$ of $L^2([0,\pi])$ comprised of eigenfunctions of A with Neumann boundary conditions, i.e., $\varphi_j(x) = \sqrt{2\pi^{-1}}\cos(jx)$ for $j \in \mathbb{N}$ and $\varphi_0(x) = \sqrt{\pi^{-1}}$. We further denote by $\lambda_j = -j^2$ the corresponding eigenvalues, for each $j \in \mathbb{N}_0$.

Lemma 2.1. *The semiflow $u(t)$ generated by* (1.1) *is slowly non-dissipative.*

Proof. Firstly, note that since b, f are bounded, then the semigroup $u(t)$ is bounded for any given time. Therefore, one can extend such solution indefinitely and its maximal time of existence is $T = \infty$. Hence, finite time blow-up can not occur.

Decompose a solution $u(t)$ of (1.1) into its Fourier modes as $u(t) = \sum_j u_j(t)\varphi_j$. Then we can project such semiflow in its j-component given by $u_j(t) := \langle u(t), \varphi_j\rangle$, yielding

$$\dot{u}_j = \langle u_{xx} + b(x,u)u, \varphi_j\rangle + \langle f(x,u,u_x), \varphi_j\rangle.$$

Let $f_j(t) := \langle f\big(x, u(t), u_x(t)\big), \varphi_j\rangle$. The strict positivity of $b(x,u) \geq \varepsilon > 0$ implies

$$\dot{u}_j \geq [\lambda_j + \varepsilon]u_j + f_j(t)$$

The variation of constants formula yield

$$u_j(t) \geq e^{(\lambda_j+\varepsilon)t}u_j(0) + \int_0^t e^{(\lambda_j+\varepsilon)\cdot[t-s]} f_j(s)\, ds. \tag{2.1}$$

Choose the particular initial data given by

$$u_j(0) = u_j^*(0) - \int_0^\infty e^{-(\lambda_j+\varepsilon)s} f_j(s)\, ds.$$

Note f is bounded, and so is the integral above. The variation of constants (2.1) yield

$$u_j(t) \geq e^{(\lambda_j+\varepsilon)t}u_j^*(0) - \int_t^\infty e^{(\lambda_j+\varepsilon)\cdot[t-s]}f_j(s)\,ds.$$

The linear part grows exponentially as $t \to \infty$, if $\lambda_j + \varepsilon > 0$ and the initial data $u_j(0) \not\equiv 0$ for some index j, whereas the integral term with $f_j(t)$ term stays bounded. At least for $j = 0$ this condition is satisfied, since $\lambda_j + \varepsilon = \varepsilon > 0$, and hence the lemma is proved. ■

2.2. Unbounded Sturm structure. To characterize the maximal compact attractor within the unbounded attractor, one can use a cut-off function and obtain a dissipative system. Therefore, the study of bounded trajectories, namely the computation of the bounded equilibria $\mathscr{E}^b$ and their heteroclinics $\mathscr{H}^b$ is the same as the dissipative case. This is done in [14]. Therefore, we focus on the behaviour of the semiflow at infinity, which is a homothety of the vector field of the equation (1.1) emphasized as

$$\mathscr{L}(u) := u_{xx} + b(x,u)u + f(x,u,u_x). \tag{2.2}$$

In order to compactify X^α, [7] used a Poincaré projection in order to identify it with the upper hemisphere of an infinite dimensional sphere in $L^2 \times \mathbb{R}$. We explain the ideas of such construction. Consider the phase-space X^α of (1.1), identified with $X^\alpha \times \{1\}$ and a subspace of $L^2 \times \{1\}$, as the tangent space at the north pole of an unitary northern hemisphere within an infinite dimensional sphere $\mathbb{S}^\infty$ in $L^2 \times \mathbb{R}$ given by

$$\mathbb{S}^\infty_+ = \{(\chi, \sqrt{1 - \|\chi\|^2}) \in L^2 \times [0,1]\}.$$

Then for each point in $u \in X^\alpha$, as a point in L^2, consider a line that passes through u and the origin $(0,0) \in L^2 \times \mathbb{R}$. The intersection of the line with the point within the upper hemisphere of $\mathbb{S}^\infty_+$ is the projection $\mathscr{P}$ of the phase space X^α, as a subspace of L^2. Its coordinates are given by

$$(\chi, z) := \mathscr{P}(u,1) = \frac{1}{\sqrt{1 + \|u\|^2}}(u,1). \tag{2.3}$$

Note that $z = 1$ if, and only if $\chi \equiv 0$. Hence, the north pole of $\mathbb{S}^\infty_+$ is the origin of X^α. Also, z decreases to 0 if, and only if, the norm $\|u\|$ increases to ∞. When $z = 0$, the projection $\mathscr{P}$ transforms the limit of grow-up solutions into the equator $\mathbb{S}^\infty_+|_{z=0}$ of $\mathbb{S}^\infty$, which is denoted by the normalized $\|\chi\| = 1$. Hence, χ denote coordinates of the unitary sphere in L^2 which is capable of determining the dynamics of grow-up solutions. For such reason, this is was called *sphere at infinity* denoted by $\mathcal{S}$.

The induced flow of $L^2 \times \{1\}$ into $\mathbb{S}_+^\infty$ through the projection $\mathcal{P}$ is given by differentiating (2.3) with respect to time, yielding

$$\chi_t = \mathcal{L}_z(\chi) - \chi\langle\mathcal{L}_z(\chi), \chi\rangle \tag{2.4}$$

$$z_t = -\langle\mathcal{L}_z(\chi), \chi\rangle \cdot z. \tag{2.5}$$

where the projected vector field is described by $\mathcal{L}_z(\chi) := z\mathcal{L}(z^{-1}\chi)$, which is a homothety of the original vector field (2.2) with scale factor $z := (1 + \|u\|^2)^{-1/2}$. In other words, $\mathcal{L}_z(\chi) = \chi_{xx} + b_z(\chi)\chi + f_z(\chi)$ where $b_z(\chi) := b(x, z^{-1}\chi, z^{-1}\chi_x)$ and $f_z(\chi) := zf(x, z^{-1}\chi, z^{-1}\chi_x)$ are homoteties of b and f respectively.

Note that the equator at infinity is invariant since $z_t = 0$ in the limit $z \to 0$. Indeed, we firstly prove that the following is bounded,

$$\langle\mathcal{L}_z(\chi), \chi\rangle \leq \|\chi_x\|^2 + B\|\chi\|^2 + \langle f_z(\chi), \chi\rangle. \tag{2.6}$$

We only need to prove that χ and χ_x have bounded L^2 norms. By definition of χ in (2.3), it follows that $\|\chi\|$ is bounded. If we decompose u as the sum of a bounded finite dimensional term and an unbounded one, as it is done in [3], we can reduce the analysis to a finite dimensional space and use the norms equivalence to conclude that $\|\chi_x\|$ is also bounded. See [10] for the same argument in a different setting.

Therefore, the term in the right hand side of (2.6) is bounded, and in the limit $z \to 0$, the equation in the equator $\mathbb{S}_+^\infty|_{z=0}$ is given by $z_t = 0$, showing that the equator is invariant.

We want to comprehend the flow in the equator $z = 0$, that describes the dynamics at infinity $\mathbb{S}_+^\infty|_{z=0}$. Note that the limit $b_z(\chi)$ exists as z tends to 0, due to (1.2). The corresponding flow at the sphere at infinity $z = 0$ is given by

$$\chi_t = \chi_{xx} + b^\infty(\chi)\chi - \chi\langle\chi_{xx} + b^\infty(\chi)\chi, \chi\rangle. \tag{2.7}$$

Alternatively, each coordinate $\chi_j = \langle\chi, \varphi_j\rangle$ satisfies

$$(\chi_j)_t = \lambda_j\chi_j + \langle b^\infty(\chi)\chi, \varphi_j\rangle - \chi_j\langle\chi_{xx} + b^\infty(\chi)\chi, \chi\rangle. \tag{2.8}$$

This flow acts on the sphere at infinity $\mathbb{S}_+^\infty|_{z=0}$, which consists of bounded trajectories. Since $u(t)$ becomes unbounded, we define the actual sphere at infinity $\mathcal{S}$ as the preimage of $\mathbb{S}_+^\infty|_{z=0}$ through $\mathcal{P}$. In particular, the grow-up solutions $u(t)$ actually converge to the unbounded functions in $\mathcal{S}$. Similarly, any compactified solution $\chi(t) \in \mathbb{S}_+^\infty|_{z=0}$ of the equation (2.7) corresponds to an actual unbounded solution $\Phi(t) := \mathcal{P}^{-1}(\chi(t)) \in \mathcal{S}$.

Therefore, we obtain a nonlinear flow at infinity at $\mathscr{S}$, which is complicated to study without further information on b^∞. Nevertheless, we can construct a Lyapunov function at the sphere at infinity, and obtain a gradient structure within $\mathscr{S}$: equilibria points and their heteroclinic connection.

Indeed, the Lyapunov function $E^\infty : \mathbb{S}^\infty_+|_{z=0} \to \mathbb{R}$ is given by

$$E^\infty = \int_0^\pi \frac{|\chi_x|^2}{2} + \int_0^\chi b^\infty(v)v\,dv\,dx \tag{2.9}$$

which yields after integration by parts, and plugging a solution χ of (2.7),

$$\frac{dE^\infty}{dt} = -\int_0^\pi \langle \chi_t, \chi_{xx} + b^\infty(\chi)\chi \rangle\,dx \tag{2.10}$$

$$= -\int_0^\pi \|\chi_{xx} + b^\infty(\chi)\chi\|^2 - \langle \chi, \chi_{xx} + b^\infty(\chi)\chi \rangle^2\,dx \leq 0. \tag{2.11}$$

where the last inequality holds due to Cauchy-Schwartz and that χ lies in the sphere at infinity, i.e., $\|\chi\| = 1$. Moreover, $\dot{E}^\infty$ vanishes if, and only if χ is an equilibria.

Lastly, the dynamics in the attractor $\mathscr{A} \subseteq X^\alpha$ is contained in a finite dimensional inertial manifold, as in [12], which exists in case we have a spectral gap condition. This is satisfied for instance if we assume f has small Lipschitz constant in u_x. That ensures compactness for trajectories in the upper hemisphere, which implies grow-up solutions converge to equilibria at the sphere at infinity.

2.3. Linear unbounded Sturm structure. In this section, we explore the case when the reaction term b is asymptotically linear, yielding a linear flow at the sphere at infinity. We gather all the tools developed in the previous sections, in order to construct the heteroclinics within the unbounded structure $\mathscr{S}$ of the unbounded attractor $\mathscr{A}$ for the parabolic equation (1.1). Firstly we describe the unbounded equilibria $\mathscr{E}^\infty$. Secondly, we use the y-map to describe grow-up solutions $\mathscr{H}^{up}$, which are seen as heteroclinics from bounded to unbounded equilibria. Thirdly, we describe the dynamics between unbounded equilibria given by $\mathscr{H}^\infty$.

The next result describes $\mathscr{E}^\infty$ and $\mathscr{H}^\infty$.

Lemma 2.2. *There are finitely many unbounded equilibria within the attrator, denoted by* $\mathscr{E}^\infty = \{\pm\Phi_j\}_{j=0}^{N^\infty}$ *where* $N^\infty := \lfloor \sqrt{b^\infty} \rfloor$ *and* $z(\pm\Phi_j) = j$. *Moreover, they are connected through a Chafee–Infante type structure.*

Proof. Let's describe the objects $\{\pm\Phi_j\}_{k=0}^{N^\infty}$ and show it plays the role of equilibria at infinity. As in Section 2.2, we want to compactify such infinite dimensional space so that it is easier to study the behaviour of grow-up solutions.

Let $u(t)$ be a grow solution, then the following limit holds in L^2-norm

$$\lim_{t\to\infty} \frac{u(t)}{\|u(t)\|} = \varphi_j \tag{2.12}$$

if, and only if, $\lim_{t\to\infty} \frac{u_j(t)}{\|u(t)\|} = 1$. This follows from direct calculation,

$$\left\| \frac{u(t)}{\|u(t)\|} - \varphi_j \right\|^2 = 2 - 2\frac{u_j(t)}{\|u(t)\|}. \tag{2.13}$$

From now on, we study the growth of each u_j and compare with its adjacent mode u_{j+1} so that we know for which indices j we have $u_j(t)/\|u(t)\| \to 1$ as $t \to \infty$.

We take $\delta > 0$ sufficiently small that will be specified later. Then, since $u(t)$ grows-up, for sufficiently big times the solution lies outside a ball of radius R and hence

$$u_{xx} + (b^\infty - \delta)u + f(x, u, u_x) \leq u_t \leq u_{xx} + (b^\infty + \delta)u + f(x, u, u_x). \tag{2.14}$$

We follow the idea from Lemma 2.1 together with the assumption (1.3) in order to project a grow-up solution in the φ_j direction, namely $u_j := \langle u, \varphi_j \rangle$. Then,

$$e^{[\lambda_j + b^\infty - \delta]t} u_j^{h,-}(0) + I_j^-(t) \leq u_j(t) \leq e^{[\lambda_j + b^\infty + \delta]t} u_j^{h,+}(0) + I_j^+(t) \tag{2.15}$$

where $I_j^\pm(t) := \int_\infty^t e^{[\lambda_j + b^\infty \pm \delta][t-s]} f_j(s)\, ds$ is the integral term in the variation of constants formula, $f_j(t) := \langle f(x, u, u_x), \varphi_j \rangle$ and $u_j^{h,\pm}(0) := u_j(0) + \int_0^\infty e^{-[\lambda_j + b^\infty \pm \delta]s} f_j(s)\, ds$.

Now choose the indices j such that $\lambda_j + b^\infty - \delta > 0$ and we compare the growth rates of u_j and u_{j+1}. This is done by checking that the lower bound of u_j is greater than the upper bound of u_{j+1}. Note that the integral terms $I_j^\pm(t)$ are bounded and will not contribute to the growth. Hence, we prove that

$$\lambda_{j+1} + b^\infty + \delta < \lambda_j + b^\infty - \delta. \tag{2.16}$$

Indeed, we can take $\delta > 0$ sufficiently small so that

$$\delta < \frac{\lambda_j - \lambda_{j+1}}{2} = \frac{2j+1}{2} \tag{2.17}$$

which is equivalent to (2.16). Note that there exists such δ because we only consider the indices j that guarantee grow-up, namely the ones such that $\lambda_j + b^\infty - \delta > 0$, which is then implies that the lower bound of (2.15) has growth. There are finitely many j so that the modes λ_j imply growth. Those are precisely $j \leq \sqrt{b^\infty - \delta}$, i.e., the ones that $j \leq \lfloor \sqrt{b^\infty - \delta} \rfloor =: N^\infty$.

Therefore, the limit of (2.12) for a grow-up solutions only holds for one fixed index j^* which is the smallest index j such that $u_{j^*}^-(0) \neq 0$.

We can then simplify the equation (2.8) and obtain

$$(\chi_j)_t = [\lambda_j + \|\chi_n\|]\chi_j. \tag{2.18}$$

The last term in equation (2.18) is nonlocal, and understanding such dynamics in the sphere at infinity is inviable. This is due to the projected flow lies in a curved space. So, we consider a secondary projection so that the induced flow lies in a planar space.

Consider a grow-up solution $u(t)$ such that its fastest growing mode with non-zero initial data is j^*. Consider also the hyperplane C_{j^*} which is tangent to the equator $\mathbb{S}_+^\infty|_{z=0}$ at the eigenfunction $(\varphi_{j^*}, 0) \in L^2 \times \mathbb{R}$.

Similarly to the projection $\mathscr{P}$, we consider any point $u \in X^\alpha \subseteq L^2$ and a line that passes through $(u, 1)$ and the origin $(0, 0)$, in $L^2 \times \mathbb{R}$. The intersection of the line with the plane C_{j^*} is the projection $\tilde{\mathscr{P}}_k$ of the phase space X^α. The coordinates of the projection $\tilde{\mathscr{P}}_k(u, 1)$ are (ξ, ζ) and can be computed as

$$(\xi, \zeta) := \tilde{\mathscr{P}}_k(u, 1) = \frac{1}{\langle u, \varphi_k \rangle}(u, 1). \tag{2.19}$$

The plane C_{j^*} can be written in its own coordinates (ξ, ζ) as

$$C_{j^*} := \{(\xi, \zeta) \in L^2 \times \mathbb{R} \,|\, \xi_{j^*} = +1,\ \xi_j \in \mathbb{R} \text{ for all } j \in \mathbb{N}_0\}. \tag{2.20}$$

We differentiate (2.19) with respect to t to obtain the flow in the plane C_{j^*}

$$\xi_t = \mathscr{L}_\zeta(\xi) - \langle \mathscr{L}_\zeta(\xi), \varphi_{j^*} \rangle \xi \tag{2.21}$$

$$\zeta_t = -\langle \mathscr{L}_\zeta(\xi), \varphi_{j^*} \rangle \zeta. \tag{2.22}$$

Since we are interested in the semiflow at infinity, we take the limit of (2.21) and (2.22) as $\zeta \to 0$. Note that the right hand side of the equation (2.22) vanishes, because the inner product is bounded:

$$\begin{aligned} \langle \mathscr{L}_\zeta(\xi), \varphi_{j^*} \rangle &\leq \langle \xi_{xx} + B\xi + f_\zeta(\xi), \varphi_{j^*} \rangle \\ &\leq \lambda_{j^*} + B + \langle f_\zeta(\xi), \varphi_{j^*} \rangle \end{aligned} \tag{2.23}$$

using that b is bounded; in C_{j^*} we have that $\xi_{j^*} = 1$ since $u_{j^*}(t)$ is the mode that grows the most; and f is bounded. Hence, the right hand side of (2.22) vanishes as $\zeta \to 0$ and the equation in the plane $C_{j^*}|_{\zeta=0}$ is given by $\zeta_t = 0$, showing that this plane is invariant.

In order to study the flow of the equation (2.21), we alternatively write each coordinate $\xi_j := \langle \xi, \varphi_j \rangle$ and its induced flow in C_{j^*} as

$$(\xi_j)_t = \langle \xi_{xx} + b_\zeta(\xi)\xi + f_\zeta(\xi), \varphi_j \rangle - \langle \xi_{xx} + b_\zeta(\xi)\xi + f_\zeta(\xi), \varphi_{j^*} \rangle \xi_j. \quad (2.24)$$

In the limit as $\zeta \to 0$, we obtain

$$(\xi_j)_t = (\lambda_j - \lambda_{j^*})\xi_j, \quad (2.25)$$

since in (1.3) we assumed that $b \to b^\infty \in \mathbb{R}_+$ outside large balls in X^α, which implies that $b_\zeta \to b^\infty$ as $\zeta \to 0$; $f_\zeta(\xi) \to 0$ since f is bounded; and $\xi_{j^*} = 1$ in the plane C_{j^*}.

Therefore, the asymptotic grow-up behaviour of the solutions $u(t)$ in the projected coordinates (ξ, ζ) within the planes C_{j^*} yield the linear flow (2.25). In particular, it can be seen that the unbounded equilibria within the sphere at infinity in the (ξ, ζ) coordinates are exactly the eigenfunctions

$$\pm\phi_{j^*} = \{(\xi, 0) \in \mathbb{S}^\infty : \xi_{j^*} = \pm 1 \text{ and } \xi_j = 0 \ \forall j \neq j^*\}, \quad (2.26)$$

for all $j^* \in \mathbb{N}_0$.

Using colinearity, we can relate the coordinates (χ, z) to the corresponding coordinates (ξ, ζ) through

$$(\xi, \zeta) = \frac{1}{\langle \chi, \phi_k \rangle}(\chi, z). \quad (2.27)$$

In particular, the equilibria in both coordinates (χ, z) and (ξ, ζ) coincide:

$$\pm\phi_{j^*} = \{(\chi, 0) \in \mathbb{S}^\infty : \chi_{j^*} = \pm 1 \text{ and } \chi_j = 0 \ \forall j \neq j^*\}, \quad (2.28)$$

Note the linear flow (2.7) in the hyperplanes $C_{j^*}|_{\zeta=0}$ and the nonlinear flow (2.25) in the sphere at infinity $\mathbb{S}^\infty_+|_{z=0}$ are topologically equivalent through the diffeomorphism (2.27), since the flow of (χ, z) and (ξ, ζ) are projections of the same semiflow $u(t)$. Hence, $\mathscr{P} \circ \tilde{\mathscr{P}}_{j^*}^{-1} : C_{j^*} \to \mathbb{S}^\infty_+$ is an equivalence relation of the flows. Therefore, they display the same dynamics. In particular, if there is a heteroclinic in the $C_{j^*}|_{\zeta=0}$ hyperplanes, there is also a heterolinic in the sphere at infinity $\mathbb{S}^\infty_+|_{z=0}$.

Note that the compactified sphere at infinity $\mathbb{S}^\infty_+|_{z=0}$ consists of bounded trajectories. Since $u(t)$ becomes unbounded, we define the actual sphere at infinity $\mathscr{S}$ as the preimage of $\mathbb{S}^\infty_+|_{z=0}$ through $\mathscr{P}$. In particular, the grow-up solutions $u(t)$ actually converge to the unbounded functions

$$\Phi_k(x) := \lim_{t \to \infty} t \cdot \varphi_k(x) \in \mathscr{S} \quad (2.29)$$

which is unbounded in all points that $\varphi_k(x) \neq 0$, and has the same zeros as φ_k. Similarly, any solution $\chi(t) \in \mathbb{S}^\infty_+|_{z=0}$ of the equation (2.7) corresponds to an actual unbounded solution $\Phi(t) := \mathscr{P}^{-1}(\chi(t)) \in \mathscr{S}$.

Note that the zero numbers of $\pm\Phi_k$ are well defined, even though they are all unbounded in the boundary. Moreover, one can define the zero number of a difference of unbounded equilibria $\Phi_k - \Phi_j$ as the zero number of the difference of its corresponding eigenfunctions $\varphi_k - \varphi_j$.

We now discuss the intra-infinite heteroclinics $\mathscr{H}^\infty$ in the attractor. Given an equilibrium with $j \in \{1, \dots, N^\infty\}$, we want to show that there is a heteroclinic connections from equilibria $\pm\Phi_j$ to $\pm\Phi_k$ for each $k \in \{0, 1, \dots, j-1\}$.

Indeed, we look at the evolution of $\xi_k(t)$ at the plane C_j restricted to $\zeta = 0$, which is tangent to the equator $\mathbb{S}^\infty_+|_{\zeta=0}$. This yields an expansion in the ξ_k direction of the equilibria $\pm\Phi_j \in C_j$, since its flow is given by

$$(\xi_k)_t = (\lambda_k - \lambda_j)\xi_k$$

and $\lambda_k > \lambda_j$. Since this is a linear expansion, for some t_*, we have that $\xi_k(t_*) = 1$, that is, $\xi_k(t_*)$ intersects the plane C_k.

On the other hand, the evolution of $\xi_j(t)$ in the plane C_k restricted to $\zeta = 0$ yields a contraction in the ξ_j direction of the equilibria $\pm\Phi_k \in C_k$, since its flow is given by

$$(\xi_j)_t = (\lambda_j - \lambda_k)\xi_j$$

and $\lambda_k > \lambda_j$.

Lastly, note these expansion and contraction occur in the C_j and C_k planes when $\zeta = 0$, respectively. Moreover, those are projections of the flow that occur in the equator $\mathbb{S}^\infty_+|_{\zeta=0}$, since it is obtained through the projection $\tilde{\mathscr{P}}_j$ that describes the dynamics in sphere at infinity given by such equator $\mathbb{S}^\infty_+|_{\zeta=0}$, and where intra-infinity heteroclinics actually occur. ■

Next we address that grow-up orbits $\mathscr{H}^{up}$. We first prove that those with a fixed number of zeros for larges times, cannot have zero dropping at $t = \infty$. In particular, a grow-up solution $u(t)$ converges to a solution with a fixed number of zeros. Later we recall the blocking and liberalism principles for unbounded solutions.

Lemma 2.3. *Let $u(t)$ be a grow-up solution in the unstable manifold of an equilibrium $e_j \in \mathscr{E}^b$. Suppose the following conditions hold*

$$z(u(t) - e_j) = k, \qquad sign(u(t,0) - e_j(0)) = \pm 1 \tag{2.30}$$

for all sufficiently large times t. Then $u(t)$ converges to $\pm\Phi_k \in \mathscr{E}^\infty$.

Proof. Without loss of generality, we suppose that $sign\big(u(t,0) - e_j(0)\big) = +1$. Comparison implies that $u(t,0) > e_j(0)$ for all $t > 0$. Therefore, $\lim_{t\to\infty} u(t,0) > 0$. Hence, $u(t)$ has to converge to some $\Phi_l \in \mathscr{E}^\infty$, that is

$$\lim_{t\to\infty} \left\| \frac{u(t)}{\|u(t)\|_{L^2}} - \varphi_l \right\|_{L^2} = 0 \tag{2.31}$$

by (2.12).

In order to obtain the desired statement, it is sufficient to prove that the convergence above also holds in the C^1-norm. Indeed, if such convergence (2.31) holds, then by hypothesis (2.30), the limit of $u(t)$ has a constant number of zeros for large time t, given by $z\big(u(t)\big) = z\big(u(t) - e_j\big)$, and does not drop at $t = \infty$, since the convergence is in C^1. Hence, $l = k$.

Any solution u lies on the finite dimensional inertial manifold and, as consequence, we can apply again norm equivalence to conclude that (2.31) implies

$$\lim_{t\to\infty} \left\| \frac{u(t)}{\|u(t)\|_{L^2}} - \varphi_l \right\|_{C^1} = 0.$$

Then, the lemma is proved. ∎

Similarly to the particular case of constant $b(x,u)$, we can establish blocking and liberalism results for unbounded solutions. Those results are stated below, and we refer to [14] for the proofs.

Lemma 2.4 (Infinite blocking). *If the equilibria $e_j \in \mathscr{E}^b$ and $\pm\Phi_k \in \mathscr{E}^\infty$ are not adjacent, then they are not connected by a heteroclinic orbit.*

Lemma 2.5 (Infinite Liberalism). *If the equilibria $e_j \in \mathscr{E}^b$ and $\pm\Phi_k \in \mathscr{E}^\infty$ are adjacent, then they are connected by a heteroclinic orbit.*

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