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The obstacle problem for noncoercive elliptic equations with variable growth and L^1-data

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Abstract. The aim of this paper is to study the obstacle problem associated with noncoercive elliptic equations with variable exponents and L^1-data. After proving the existence and regularity of entropy solutions, we have extended the Lewy–Stampacchia inequalities to the case of noncoercive elliptic operators.

1. Introduction

In the recent decades, the topic of nonlinear partial differential equations with nonstandard growth conditions has captured an increasing attention because of its applications to the mathematical modelling of numerous real world phenomena. Particularly, a number of papers has focused on elliptic obstacle problems involving variable exponents; for instance, see [15, 16, 21, 23, 25, 26] and the references therein.

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$ and $f \in L^1(\Omega)$. Hereinafter, for any two bounded measurable functions $r(\cdot), s(\cdot) : \Omega \to \mathbb{R}$, we set

$$\underline{r} = \operatorname*{ess\,inf}_{x\in\Omega} r(x) \quad \text{and} \quad \overline{r} = \operatorname*{ess\,sup}_{x\in\Omega} r(x),$$

and we write

$$r(\cdot) \ll s(\cdot) \quad \text{if} \quad \operatorname*{ess\,inf}_{x\in\Omega}(s(x) - r(x)) > 0.$$

Let $p(\cdot) : \overline{\Omega} \to (1, +\infty)$ be a continuous function, and $\gamma(\cdot) : \Omega \to [0, +\infty)$ be a measurable function such that

$$1 < \underline{p} \leq \overline{p} < N, \tag{1.1}$$

$$0 \leq \overline{\gamma} < \underline{p} - 1. \tag{1.2}$$

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Our main goal is to prove the existence of entropy solutions for the obstacle problem associated with the following nonlinear noncoercive elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{a(x,\nabla u)}{(1+|u|)^{\gamma(x)}}\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector function satisfying for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$, the assumptions

$$|a(x,\xi)| \le \beta |\xi|^{p(x)-1}, \tag{1.4}$$

$$a(x,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)}, \tag{1.5}$$

$$\big[a(x,\xi) - a(x,\xi')\big]\cdot\big[\xi-\xi'\big] > 0, \tag{1.6}$$

where $\alpha > 0$, and $\beta > 0$.

We define, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the nonlinear elliptic operator

$$\mathcal{A}(u) = -\operatorname{div}\Big(\frac{a(x,\nabla u)}{(1+|u|)^{\gamma(x)}}\Big),$$

which, thanks to (1.4) and (1.5), maps $W_0^{1,p(\cdot)}(\Omega)$ into its dual space $W^{-1,p'(\cdot)}(\Omega)$, but its coercivity can degenerate when u is too big. Due to the lack of coercivity, the variational methods of Leray–Lions (see, for instance, [19]) cannot be applied even if the data f is sufficiently regular.

For a given function $\psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we define the convex set

$$\mathcal{K}_\psi = \big\{v \in W_0^{1,p(\cdot)}(\Omega) : v \ge \psi \quad \text{a.e. in } \Omega\big\}.$$

The unilateral problem relative to $\mathcal{A}$, f, and the obstacle ψ (denoted by $(\mathcal{A}, f, \psi)$) can be formulated, using the duality between $W_0^{1,p(\cdot)}(\Omega)$ and $W^{-1,p'(\cdot)}(\Omega)$, in terms of the variational inequality

$$\begin{cases} u \in \mathcal{K}_\psi, \\ \displaystyle\int_\Omega \frac{a(x,\nabla u)\cdot\nabla(u-v)}{(1+|u|)^{\gamma(x)}}\,dx \le \langle f, u-v\rangle, \quad \forall v \in \mathcal{K}_\psi, \end{cases} \tag{1.7}$$

whenever $f \in W^{-1,p'(\cdot)}(\Omega)$. In the case $f \in L^1(\Omega)$, the last fact holds only if $p(\cdot) > N$ (thanks to the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^\infty(\Omega)$). Unfortunately, if $f \in L^1(\Omega)$ and $1 < p(\cdot) < N$, then the solution u of (1.3) does not belong to the space $W_0^{1,p(\cdot)}(\Omega)$ (see [28]). Therefore, the formulation (1.7) does not remain valid since both sides of inequality (1.7) are maybe meaningless. Following [9, 25], this leads to introducing a more general formulation of the obstacle problem $(\mathcal{A}, f, \psi)$ using the truncation function at level $k > 0$, $T_k : \mathbb{R} \to \mathbb{R}$ defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

Definition 1.1. An entropy solution of the obstacle problem $(\mathcal{A}, f, \psi)$ associated to problem (1.3) is a measurable function u such that

$$\begin{cases} u \ge \psi \quad \text{a.e. in } \Omega, \\ T_k(u) \in W_0^{1,p(\cdot)}(\Omega), \quad \forall k > 0, \\ \displaystyle\int_\Omega \frac{a(x, \nabla u) \cdot \nabla T_k(u - v)}{(1 + |u|)^{\gamma(x)}}\, dx \le \int_\Omega f\, T_k(u - v)\, dx, \quad \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{1.8}$$

We point out that, in the case of constant exponents p and γ, the existence and regularity of entropy solutions to the obstacle problem (1.8) were obtained in [5–7, 11, 30]. Also in [1, 29], the authors studied the existence of entropy solutions to the obstacle problem associated with the operator $\mathcal{A}$ with additional lower-order terms. In the coercive case, i.e. $\gamma(\cdot) \equiv 0$, the obstacle problem $(\mathcal{A}, f, \psi)$ has been considered by many authors (see, among others, [10, 13, 25]).

The inequality of Lewy–Stampacchia in the general framework has been considered in numerous papers, see for example [15, 21–26]. In particular, the authors of [23] and [22] have proved the Lewy–Stampacchia inequalities for pseudomonotone elliptic operators in the context of variable exponent Sobolev spaces.

In this paper, in the case when $p(\cdot)$ is log-Hölder continuous (see Remark 2.5 below) using the techniques of [25, 27], we establish the existence of an entropy solution u to the obstacle problem (1.8) such that $|u|^{q(\cdot)} \in L^1(\Omega)$ for all $0 \ll q(\cdot) \ll q_0(\cdot)$, and $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$ for all $0 \ll q(\cdot) \ll q_1(\cdot)$, where

$$q_0(\cdot) = p^*(\cdot)\Big(1 - \frac{1 + \overline{\gamma}}{\underline{p}}\Big) \quad \text{and} \quad q_1(\cdot) = \frac{p(\cdot) q_0(\cdot)}{q_0(\cdot) + 1 + \gamma(\cdot)},$$

where $p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)}$. In particular, if $\underline{p} > 2 - \frac{1 - \overline{\gamma}(N-1)}{N}$ then

$$u \in W_0^{1,q(\cdot)}(\Omega), \quad \text{for all } 1 \le q(\cdot) \ll q_1(\cdot).$$

Furthermore, if $p(\cdot) - 1 \ll q_1(\cdot)$, then $\mathcal{A}(u) \in L^1(\Omega)$ and the following Lewy–Stampacchia inequalities hold:

$$f \le \mathcal{A}(u) \le f + (\mathcal{A}(\psi) - f)^+ \quad \text{a.e. in } \Omega.$$

And in the case of a merely continuous $p(\cdot)$ on $\overline{\Omega}$ satisfying $2-\frac{1-\overline{\gamma}(N-1)}{N}<\underline{p}\leq\overline{p}<N$, by taking advantage of the method of [12], we prove that $u\in W_0^{1,q(\cdot)}(\Omega)$ for all continuous function $q(\cdot)$ on $\overline{\Omega}$ satisfying

$$1\leq q(x)<\frac{N(p(x)-1-\overline{\gamma})}{N-1-\overline{\gamma}}\quad \text{in } \overline{\Omega}.$$

The rest of the paper is organised as follows. In Section 2, we recall the definitions of Lebesgue, Marcinkiewicz and Sobolev spaces with variable exponent and some of their properties. In Section 3, we state our main results. In Section 4, we consider the approximating obstacle problems, and establish the uniform estimates of solutions for the approximation problems. In Section 5, we prove the strong convergence of the truncations of these approximating solutions. Finally, in Sections 6 and 7, we establish the existence results and we show that Lewy–Stampacchia inequalities hold true in the context of log-Hölder continuous exponent $p(\cdot)$.

2. Mathematical preliminaries

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces. For further details on this topic, we refer to [4, 14, 17] and references therein. Hereinafter, we write

$$\mathcal{P}_0(\Omega)=\big\{h\in L^\infty(\Omega):\underline{h}>0\big\}\quad\text{and}\quad \mathcal{P}_1(\Omega)=\big\{h\in L^\infty(\Omega):\underline{h}\geq 1\big\}.$$

For any $p\in\mathcal{P}_0(\Omega)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u)=\int_\Omega |u|^{p(x)}\,dx,$$

is finite. The space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg–Nakano quasi-norm

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\Big\{\lambda>0:\rho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big)\leq 1\Big\},$$

is a quasi-Banach space (see [2,14]). In particular, if $p\in\mathcal{P}_1(\Omega)$ then the above expression defines a norm in $L^{p(\cdot)}(\Omega)$. In this case, the space $L^{p(\cdot)}(\Omega)$ becomes a separable Banach space (see e.g. [17]). Moreover, if $\underline{p}>1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual space can be identified with $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$, and for all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, the Hölder-type inequality holds (see [17])

$$\left|\int_\Omega uv\,dx\right|\leq 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}. \tag{2.1}$$

The norm and the modular are related by the following inequalities.

Proposition 2.1 ([14]). *Let $p \in \mathcal{P}_0(\Omega)$. Then, for every $u \in L^{p(\cdot)}(\Omega)$, one has $\rho_{p(\cdot)}(u) < 1$ (> 1; $= 1$) if and only if $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ (> 1; $= 1$); further,*

$$\text{if} \quad \|u\|_{L^{p(\cdot)}(\Omega)} < 1 \quad \text{then} \quad \|u\|^{\overline{p}}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{\underline{p}}_{L^{p(\cdot)}(\Omega)}, \tag{2.2}$$

$$\text{if} \quad \|u\|_{L^{p(\cdot)}(\Omega)} > 1 \quad \text{then} \quad \|u\|^{\underline{p}}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{\overline{p}}_{L^{p(\cdot)}(\Omega)}. \tag{2.3}$$

The above proposition states that the norm convergence and modular convergence are equivalent, that is to say, if $u_n, u \in L^{p(\cdot)}(\Omega)$, then

$$\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \to 0 \quad \text{if and only if} \quad \rho_{p(\cdot)}(u_n - u) \to 0.$$

Next we define Marcinkiewicz (weak Lebesgue) spaces with variable exponent and we investigate their relation with variable exponent Lebesgue spaces.

Definition 2.2 ([2]). Let $p \in \mathcal{P}_0(\Omega)$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ if

$$\|u\|_{\mathcal{M}^{p(\cdot)}(\Omega)} = \sup_{\lambda > 0} \lambda \|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty, \tag{2.4}$$

where χ_E denotes the characteristic function of a measurable set E.

Note that inequalities (2.2) and (2.3) imply that (2.4) is equivalent to say that there exists a positive constant M such that

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)}\, dx \le M, \quad \forall \lambda > 0. \tag{2.5}$$

If $p, q \in \mathcal{P}_0(\Omega)$ with $q \le p$, then we have the following two inclusions (see [14]):

$$L^{p(\cdot)}(\Omega) \subseteq L^{q(\cdot)}(\Omega) \quad \text{and} \quad L^{p(\cdot)}(\Omega) \subset \mathcal{M}^{p(\cdot)}(\Omega) \subseteq \mathcal{M}^{q(\cdot)}(\Omega).$$

The following result is from [27, Proposition 2.5].

Proposition 2.3. *Let $p, q \in \mathcal{P}_0(\Omega)$ such that $q(\cdot) \ll p(\cdot)$, then*

$$\mathcal{M}^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega).$$

We will need the following results proved in [20].

Lemma 2.4. *Let $u \in \mathcal{M}^{p(\cdot)}(\Omega)$ with $p \in \mathcal{P}_0(\Omega)$. Then there exists a constant $c > 0$ such that*

$$\text{meas}\{|u| > \lambda\} \le \frac{c}{\lambda^{\underline{p}}}, \quad \forall \lambda > 0.$$

For any $p \in \mathcal{P}_1(\Omega)$, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We define $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the above norm. The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable Banach spaces. If $\underline{p} > 1$ they are reflexive and the dual space of $W_0^{1,p(\cdot)}(\Omega)$ will be denoted by $W^{-1,p'(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in \mathcal{P}_1(\Omega)$, the Poincaré inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

holds for some $c > 0$ which depends on Ω and p. Therefore, $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ and $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.5. The smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$ but if p is log-Hölder continuous, that is, there exists a positive constant L such that

$$|p(x) - p(y)| \le \frac{L}{-\ln|x-y|}, \quad \forall x, y \in \overline{\Omega}, 0 < |x-y| \le \frac{1}{2}, \tag{2.6}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$.

We have the following embedding result which can be found in [4].

Proposition 2.6. *If $p, q \in C(\overline{\Omega})$ with $1 < \underline{p} \le \overline{p} < N$ and $1 \le q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ in $\overline{\Omega}$, then for every $u \in W_0^{1,p(\cdot)}(\Omega)$*

$$\|u\|_{L^{q(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{2.7}$$

where c is some positive constant independent of u. The embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact. Moreover, if p satisfies (2.6), then the Sobolev inequality (2.7) holds also for $q(\cdot) = p^*(\cdot)$.

We denote by $\mathcal{T}_0^{1,p(\cdot)}(\Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ for any $k > 0$. Note that $\mathcal{T}_0^{1,p(\cdot)}(\Omega)$ is not contained in the Sobolev space $W_0^{1,1}(\Omega)$. However, the following proposition gives a sense to the gradient of $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$.

Proposition 2.7 ([27]). *Let $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\le k\}} \quad \text{a.e. in } \Omega, \text{ for any } k > 0.$$

Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

3. Main results

Our first result is the following theorem.

Theorem 3.1. *Assume that hypotheses* (1.1)*,* (1.2)*,* (1.4)*–*(1.6) *and* (2.6) *hold. Then there exists at least one entropy solution u to the obstacle problem* (1.8)*. Moreover, $u \in L^{q(\cdot)}(\Omega)$, for any $q \in \mathcal{P}_0(\Omega)$ with $q(\cdot) \ll q_0(\cdot)$, and $|\nabla u| \in L^{q(\cdot)}(\Omega)$, for any $q \in \mathcal{P}_0(\Omega)$ with $q(\cdot) \ll q_1(\cdot)$, where*

$$q_0(\cdot) = p^*(\cdot)\Big(1 - \frac{1+\overline{\gamma}}{\underline{p}}\Big) \quad \text{and} \quad q_1(\cdot) = \frac{p(\cdot)q_0(\cdot)}{q_0(\cdot)+1+\gamma(\cdot)}. \tag{3.1}$$

In particular, if $\underline{p} > 2 - \frac{1-\overline{\gamma}(N-1)}{N}$ then $u \in W_0^{1,q(\cdot)}(\Omega)$, for any $q \in \mathcal{P}_1(\Omega)$ with $q(\cdot) \ll q_1(\cdot)$.

Moreover, if $p(\cdot) - 1 \ll q_1(\cdot)$ then $\mathcal{A}(u) \in L^1(\Omega)$ and the following Lewy–Stampacchia inequalities hold:

$$f \le \mathcal{A}(u) \le f + (\mathcal{A}(\psi) - f)^+ \quad \text{a.e. } in\ \Omega. \tag{3.2}$$

Remark 3.2. When $\gamma(\cdot) \equiv 0$, the statement of Theorem 3.1 coincides with that of [25, Theorem 2.1].

Remark 3.3. We note that in Theorem 3.1, we need to use the continuous Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, and this requires us to assume that the exponent $p(\cdot)$ is log-Hölder continuous.

In our second result, which is a direct generalisation of [6, Theorem 2.4] to the context of variable exponents, we will assume $p(\cdot)$ to be a merely continuous function.

Theorem 3.4. *Assume* (1.1)*,* (1.2)*,* (1.4)*–*(1.6) *and that*

$$\underline{p} > 2 - \frac{1-\overline{\gamma}(N-1)}{N}. \tag{3.3}$$

Then there exists at least one entropy solution u to the obstacle problem (1.8)*. Moreover, $u \in W_0^{1,q(\cdot)}(\Omega)$ for all continuous function $q(\cdot)$ on $\overline{\Omega}$ satisfying*

$$1 \le q(x) < \frac{N(p(x)-1-\overline{\gamma})}{N-1-\overline{\gamma}} \quad in\ \overline{\Omega}. \tag{3.4}$$

Remark 3.5. Note that condition (3.3) implies that $1 \ll \frac{N(p(x)-1-\overline{\gamma})}{N-1-\overline{\gamma}} \ll p(\cdot)$ and $1 \ll q_1(\cdot)$. When $\gamma(\cdot) \equiv 0$, it corresponds to the well-known condition $\underline{p} > 2 - \frac{1}{N}$.

Remark 3.6. We remark that if $\gamma(x) = \theta(p(x) - 1)$, condition (1.2) is nothing else than $0 \le \theta < \frac{\underline{p}-1}{\overline{p}-1}$ which is a crucial condition for the existence of the solutions for problem (1.3) (see [28]).

Remark 3.7. Notice that in the case of constant exponents p and γ, Theorem 3.4 becomes a special case of Theorem 3.1, and we have

$$q_0 = \frac{N(p-1-\gamma)}{N-p} \quad \text{and} \quad q_1 = \frac{N(p-1-\gamma)}{N-1-\gamma},$$

which coincide with the exponents obtained in [3, 18] while studying the existence of entropy solutions for the Dirichlet problem (1.3) in the framework of constant exponents.

Remark 3.8. It is worth noting that if $\gamma(\cdot) = \gamma$ is constant, then we get

$$q_1(x) \leq \frac{N(p(x)-1-\gamma)}{N-1-\gamma} \quad \text{in } \overline{\Omega}.$$

Therefore, the particular case of Theorem 3.1 when $\underline{p} > 2 - \frac{1-\overline{\gamma}(N-1)}{N}$ is included in Theorem 3.4 as a special case. However, when $\gamma(\cdot)$ is variable, we can not compare $q_1(\cdot)$ with $\frac{N(p(\cdot)-1-\overline{\gamma})}{N-1-\overline{\gamma}}$.

4. Approximate problem and uniform estimates

To prove our main results, let us consider the sequence of approximate problems

$$\begin{cases} u_n \in \mathcal{K}_\psi, \\ \displaystyle\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla(u_n-v)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \leq \int_\Omega f_n\,(u_n-v)\,dx, \quad \forall v \in \mathcal{K}_\psi, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$. Since $f_n \in L^\infty(\Omega)$, it follows from the result of [31, Theorem 3.1] that, for fixed $n \in \mathbb{N}$, problem (4.1) has at least one solution $u_n \in \mathcal{K}_\psi$.

In the rest of this section, let u_n be a solution of (4.1). We prove some uniform estimates for the solutions of (4.1) in the Marcinkiewicz spaces with variable exponent.

Lemma 4.1. *Assume (1.1), (1.2), (1.4)–(1.6). Then there exist two positive constants c_1 and c_2, not depending on n, such that*

$$\int_{B_k^n} |\nabla u_n|^{p(x)}\,dx \leq c_1(2+k)^{\overline{\gamma}}, \qquad \forall k \geq \|\psi\|_{L^\infty(\Omega)}, \tag{4.2}$$

$$\int_{A_k^n} |\nabla u_n|^{p(x)}\,dx \leq c_2(1+k)^{1+\overline{\gamma}}, \quad \forall k > 0, \tag{4.3}$$

where $A_k^n = \{x \in \Omega : |u_n(x)| < k\}$ and $B_k^n = \{x \in \Omega : k \leq |u_n(x)| < k+1\}$.

Proof. For a fixed integer $k > 0$, we introduce the function $\varphi_k : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_k(s) = T_{k+1}(s) - T_k(s) = \begin{cases} 1, & \text{if } s \geq k+1, \\ s-k, & \text{if } k \leq s < k+1, \\ 0, & \text{if } 0 \leq s < k, \\ -\varphi_k(-s), & \text{if } s < 0. \end{cases} \tag{4.4}$$

Let $k \geq \|\psi\|_{L^\infty(\Omega)}$, using

$$v = u_n - \varphi_k(u_n),$$

as a test function in (4.1), we obtain

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi_k(u_n)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \leq \int_\Omega f_n\varphi_k(u_n)\,dx, \quad \forall k \geq \|\psi\|_{L^\infty(\Omega)},$$

which implies

$$\int_{B_k^n} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \leq \|f\|_{L^1(\Omega)}.$$

Thanks to condition (1.5) and since $|T_n(u_n)| \leq k+1$ on B_k^n, we have

$$\begin{aligned} \alpha\int_{B_k^n} \frac{|\nabla u_n|^{p(x)}}{(2+k)^{\overline{\gamma}}}\,dx &\leq \alpha\int_{B_k^n} \frac{|\nabla u_n|^{p(x)}}{(1+|T_k(u_n)|)^{\overline{\gamma}}}\,dx \\ &\leq \alpha\int_{B_k^n} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\ &\leq \int_{B_k^n} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\ &\leq \|f\|_{L^1(\Omega)}. \end{aligned}$$

Let

$$v = u_n - T_k(u_n - \psi), \quad k > 0.$$

It is easy to see that $v \in \mathcal{K}_\psi$. Hence, taking v as a test function in problem (4.1), we obtain

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_k(u_n-\psi)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \leq \int_\Omega f_n T_k(u_n-\psi)\,dx.$$

By assumption (1.5), we get

$$\begin{aligned} &\alpha\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\ &\qquad \leq k\|f\|_{L^1(\Omega)} + \int_{\{u_n-\psi<k\}} \frac{a(x,\nabla u_\varepsilon)\cdot\nabla\psi}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx. \end{aligned} \tag{4.5}$$

Now we estimate the second term on the right-hand side of (4.5) using (1.4) and Young's inequality with $\eta > 0$

$$
\begin{aligned}
\int_{\{u_n-\psi<k\}} & \frac{a(x,\nabla u_n)\cdot\nabla\psi}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\le \beta\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)-1}|\nabla\psi|}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\le \beta\eta\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\quad + C(\eta)\int_{\{u_n-\psi<k\}} \frac{|\nabla\psi|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\le \beta\eta\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx + C. \qquad (4.6)
\end{aligned}
$$

Combining (4.5) and (4.6) gives

$$
\begin{aligned}
\alpha\int_{\{u_n-\psi<k\}} & \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\le k\|f\|_{L^1(\Omega)} + \beta\eta\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx + C.
\end{aligned}
$$

Choosing η such that $\alpha = 2\beta\eta$, we find

$$
\frac{\alpha}{2}\int_{\{u_n-\psi<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \le k\|f\|_{L^1(\Omega)} + C. \qquad (4.7)
$$

Replacing k by $k + \|\psi\|_{L^\infty(\Omega)}$ in (4.7) and noting that $\{|u_n| < k\} \subset \{|u_n - \psi| < k + \|\psi\|_{L^\infty(\Omega)}\}$, we get

$$
\frac{\alpha}{2}\int_{\{|u_n|<k\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \le C(1+k),
$$

which yields (4.3). Therefore, Lemma 4.1 is completely proved. ∎

Lemma 4.2. *Under the assumptions of Theorem 3.1, there exists a positive constant M independent of n, such that*

$$
\int_{\{|u_n|>k\}} k^{q(x)}\,dx \le M, \quad \forall k > 0,
$$

with $q(x) = p^*(x)(1 - \frac{1+\overline{\gamma}}{\underline{p}})$.

Proof. We follow the techniques used in [27] with some improvements based on the degenerate coercivity.

Case (1): $0 < k \leq 1$. In this case, it is clear that $u_n \in \mathcal{M}^{q(\cdot)}(\Omega)$ and

$$\int_{\{|u_n|>k\}} k^{q(x)}\, dx \leq \text{meas}(\Omega).$$

Case (2): $k \geq 1$. Thanks to (4.3), we have $T_k(u_n) \in W_0^{1,p(\cdot)}(\Omega)$. Let $\varphi = \frac{T_k(u_n)}{k}$, then

$$\begin{aligned} \int_{\{|u_n|>k\}} k^{q(x)}\, dx &= \int_{\{|\varphi|=1\}} k^{q(x)} |\varphi|^{p^*(x)}\, dx \\ &\leq \int_\Omega (k^\sigma |\varphi|)^{p^*(x)}\, dx, \end{aligned} \tag{4.8}$$

where $\sigma = (1 - \frac{1+\overline{\gamma}}{\underline{p}})$. The last term in (4.8) can be estimated using the Sobolev inequality (2.7) and Proposition 2.1

$$\begin{aligned} \int_\Omega (k^\sigma |\varphi|)^{p^*(x)}\, dx &\leq \|k^\sigma \varphi\|^\eta_{L^{p^*(\cdot)}(\Omega)} \\ &\leq C \|\nabla(k^\sigma \varphi)\|^\eta_{L^{p(\cdot)}(\Omega)} \\ &\leq C \left(\int_\Omega |\nabla(k^\sigma \varphi)|^{p(x)}\, dx \right)^{\frac{\eta}{\delta}} \\ &\leq C \left(\int_\Omega \frac{|\nabla T_k(u_n)|^{p(x)}}{k^{1+\overline{\gamma}}} k^{p(x)(\sigma-1)+1+\overline{\gamma}}\, dx \right)^{\frac{\eta}{\delta}}, \end{aligned} \tag{4.9}$$

where

$$\eta = \begin{cases} \overline{p^*}, & \text{if } \|k^\sigma \varphi\|_{L^{p^*(\cdot)}(\Omega)} \geq 1, \\ \underline{p^*}, & \text{if } \|k^\sigma \varphi\|_{L^{p^*(\cdot)}(\Omega)} \leq 1, \end{cases} \qquad \delta = \begin{cases} \overline{p}, & \text{if } \|\nabla(k^\sigma \varphi)\|_{L^{p(\cdot)}(\Omega)} \leq 1, \\ \underline{p}, & \text{if } \|\nabla(k^\sigma \varphi)\|_{L^{p(\cdot)}(\Omega)} \geq 1. \end{cases}$$

Note that $k^{p(x)(\sigma-1)+1+\overline{\gamma}} \leq 1$ since $k > 1$ and $p(x)(\sigma - 1) + 1 + \overline{\gamma} \leq 0$. Finally, combining (4.8) and (4.9) leads to the desired result, which completes the proof of Lemma 4.2. ■

The following lemma is proved in much the same way as [20, Lemma 2.2], so we shall omit the proof.

Lemma 4.3. *If there is a positive constant M, independent of n, such that*

$$\int_{\{|u_n|>k\}} k^{q(x)}\, dx \leq M, \quad \forall k > 0,$$

for some $q \in \mathcal{P}_0(\Omega)$*, then, under estimate* (4.3)*, there holds* $|\nabla u_n|^{\xi(x)} \in \mathcal{M}^{q(\cdot)}(\Omega)$*, where* $\xi(x) = \frac{p(x)}{q(x)+1+\gamma(x)}$*. Moreover, there exists a positive constant* M_1*, independent of* n*, such that*

$$\int_{\{|\nabla u_n|^{\xi(x)}>k\}} k^{q(x)}\,dx \le M_1, \quad \forall k > 0. \tag{4.10}$$

Lemma 4.4. *Under the assumptions of Theorem* 3.4*, there exists a positive constant* C*, independent of* n*, such that*

$$\|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \le C, \tag{4.11}$$

$$\|u_n\|_{L^{q^*(\cdot)}(\Omega)} \le C, \tag{4.12}$$

for all continuous functions $q(\cdot)$ *on* $\overline{\Omega}$ *satisfying* (3.4).

Proof. Here, our technique follows [8, 12]. First, we note that assumption (3.3) implies that

$$1 < \frac{N(p(x)-1-\overline{\gamma})}{N-1-\overline{\gamma}}, \quad \forall x \in \overline{\Omega}.$$

From (1.1) and (3.4), we deduce

$$q(x) < p(x), \quad \forall x \in \overline{\Omega}.$$

The proof proceeds in two steps.

Step 1: Suppose that $\overline{q}$ satisfies $1 \le \overline{q} < \frac{N(\underline{p}-1-\overline{\gamma})}{N-1-\overline{\gamma}}$. Then it follows that $\overline{q} < \underline{p}$ and

$$\frac{\underline{p}-\overline{q}}{\overline{p}}\overline{q}^* - \overline{\gamma} > 1. \tag{4.13}$$

In view of the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$, we deduce from (4.2) and Hölder's inequality

$$\begin{aligned}\int_{B_k^n} |\nabla u_n|^{\overline{q}}\,dx &\le \left(\int_{B_k^n} |\nabla u_n|^{\underline{p}}\,dx\right)^{\frac{\overline{q}}{\underline{p}}} (\text{meas}(B_k^n))^{1-\frac{\overline{q}}{\underline{p}}} \\ &\le c_1(2+k)^{\frac{\overline{q\gamma}}{\underline{p}}} (\text{meas}(B_k^n))^{1-\frac{\overline{q}}{\underline{p}}}.\end{aligned} \tag{4.14}$$

Now, let $k_0 \ge \max(2, \|\psi\|_{L^\infty(\Omega)})$, then for all $k \ge k_0$, we have

$$\int_{B_k^n} |\nabla v|^{\overline{q}}\,dx \le c_2 k^{\frac{\overline{q\gamma}}{\underline{p}}} (\text{meas}(B_k^n))^{1-\frac{\overline{q}}{\underline{p}}}.$$

Using this and (4.3), we get

$$\int_\Omega |\nabla u_n|^{\overline{q}}\,dx \le \sum_{k=0}^{k_0-1}\int_{B_k^n}|\nabla u_n|^{\overline{q}}\,dx + \sum_{k=k_0}^{+\infty}\int_{B_k^n}|\nabla u_n|^{\overline{q}}\,dx$$
$$\le c_3 + c_4\sum_{k=k_0}^{+\infty} k^{\frac{\overline{q}\overline{\gamma}}{\underline{p}}}(\mathrm{meas}(B_k^n))^{1-\frac{\overline{q}}{\underline{p}}}. \quad (4.15)$$

Clearly, $\mathrm{meas}(B_k^n) \le \frac{1}{k^{\overline{q}^*}}\int_{B_k^n}|u_n|^{\overline{q}^*}\,dx$ for all $k \ge k_0$. From this estimate and invoking Hölder's inequality again, we obtain

$$\int_\Omega |\nabla u_n|^{\overline{q}}\,dx \le c_3 + c_4\sum_{k=k_0}^{+\infty}\frac{1}{k^{\frac{\underline{p}-\overline{q}}{\underline{p}}\overline{q}^*-\frac{\overline{q}\overline{\gamma}}{\underline{p}}}}\left(\int_{B_k^n}|\nabla u_n|^{\overline{q}^*}\,dx\right)^{\frac{\underline{p}-\overline{q}}{\underline{p}}}$$
$$\le c_3 + c_4\left(\sum_{k=k_0}^{+\infty}\frac{1}{k^{\frac{\underline{p}-\overline{q}}{\overline{p}}\overline{q}^*-\overline{\gamma}}}\right)^{\frac{\overline{q}}{\underline{p}}}\left(\sum_{k=k_0}^{+\infty}\int_{B_k^n}|u_n|^{\overline{q}^*}\,dx\right)^{\frac{\underline{p}-\overline{q}}{\underline{p}}}. \quad (4.16)$$

Thanks to (4.13), we deduce from (4.16) that

$$\int_\Omega |\nabla u_n|^{\overline{q}}\,dx \le c_3 + c_5\|u_n\|_{L^{\overline{q}^*}(\Omega)}^{\frac{\underline{p}-\overline{q}}{\underline{p}}\overline{q}^*}.$$

The Sobolev inequality gives

$$\|u_n\|_{L^{\overline{q}^*}(\Omega)}^{\overline{q}} \le c_6 + c_7\|u_n\|_{L^{\overline{q}^*}(\Omega)}^{\sigma} \quad \text{with } \sigma = \frac{\underline{p}-\overline{q}}{\underline{p}}\overline{q}^*. \quad (4.17)$$

It is easy to verify that the condition $\underline{p} < N$ implies $\sigma < \overline{q}$. Thus, we conclude that

$$\|u_n\|_{W_0^{1,\overline{q}}(\Omega)} \le C \quad \text{for all } 1 \le \overline{q} < \frac{N(\underline{p}-1-\overline{\gamma})}{N-1-\overline{\gamma}}. \quad (4.18)$$

In particular, there exists a constant $C' > 0$, independent of n, such that

$$\|u_n\|_{L^1(\Omega)} \le C'. \quad (4.19)$$

Step 2: Let $q \in C(\overline{\Omega})$ satisfying (3.4). The continuity of $p(\cdot)$ and $q(\cdot)$ on $\overline{\Omega}$ guarantees the existence of a constant $\delta > 0$ such that

$$\max_{y\in B(x,\delta)\cap\Omega} q(y) < \min_{y\in B(x,\delta)\cap\Omega}\frac{N(\overline{p}(y)-1-\overline{\gamma})}{N-1-\overline{\gamma}} \quad \text{for all } x \in \Omega, \quad (4.20)$$

where $B(x,\delta)$ denotes the open ball of centre x and radius δ. Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of open balls $(B_j)_{j=1,\dots,m}$. Moreover, there exists a constant $\nu > 0$, such that

$$\delta > |\Omega_j| = \text{meas}(\Omega_j) > \nu, \quad \Omega_j = B_j \cap \Omega \quad \text{for all } j = 1,\dots,m. \tag{4.21}$$

We denote by $\overline{q_j}$ the local maximum of $q(\cdot)$ on $\overline{\Omega_j}$ (respectively $\underline{p_j}$ the local minimum of $p(\cdot)$ on $\overline{\Omega_j}$). Therefore, (4.21) implies that

$$\overline{q_j} < \frac{N(\underline{p_j} - 1 - \overline{\gamma})}{N - 1 - \overline{\gamma}} \quad \text{for all } j = 1,\dots,m. \tag{4.22}$$

Now locally, using the same arguments as before, we find that the estimates (4.14) and (4.15) hold on $\Omega_j \cap B_k^n$ and Ω_j, respectively. In particular, it is easy to check that, instead of the global estimate (4.16), we find

$$\begin{aligned}\int_{\Omega_j} |\nabla u_n|^{\overline{q_j}}\,dx \\ &\leq c_3 + c_4\left(\sum_{k=k_0}^{+\infty} \frac{1}{k^{((\underline{p_j}-\overline{q_j})/p_j^+)\overline{q_j}^*-\overline{\gamma}}}\right)^{\frac{\overline{q_j}}{\underline{p_j}}}\left(\sum_{k=k_0}^{+\infty}\int_{\Omega_j\cap B_k^n}|u_n|^{\overline{q_j}^*}\,dx\right)^{\frac{\underline{p_j}-\overline{q_j}}{\underline{p_j}}} \\ &\leq c_3 + c_5\|u_n\|_{L^{\overline{q_j}^*}(\Omega_j)}^{\frac{\underline{p_j}-\overline{q_j}}{\underline{p_j}}\overline{q_j}^*}.\end{aligned} \tag{4.23}$$

We denote by $\tilde{u}_{nj}$ the average of u_n over Ω_j

$$\tilde{u}_{nj} = \frac{1}{\text{meas}(\Omega_j)}\int_{\Omega_j} u_n(x)\,dx.$$

From (4.19) and (4.21), we have

$$|\tilde{u}_{nj}| \leq \frac{C'}{\nu}. \tag{4.24}$$

By virtue of the Poincaré–Wirtinger inequality, we obtain

$$\|u_n - \tilde{u}_{nj}\|_{L^{\overline{q_j}^*}(\Omega_j)} \leq c_6\|\nabla u_n\|_{L^{\overline{q_j}}(\Omega_j)}. \tag{4.25}$$

In view of (4.23), (4.24) and (4.25), we deduce

$$\|u_n\|_{L^{\overline{q_j}^*}(\Omega_j)}^{\overline{q_j}} \leq c_7 + c_8\|u_n\|_{L^{\overline{q_j}^*}(\Omega_j)}^{\sigma} \quad \text{with } \sigma = \frac{\underline{p_j} - \overline{q_j}}{\underline{p_j}}\overline{q_j}^*. \tag{4.26}$$

Clearly, (4.22) and the condition $\underline{p} < N$ imply $\sigma < \overline{q}$, and we can therefore conclude that

$$\|u_n\|_{L^{\overline{q_j}^*}(\Omega_j)} \leq c_9, \quad \text{for all } j = 1,\dots,m. \tag{4.27}$$

Note that $q(x) \le \overline{q_j}$ and $q(x) \le q^*(x) \le \overline{q_j}^*$ for all $x \in \overline{\Omega}_j$ and for all $j = 1, \dots, m$. Thus, from (4.27) and (4.23), we have the desired result

$$\|u_n\|_{L^{q^*(\cdot)}(\Omega)} + \|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \le c_{10}.$$

This finishes the proof of Lemma 4.4. ∎

5. The strong convergence of the truncations

Employing the uniform estimates obtained in the previous section, we are able to get the strong compactness of the truncations.

Proposition 5.1. *Assume that hypotheses* (1.1), (1.2), (1.4)–(1.6) *hold true and let u_n be a sequence of solutions to* (4.1). *Then, there exists a subsequence of u_n (still denoted by u_n) and a function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ such that $u \ge \psi$ and*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } W_0^{1,p(\cdot)}(\Omega), \tag{5.1}$$

as $n \to +\infty$, for every $k > 0$.

Proof. We will proceed with the proof in two steps.

Step 1: The almost everywhere convergence of u_n in Ω. We claim that (u_n) is a Cauchy sequence in measure. Indeed, let $\delta > 0$, we have

$$\{|u_n - u_m| > \delta\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > \delta\},$$

which implies that

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\epsilon > 0$, by invoking Lemma 2.4, we may choose $k = k(\epsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\epsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \le \frac{\epsilon}{3}. \tag{5.2}$$

From estimate (4.3), it follows that the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Then, up to a subsequence (not relabelled)

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega) \quad \text{as } n \to +\infty.$$

Thanks to the compact embedding (2.7), we get

$$T_k(u_n) \to \eta_k \quad \text{strongly in } L^{r(\cdot)}(\Omega) \text{ and a.e in } \Omega,$$

for all $r \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*(x)$ in $\overline{\Omega}$. Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure. Thus,

$$\text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \delta\big\} \leq \frac{\epsilon}{3} \quad \text{for all } m, n \geq n_0(\epsilon, \delta). \tag{5.3}$$

Combining this with (5.2) yields

$$\forall \delta, \epsilon > 0, \exists n_0(\epsilon, \delta) \in \mathbb{N}, \forall n, m \geq n_0(\epsilon, \delta) : \text{meas}\{|u_n - u_m| > \delta\} \leq \epsilon,$$

which proves that the sequence (u_n) is a Cauchy sequence in measure and then it converges almost everywhere to some measurable function u, thus

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega). \tag{5.4}$$

Step 2: The strong convergence of the truncation $T_k(u_n)$. Let $m > k \geq \|\psi\|_\infty$, we set $h_m(s) = 1 - |\varphi_m(s)|$ and $v = u_n - h_m(u_n)(T_k(u_n) - T_k(u))$ where φ_m is defined as in (4.4). Since $v \in W_0^{1,p(\cdot)}(\Omega)$ and $v \geq \psi$, v is an admissible test function to the approximate problem (4.1), so we have

$$\overbrace{\int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla u_n h'_m(u_n)(T_k(u_n) - T_k(u))}{(1 + |T_n(u_n)|)^{\gamma(x)}}\, dx}^{I_1}$$
$$+ \overbrace{\int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u)) h_m(u_n)}{(1 + |T_n(u_n)|)^{\gamma(x)}}\, dx}^{I_2}$$
$$\leq \overbrace{\int_\Omega T_n(f) h_m(u_n)(T_k(u_n) - T_k(u))\, dx}^{I_3} .$$

Hereafter, we denote $\omega(n, m)$ (as in [20]) for all quantities, possibly different, such that $\lim_{m\to+\infty} \lim_{n\to+\infty} \omega(n, m) = 0$. That is to say, in the limit process for $\omega(n, m)$, first let $n \to +\infty$ for fixed m, then let m tend to infinity. Similarly, the notation $\omega(n)$ represents all quantities, maybe different, such that $\lim_{n\to+\infty} \omega(n) = 0$. Our aim is to prove that for all $k > 0$

$$\lim_{n\to+\infty} \int_\Omega \big[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))\big] \cdot \nabla(T_k(u_n) - T_k(u))\, dx = 0. \tag{5.5}$$

First of all, using the fact that $h_m(u_n)(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$, we get $I_3 = \omega(n)$.

Next, we take $v = u_n - \varphi_m(u_n)$ as a test function in (4.1). The almost everywhere convergence of u_n to u implies $\varphi_m(u_n) \to \varphi_m(u)$ as $n \to +\infty$, and $\varphi_m(u_n) \to 0$ as

$m \to +\infty$. Therefore, thanks to (1.5) we obtain

$$
\begin{aligned}
\alpha \int_\Omega \frac{|\nabla \varphi_m(u_n)|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx &\le \int_\Omega f_n \varphi_m(u_n)\,dx \\
&= \int_\Omega (f_n - f)\varphi_m(u_n)\,dx \\
&\quad + \int_\Omega f(\varphi_m(u_n) - \varphi_m(u))\,dx + \int_\Omega f\varphi_m(u)\,dx \\
&= \omega(n,m). \qquad (5.6)
\end{aligned}
$$

We employ (1.4) and (5.6) to obtain, using Young's inequality,

$$
\begin{aligned}
|I_1| &\le \int_\Omega \frac{|h'_m(u_n)||\nabla u_n|^{p(x)}|T_k(u_n) - T_k(u)|}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&\le 2k \int_\Omega \frac{|\nabla \varphi_m(u_n)|^{p(x)}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\
&= \omega(n,m).
\end{aligned}
$$

We are left with the estimate of I_2 which can be split as follows

$$
\begin{aligned}
I_2 = &\overbrace{\int_\Omega \frac{a(x,\nabla T_k(u_n))\cdot \nabla (T_k(u_n) - T_k(u))h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx}^{J_1} \\
&- \overbrace{\int_{\{|u_n|>k\}} \frac{a(x,\nabla u_n)\cdot \nabla T_k(u) h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx}^{J_2}.
\end{aligned}
$$

Let $n > m+1$, since h_m has compact support, the integral J_2 is taken on the subset $\{|u_n| \le m+1\}$, so J_2 can be written as follows:

$$
J_2 = \int_\Omega \frac{a(x,\nabla T_{m+1}(u_n))\cdot \nabla T_k(u) h_m(u_n)\chi_{\{|u_n|>k\}}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx.
$$

From (5.4), we deduce that the sequence

$$
\left\{ \frac{a(x,\nabla T_{m+1}(u_n))h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma(x)}} \right\}_n,
$$

weakly converges in $(L^{p'(\cdot)}(\Omega))^N$. On the other hand $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ strongly converges to zero in $(L^{p(\cdot)}(\Omega))^N$, so that we get

$$
J_2 = \omega(n).
$$

Noting that for $n > m+1 > m > k$, $h_m(u_n) = 1$ on the set $\{|u_n| \le k\}$, then J_1 simplifies to

$$\begin{aligned} J_1 &= \int_\Omega \frac{a(x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \\ &= \overbrace{\int_{\{|u_n|\le k\}} \frac{[a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx}^{K_1} \\ &\quad + \overbrace{\int_\Omega \frac{a(x,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))\chi_{\{|u_n|\le k\}}}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx}^{K_2}. \end{aligned}$$

In view of (1.4) and using Lebesgue's dominated convergence theorem, we conclude that

$$\left\{\frac{a(x,\nabla T_k(u))}{(1+|T_n(u_n)|)^{\gamma(x)}}\right\}_n,$$

converges strongly in $(L^{p'(\cdot)}(\Omega))^N$, by (5.4) we also deduce that $K_2 = \omega(n)$.

Based on the previous estimates, by (1.6) and the fact that the integral K_1 is taken on the subset $\{|u_n| \le k\}$, we finally get

$$\begin{aligned} \omega(n,m) &= K_1 \\ &\ge \frac{1}{(1+k)^{\overline{\gamma}}}\int_\Omega \big[a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\big]\cdot\nabla(T_k(u_n)-T_k(u))\,dx \\ &\ge 0. \end{aligned}$$

Therefore, (5.5) is proved. Under assumptions (1.4)–(1.6), it is well known that (5.5) implies

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega) \text{ for all } k > 0.$$

This affirms that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \qquad \blacksquare$$

6. Existence of entropy solutions

Let u_n be a solution to (4.1) and let $v \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $v(x) \ge \psi(x)$ in Ω. For a fixed $k > 0$, the function

$$u_n - T_k(u_n - v),$$

is an admissible test function in (4.1). With this choice of test function we get

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_k(u_n - v)}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \le \int_\Omega f_n T_k(u_n - v)\,dx. \tag{6.1}$$

Since on the set $\{x \in \Omega; |u_n - v| < k\}$ we have $|u_n| \le h = k + \|v\|_{L^\infty(\Omega)}$, therefore, (6.1) can be written as

$$\int_\Omega \chi_n \frac{a(x,\nabla u_n)\cdot\nabla u_n}{\big(1+|T_n(u_n)|\big)^{\gamma(x)}}\,dx + \int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla v\,dx \le \int_\Omega f_n T_k(u_n - v)\,dx, \tag{6.2}$$

where $\chi_n(x) = \chi_{\{|u_n - v|<k\}}(x)$, and $A(x,u_n,\nabla u_n) = \frac{a(x,\nabla T_h(u_n))}{(1+|T_n(u_n)|)^{\gamma(x)}}\chi_n$. Let us pass to the limit in (6.2). On the right-hand side, it is easy since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n - v)$ converges to $T_k(u-v)$ weakly$*$ in $L^\infty(\Omega)$. As for the first term on the left-hand side, we have, by using Fatou's lemma

$$\int_\Omega \chi\,\frac{a(x,\nabla u)\cdot\nabla u}{(1+|u|)^{\gamma(x)}}\,dx \le \liminf_{n\to+\infty}\int_\Omega \chi_n\,\frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx,$$

where $\chi(x) = \chi_{\{|u-v|<k\}}(x)$. In the second term of the left-hand side we have, by using (1.4), the boundedness of the sequence $T_h(u_n)$ in $W_0^{1,p}(\Omega)$, and the almost everywhere convergence of ∇u_n in Ω to ∇u, we deduce that

$$\|A(x,u_n,\nabla u_n)\|_{(L^{p'(\cdot)}(\Omega))^N} \le C, \tag{6.3}$$

and

$$A(x,u_n,\nabla u_n) \to A(x,u,\nabla u) \quad \text{a.e. in } \Omega, \tag{6.4}$$

where $A(x,u,\nabla u) = \frac{a(x,\nabla T_h(u))}{(1+|u|)^{\gamma(x)}}\chi(x)$. By (6.3), (6.4), and using Vitali's theorem, we can conclude that

$$A(x,u_n,\nabla u_n) \to A(x,u,\nabla u) \quad \text{weakly in } (L^{p'(\cdot)}(\Omega))^N.$$

Hence,

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla v\,dx \to \int_\Omega A(x,u,\nabla u)\cdot\nabla v\,dx.$$

Letting n tend to infinity in (6.2) yields

$$\int_\Omega \chi\frac{a(x,\nabla u)\cdot\nabla u}{(1+|u|)^{\gamma(x)}}\,dx + \int_\Omega \chi\frac{a(x,\nabla T_h(u))\cdot\nabla v}{(1+|u|)^{\gamma(x)}}\,dx \le \int_\Omega f\,T_k(u-v)\,dx.$$

Since $T_h(u) = u$ on the set $\{x \in \Omega : |u - v| < k\}$, the previous inequality can be rewritten as

$$\int_\Omega \frac{a(x,\nabla u)\cdot\nabla T_k(u-v)}{(1+|u|)^{\gamma(x)}}\,dx \le \int_\Omega f\,T_k(u-v)\,dx.$$

This proves that u is an entropy solution of the obstacle problem $(\mathcal{A}, f, \psi)$.

7. Lewy–Stampacchia inequalities

In this section we assume the hypotheses of Theorem 3.1 are fulfilled.

Let us now consider the sequence of operators $\mathcal{A}_n : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ defined by

$$\mathcal{A}_n(v) = -\mathrm{div}\Big(\frac{a(x,\nabla v)}{(1+|T_n(v)|)^{\gamma(x)}}\Big),$$

with $p(\cdot)$ is log-Hölder continuous and verify $p(\cdot)-1 \ll q_1(\cdot)$.

Let u_n be a solution of the approximate obstacle problem (4.1), then using the same arguments as in [27], we can prove easily that

$$\frac{a(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\gamma(x)}} \to \frac{a(x,\nabla u)}{(1+|u|)^{\gamma(x)}} \quad \text{strongly in } (L^1(\Omega))^N.$$

Note that for each $n \in \mathbb{N}$, the operator $\mathcal{A}_n$ is a pseudomonotone coercive operator and satisfying to the hypothesis of [22], so it follows that

$$f_n \le \mathcal{A}_n(u_n) \le f_n + (\mathcal{A}_n(\psi) - f_n)^+ \quad \text{in } W^{-1,p'(\cdot)}(\Omega).$$

In particular, the previous inequality holds in the sense of distributions.

Let $0 \le \varphi \in \mathcal{C}_0^\infty(\Omega)$, then

$$\int_\Omega f_n\,\varphi\,dx \le \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi}{(1+|T_n(u_n)|)^{\gamma(x)}}\,dx \le \int_\Omega \big(f_n + (\mathcal{A}_n(\psi)-f_n)^+\big)\varphi\,dx.$$

Since f_n converges to f in $L^1(\Omega)$ and

$$\frac{a(x,\nabla \psi)}{(1+|T_n(\psi)|)^{\gamma(x)}} \to \frac{a(x,\nabla \psi)}{(1+|\psi|)^{\gamma(x)}} \quad \text{strongly in } (L^{p'(\cdot)}(\Omega))^N,$$

by letting $n \to +\infty$ in the above inequality we obtain

$$f \le \mathcal{A}(u) \le f + (\mathcal{A}(\psi) - f)^+ \quad \text{in } \mathcal{D}'(\Omega).$$

Finally, since f and $(\mathcal{A}(\psi)-f)^+$ belong to $L^1(\Omega)$, we conclude also that $\mathcal{A}(u) \in L^1(\Omega)$ and (3.2) follows.

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