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The b-function with respect to weights of hypergeometric ideals of codimension one

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Abstract. We study the b-function with respect to weights of hypergeometric ideals $H_A(\beta)$, where A is an $n \times (n+1)$ matrix of maximal rank. We describe the b-function for any weight vector $(-\omega, \omega)$ with $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ and any parameter vector $\beta \in \mathbb{C}^n$.

1. Introduction

Hypergeometric ideals were introduced by Gel'fand, Graev, Kapranov and Zelevinskii in a series of papers [6–8], as a generalization of classical hypergeometric functions like the Gauss' function. These ideals are defined in terms of a full-rank matrix $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$ and a vector of parameters $\beta \in \mathbb{C}^d$. In [13] Saito, Sturmfels and Takayama study these ideals from an algorithmic point of view. In this computer algebra framework, they devote one section to the study of the b-function of holonomic ideals with respect to weights as a tool for some algorithms in the Weyl algebra. See also [2] for a clear exposition on this topic.

The b-function with respect to weights was studied in [13] for the case of regular hypergeometric ideals and generic parameters. In [12] it is studied for general hypergeometric systems but for particular choices of the weights, more precisely, weights along coordinate hyperplanes. In this article we describe the b-function for any weight, in the case of hypergeometric ideals associated with a matrix $A \in \mathcal{M}_{n\times(n+1)}(\mathbb{Z})$ of rank n. Such ideals are not necessarily regular. We give an explicit description of the b-function of such ideals in Theorem 3.19, which is the main result of the paper. The first step to compute the b-function is the description of the Gröbner deformations of the hypergeometric ideal, given in Proposition 3.16. For a direct computation of the b-function we need a Gröbner basis of this Gröbner deformations, but we avoid these long computations by considering different ideals, for which the computations of Gröbner basis are simpler. In this way we provide a multiple and a

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divisor of the b-function. It is not hard to see that, generically in β, they coincide. For the non-generic case, the arguments are on the degree of the b-function.

In the computation of the b-function we are led to solve certain linear systems of equations. In the case under consideration the linear systems we need to solve are the kind of systems studied in [4]. They depend on the Stirling numbers of second kind $S(\ell, k)$, which appear naturally in this context, since in the Weyl algebra we have the following relation

$$(x\partial_x)^\ell = \sum_{k=0}^{\ell} S(\ell, k) x^k \partial_x^k. \tag{1.1}$$

The choice of the matrix $A \in \mathcal{M}_{n\times(n+1)}(\mathbb{Z})$ is equivalent to the toric ideal I_A being principal, which simplifies a lot the computations. If we consider matrices such that I_A is neither principal nor homogeneous, then the computation of the b-function for any weight ω is much more involved, as can be seen in [3] for the case of a family of non-homogeneous matrices A such that $I_A \subseteq \mathbb{C}[\partial_1, \partial_2, \partial_3]$ is generated by two operators.

The computation of the b-function with respect to weights is implemented in the computer algebra system Singular [9], and we have used it to compute inspiring examples.

2. The weighted b-function of a holonomic ideal

We introduce the b-function with respect to weights of a holonomic ideal following [13]. We denote by D_n the n-th Weyl algebra over the field $\mathbb{C}$, i.e., the ring

$$\mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$$

subject to the relations

$$\partial_i x_j = x_j \partial_i + \delta_{ij}, \quad x_i x_j = x_j x_i, \quad \partial_i \partial_j = \partial_j \partial_i$$

for $1 \le i, j \le n$.

Every operator P in the Weyl algebra has a unique expansion of the form

$$P = \sum_{(\alpha,\beta)\in\mathbb{N}^{2n}} c_{\alpha,\beta} \mathbf{x}^\alpha \partial^\beta,$$

where only a finite number of coefficients $c_{\alpha,\beta}$ are not zero. We use here the usual multi-index notation, where $\mathbf{x}^\alpha$ stands for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and ∂^β for $\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

Let $<$ be a term order on D_n and let $\mathcal{G}$ be a Gröbner basis with respect to $<$. By the division algorithm in D_n, the remainder of dividing an operator $P \in D_n$ with respect to the elements in $\mathcal{G}$ is well defined. It is called the normal form of P with respect to $\mathcal{G}$, and it is denoted by $NF(P, \mathcal{G})$. Let $I \subseteq D_n$ be an ideal, and $\mathcal{G}$ a Gröbner basis of I with respect to $<$, then $P \in I$ if and only if $NF(P, \mathcal{G}) = 0$.

Definition 2.1. A vector $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ is called a *weight vector* (for the Weyl algebra) if

$$u_i + v_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n.$$

Definition 2.2. Let $u, v \in \mathbb{R}^n$. For a non-zero operator $P = \sum c_{\alpha,\beta} \mathbf{x}^\alpha \partial^\beta \in D_n$, *the initial form* of P with respect to (u, v) is

$$\mathrm{in}_{(u,v)}(P) := \sum_{\alpha u + \beta v = m} c_{\alpha,\beta} \mathbf{x}^\alpha \partial^\beta,$$

where $m = \max \{\alpha \cdot u + \beta \cdot v \mid c_{\alpha,\beta} \neq 0\}$.

This definition extends straightforwardly to ideals $I \subseteq D_n$,

$$\mathrm{in}_{(u,v)}(I) := D_n \cdot \left\{\mathrm{in}_{(u,v)}(P) \mid P \in I\right\}.$$

Definition 2.3. Given any ideal $I \subseteq D_n$, *the characteristic variety* $\mathrm{Char}(I)$ is the affine variety in $\mathbb{C}^{2n}$ defined by the characteristic ideal $\mathrm{in}_{(0,e)}(I)$, where $e = (1, \ldots, 1)$.

Theorem 2.4 ([14]). *Let I be a proper ideal in D_n. Every irreducible component of* $\mathrm{Char}(I)$ *has dimension at least n.*

Definition 2.5. An ideal $I \subseteq D_n$ is said to be *holonomic* if $\dim(\mathrm{Char}(I)) = n$, i.e., if the dimension of the characteristic variety in $\mathbb{C}^{2n}$ is as small as possible.

Let I be a left ideal in D_n and a non-zero weight vector of the form $(-\omega, \omega)$ with $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \setminus \{0\}$. For $s := \omega_1 x_1 \partial_1 + \cdots + \omega_n x_n \partial_n$ we consider the intersection

$$\mathrm{in}_{(-\omega,\omega)}(I) \cap \mathbb{C}[s],$$

which is an ideal in the principal ideal domain $\mathbb{C}[s]$.

Definition 2.6. The monic generator of $\mathrm{in}_{(-\omega,\omega)}(I) \cap \mathbb{C}[s]$ is called the b-function of the ideal I with respect to ω. It is denoted by $b_{I,\omega}(s)$.

Theorem 2.7 (see [10]). *The b-function of a holonomic ideal $I \subseteq D_n$ is not the zero polynomial for any $\omega \in \mathbb{R}^n \setminus \{0\}$.*

By definition there are two main steps in the computation of the b-function of an ideal $I \subseteq D_n$ with respect to a weight ω:

(i) First we need to compute the initial ideal $\mathrm{in}_{(-\omega,\omega)}(I)$.

(ii) Then we have to compute the intersection of $\mathrm{in}_{(-\omega,\omega)}(I)$ with the subalgebra $\mathbb{C}[s]$.

Step (ii) can be tackled by elimination, computing a Gröbner basis with respect to an appropriate elimination order. Another way is to apply the method of indeterminate coefficients to compute the minimal polynomial of

$$s = \omega_1 x_1 \partial_1 + \omega_2 x_2 \partial_2 + \cdots + \omega_n x_n \partial_n$$

as an endomorphism of $D_n/\mathrm{in}_{(-\omega,\omega)}(I)$ (see [11]). For this we need a Gröbner basis $\mathcal{G}$ of $\mathrm{in}_{(-\omega,\omega)}(I)$ with respect to a term order so that we can compute the normal form, $NF(s^k, \mathcal{G})$, for any $k \in \mathbb{Z}_{>0}$. Since the condition

$$NF(s^j, \mathcal{G}) + \sum_{k=0}^{j-1} a_k NF(s^k, \mathcal{G}) = 0$$

is equivalent to

$$s^j + \sum_{k=0}^{j-1} a_k s^k \in \mathrm{in}_{(-\omega,\omega)}(I),$$

we can use here any term order on the monomials in the Weyl algebra D_n.

We look for the smallest positive integer n such that there exists a non-trivial solution $a_0, \ldots, a_{n-1} \in \mathbb{C}$ to the equation

$$NF\big(s^n + a_{n-1}s^{n-1} + \cdots + a_0, \mathcal{G}\big) = 0.$$

Then the b-function is

$$b_{I,\omega}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0.$$

Remark 2.8. Notice that if we find a monic polynomial $b(s)$ such that

$$NF(b(s), \mathcal{G}) = 0,$$

not knowing whether $b(s)$ has minimal degree, then we can only say that $b(s)$ is a multiple of the b-function $b_{I,\omega}(s)$.

We end this section with a technical result that will simplify the computation of the normal form of powers of s. It says that when we have an element of the form $a_0 + a_1 x_1 \partial_1 + \cdots + a_n x_n \partial_n$ in a Gröbner basis $\mathcal{G}$, we can start reducing s by this element.

Lemma 2.9. *Let $<$ be any term order on D_n and let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a Gröbner basis of $D_n \cdot \{g_1, \ldots, g_m\}$ with respect to $<$, such that g_1 is of the form*

$$g_1 = a_1 x_1 \partial_1 + \cdots + a_n x_n \partial_n + a_0, \quad \textit{with } a_i \in \mathbb{C}.$$

For any $\omega \in \mathbb{R}^n \setminus \{0\}$*, let* $s = \omega_1 x_1 \partial_1 + \cdots + \omega_n x_n \partial_n$*, then,*

$$NF(s^k, \mathcal{G}) = NF\big(NF(s, g_1)^k, \mathcal{G}\big) = NF\big(NF(s, g_1)^k, \mathcal{G} \setminus \{g_1\}\big)$$

for any $k \in \mathbb{Z}_{>0}$*.*

Proof. Let us suppose that $LT_<(g_1) = a_1 x_1 \partial_1$ (hence $a_1 \neq 0$). Then, defining $\bar{s} := NF(s, \{g_1\})$, we have

$$\bar{s} = \Big(\omega_2 - \frac{a_2}{a_1}\omega_1\Big)x_2\partial_2 + \cdots + \Big(\omega_n - \frac{a_n}{a_1}\omega_1\Big)x_n\partial_n - \frac{a_0}{a_1}\omega_1.$$

The second equality of the statement is clear, since no monomial in $\bar{s}$ (and hence no monomial in $\bar{s}^k$, see equation (1.1)) is divisible by g_1.

We claim that

$$NF(s^k, \{g_1\}) = \bar{s}^k.$$

Indeed, by the multinomial theorem,

$$s^k = \sum_{\substack{i_1+\cdots+i_n=k,\\ i_j \geq 0}} \binom{k}{i_1, \ldots, i_n} (\omega_1 x_1 \partial_1)^{i_1} \cdots (\omega_n x_n \partial_n)^{i_n},$$

where

$$\binom{k}{i_1, \ldots, i_n} = \frac{k!}{i_1! \cdots i_n!},$$

because $i_1 + \cdots + i_n = k$. Since the division of s^k by g_1 occurs in fact in the commutative subring $\mathbb{C}[\theta_1, \ldots, \theta_n] \subset D_n$ (where $\theta_i = x_i \partial_i$), we have

$$\begin{aligned}
&NF(s^k, \{g_1\})\\
&\quad= \sum \binom{k}{i_1, \ldots, i_n} \Big[-\frac{a_2}{a_1}\omega_1\theta_2 - \cdots - \frac{a_n}{a_1}\omega_n\theta_n - \frac{a_0}{a_1}\omega_1\Big]^{i_1} (\omega_2\theta_2)^{i_2} \cdots (\omega_n\theta_n)^{i_n}\\
&\quad= \sum \binom{k}{i_1, \ldots, i_n} \sum \binom{i_1}{j_1, \ldots, j_n} \Big(-\frac{a_0}{a_1}\omega_1\Big)^{j_1} \Big(-\frac{a_2}{a_1}\omega_1\theta_2\Big)^{j_2} \cdots\\
&\qquad\quad \cdot \Big(-\frac{a_n}{a_1}\omega_n\theta_n\Big)^{j_n} (\omega_2\theta_2)^{i_2} \cdots (\omega_n\theta_n)^{i_n}\\
&\quad= \sum \binom{k}{i_2, \ldots, i_n, j_1, \ldots, j_n} \Big(-\frac{a_0}{a_1}\omega_1\Big)^{j_1} (\omega_2\theta_2)^{i_2} \Big(-\frac{a_2}{a_1}\omega_1\theta_2\Big)^{j_2} \cdots\\
&\qquad\quad \cdot (\omega_n\theta_n)^{i_n} \Big(-\frac{a_n}{a_1}\omega_n\theta_n\Big)^{j_n}\\
&\quad= \bar{s}^k,
\end{aligned}$$

since $j_1 + \cdots + j_n + i_2 + \cdots + i_n = k$.

Then the claim is proved and therefore

$$NF(s^k, \mathcal{G}) = NF\big(NF(s^k, \{g_1\}), \mathcal{G} \setminus \{g_1\}\big) = NF(\bar{s}^k, \mathcal{G} \setminus \{g_1\}) = NF(\bar{s}^k, \mathcal{G})$$

as we wanted. ∎

3. The b-function with weights of hypergeometric ideals of codimension one

The hypergeometric ideals were introduced by Gel'fand, Kapranov and Zelevinskii in [8] (see also [6, 7]), and are also called GKZ-systems. See [5] for a beautiful introduction on this topic.

Definition 3.1. The support of a vector $v \in \mathbb{N}^n$ is

$$\operatorname{supp}(v) = \big\{i \mid 1 \le i \le n,\ v_i > 0\big\}.$$

Any non-zero vector $v \in \mathbb{N}^n$ can be written uniquely as $v = v^+ - v^-$ where $v^+, v^- \in \mathbb{N}^n$ and have disjoint supports. Then we associate to any non-zero vector $v \in \mathbb{N}^n$ the binomial $\partial^{v^+} - \partial^{v^-} \in \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \dots, \partial_n]$.

Let $A = (a_{i,j})_{i,j}$ be a $d \times n$ matrix with integer coefficients and maximal rank $d \le n$. The toric ideal associated with A is the ideal $I_A \subseteq \mathbb{C}[\partial_1, \dots, \partial_n]$ generated by

$$\big\{\partial^{v^+} - \partial^{v^-} \mid v \in \mathbb{Z}^n,\ Av = 0\big\}.$$

Let $\beta \in \mathbb{C}^d$ be a complex vector. The matrix A and the vector β define d Euler operators

$$E_i - \beta_i := \sum_{j=1}^{n} a_{i,j} x_j \partial_j - \beta_i, \quad 1 \le i \le d.$$

The hypergeometric ideal associated with the matrix A and parameter vector β, denoted by $H_A(\beta)$, is the left ideal in the Weyl algebra

$$H_A(\beta) := D_n I_A + D_n(E_1 - \beta_1, \dots, E_d - \beta_d).$$

Remark 3.2. The hypergeometric ideal $H_A(\beta)$ gives rise to the following system of partial differential equations:

$$\sum_{j=1}^{n} (a_{i,j} x_j \partial_j - \beta_i) \bullet f = 0 \quad \text{for } i = 1, \dots, d,$$

$$(\partial^{\mathbf{u}} - \partial^{\mathbf{v}}) \bullet f = 0 \quad \text{for } \mathbf{u}, \mathbf{v} \text{ such that } A\mathbf{u} = A\mathbf{v},$$

where f is an indeterminate function and $\bullet$ stands for the natural action of differential operators on functions (i.e., $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$ and $x_i \bullet f = x_i f$ for $i = 1, \dots, n$).

Example 3.3. Let us consider the hypergeometric system associated to the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \end{pmatrix}.$$

We have that $\mathrm{Ker}(A)$ is generated by the vector $v = (4, -2, 1)$. Then we can write v as $v = v^+ - v^- = (4, 0, 1) - (0, 2, 0)$, and hence the hypergeometric ideal is

$$H_A(\beta) = D_3\big(\partial_x^4 \partial_z - \partial_y^2, x\partial_x + 3y\partial_y + 2z\partial_z - \beta_1, -x\partial_x + 4z\partial_z - \beta_2\big),$$

where we are denoting x_1, x_2, x_3 by x, y, z respectively.

Theorem 3.4 (cf. [1, 8]). *The hypergeometric ideal $H_A(\beta)$ is holonomic for any pair (A, β).*

From now on we will focus on the case of a matrix

$$A = (a_{i,j}) \in \mathcal{M}_{n\times(n+1)}(\mathbb{Z})$$

with the assumption that A has rank n. Then the kernel of A,

$$\mathrm{Ker}_{\mathbb{Z}}(A) = \big\{v \in \mathbb{Z}^{n+1} \mid Av = 0\big\},$$

is one-dimensional and hence the toric ideal is principal, of the form

$$I_A = (\partial^{v_A^+} - \partial^{v_A^-}) \subseteq \mathbb{C}[\partial],$$

for a certain vector $v_A \in \mathbb{Z}^{n+1}$. From now on we will denote v_A by v to simplify the notation.

The hypergeometric ideal $H_A(\beta)$ is then

$$H_A(\beta) = D_{n+1}\big(\partial^{v^+} - \partial^{v^-}, E_1 - \beta_1, \ldots, E_n - \beta_n\big).$$

Since the toric ideal I_A is principal these ideals are known as *hypergeometric ideals of codimension one*.

The ideal $H_A(\beta)$ is regular if and only if the operator $\partial^{v^+} - \partial^{v^-}$ is homogeneous with respect to the usual grading, i.e., if $|v^+| = |v^-|$, or equivalently if $(1, \ldots, 1)$ belongs to the $\mathbb{Q}$-row span of A.

Notation 3.5. The b-function $b_{H_A(\beta),\omega}(s)$ will be denoted by $b_{\omega,\beta}(s)$.

Definition 3.6. The *Euler space* of A is the $\mathbb{C}$-linear span of the Euler operators described by the rows of A,

$$\mathcal{E}_A = \mathbb{C} \cdot (E_1, \ldots, E_n).$$

For every $E \in \mathcal{E}_A$ there exists a complex number β_E such that $E - \beta_E \in H_A(\beta)$.

Do not be misled by the notation, while $\beta = (\beta_1, \dots, \beta_n)$ denotes a vector in $\mathbb{C}^n$, β_i and β_E are complex numbers.

Notation 3.7. For every $1 \le i \le n+1$, let $<_i$ denote any term order on D_{n+1} such that

$$x_i \partial_i <_i x_j \partial_j \tag{3.1}$$

for every $1 \le j \le n+1$ different from i.

We write

$$\mathcal{C}^+ = \{1 \le i \le n+1 \mid v_i > 0\},$$
$$\mathcal{C}^- = \{1 \le i \le n+1 \mid v_i < 0\}.$$

Notice, that for $1 \le j \le n+1$ we have $j \in \mathcal{C}^+ \cup \mathcal{C}^-$ if and only if $v_j \neq 0$. We have that

$$\mathcal{C}^+ \cup \mathcal{C}^- \subseteq \{1, \dots, n+1\},$$

but in general the inclusion is strict. Indeed, for any $1 \le j \le n+1$ we have that $j \in \mathcal{C}^+ \cup \mathcal{C}^-$ if and only if the minor of A corresponding to the columns of A except the j-th ncolumn has non-zero determinant. Let us denote this square matrix by A_j.

By hypothesis A is an n by $n+1$ matrix with maximal rank and hence

$$\mathcal{C}^+ \cup \mathcal{C}^- \neq \emptyset.$$

Remark 3.8. Notice however that it may happen that $\mathcal{C}^- = \emptyset$, and hence

$$I_A = (\partial^v - 1).$$

This occurs when the matrix A has the property that the sign of $(-1)^{i+1}|A_i|$ is constant for every $1 \le i \le n+1$ with $|A_i| \neq 0$.

Notation 3.9. For $i \in \mathcal{C}^+ \cup \mathcal{C}^-$ we denote by $A^{(i)} = (a^{(i)}_{j,k})$ the matrix equivalent to A (i.e. obtained by performing elementary operations on the rows of A) such that, when dropping the i-th column, the resulting matrix is the identity. We have that

$$A^{(i)} = A_i^{-1} A.$$

For $1 \le j \le n$ let us denote by $E^{(i)}_j$ the Euler operators corresponding to $A^{(i)}$. It is clear that every $E^{(i)}_j$ belongs to the Euler space $\mathcal{E}_A$ and hence the corresponding parameter $\beta^{(i)}_j$ is well defined.

By definition, the operators $E^{(i)}_j$ are

$$E^{(i)}_j = \begin{cases} a^{(i)}_{j,i} x_i \partial_i + x_j \partial_j & \text{if } j < i, \\ a^{(i)}_{j,i} x_i \partial_i + x_{j+1} \partial_{j+1} & \text{if } j \ge i, \end{cases}$$

where $a_{j,i}^{(i)} \in \mathbb{Q}$. We can write these operators in a compact form as follows:

$$E_j^{(i)} - \beta_j^{(i)} = a_{j,i}^{(i)} x_i \partial_i + x_{j+\delta_j^{(i)}} \partial_{j+\delta_j^{(i)}} - \beta_j^{(i)}, \tag{3.2}$$

where

$$\delta_j^{(i)} = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j \geq i. \end{cases}$$

Definition 3.10. For $i \in \mathcal{C}^+ \cup \mathcal{C}^-$ and $\beta \in \mathbb{C}^n$, we define the set

$$\mathcal{G}_{i,\beta} = \{E_1^{(i)} - \beta_1^{(i)}, \dots, E_n^{(i)} - \beta_n^{(i)}\}.$$

Lemma 3.11. *For every $i \in \mathcal{C}^+ \cup \mathcal{C}^-$, the set $\mathcal{G}_{i,\beta}$ is the reduced Gröbner basis of the ideal*

$$D_{n+1}(E_1 - \beta_1, \dots, E_n - \beta_n)$$

with respect to the term order $<_i$.

Proof. We clearly have that for any $i \in \mathcal{C}^+ \cup \mathcal{C}^-$,

$$D_{n+1}(E_1^{(i)} - \beta_1^{(i)}, \dots, E_n^{(i)} - \beta_n^{(i)}) = D_{n+1}(E_1 - \beta_1, \dots, E_n - \beta_n).$$

It is straightforward to check that the S-polynomials of elements of $\mathcal{G}_{i,\beta}$ reduce to zero (we refer to [13] for a description of the Buchberger algorithm in the Weyl algebra). Indeed, by definition of the order $<_i$, the leading term of $E_j^{(i)} - \beta_j^{(i)}$ is

$$LT_{<_i}(E_j^{(i)} - \beta_j^{(i)}) = \begin{cases} x_j \partial_j & \text{if } j < i, \\ x_{j+1}\partial_{j+1} & \text{if } j \geq i. \end{cases}$$

for $1 \leq j \leq n$. Then, for $1 \leq j < k \leq n$

$$\begin{aligned} &S(E_j^{(i)} - \beta_j^{(i)}, E_k^{(i)} - \beta_k^{(i)}) \\ &\quad = x_{k+\delta_k^{(i)}} \partial_{k+\delta_k^{(i)}} (E_j^{(i)} - \beta_j^{(i)}) - x_{j+\delta_j^{(i)}} \partial_{j+\delta_j^{(i)}} (E_k^{(i)} - \beta_k^{(i)}) \\ &\quad = a_{j,i}^{(i)} x_i \partial_i (E_k^{(i)} - \beta_k^{(i)}) - a_{k,i}^{(i)} x_i \partial_i (E_j^{(i)} - \beta_j^{(i)}) \\ &\qquad + \beta_k^{(i)} (E_j^{(i)} - \beta_j^{(i)}) - \beta_j^{(i)} (E_k^{(i)} - \beta_k^{(i)}) \longrightarrow \{0\}, \end{aligned}$$

where we are using that $j + \delta_j^{(i)} < k + \delta_k^{(i)}$ if $j < k$, and $\ell + \delta_\ell^{(i)} \neq i$ for $1 \leq \ell \leq n$. ∎

Definition 3.12. For $i \in \mathcal{C}^+ \cup \mathcal{C}^-$, we define

$$s_i(\omega, \beta) = NF(\omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}, \mathcal{G}_{i,\beta})$$

for any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\beta \in \mathbb{C}^n$.

Lemma 3.13. *For any $i \in \mathcal{C}^+ \cup \mathcal{C}^-$, and for any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\beta \in \mathbb{C}^n$, we have*

$$s_i(\omega, \beta) = \alpha_i(\omega) x_i \partial_i + \gamma_i(\omega, \beta),$$

where α_i is a linear function on ω and γ_i is a linear function on ω and β. More precisely

$$\alpha_i(\omega) = \frac{1}{v_i}\langle \omega, v \rangle$$

and

$$\gamma_i(\omega, \beta) = \langle \omega^{(i)}, A_i^{-1}\beta \rangle,$$

where $\omega^{(i)} = (\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_{n+1}) \in \mathbb{R}^n$.

Moreover, for $i, j \in \mathcal{C}^+ \cup \mathcal{C}^-$ with $i < j$ we have

$$\gamma_j(\omega, \beta) - \gamma_i(\omega, \beta) = \frac{\beta_i^{(j)}}{v_i}\langle \omega, v \rangle.$$

Proof. Recall that the leading term of $E_j^{(i)} - \beta_j^{(i)}$ with respect to $<_i$ is

$$LT_{<_i}(E_j^{(i)} - \beta_j^{(i)}) = x_{j+\delta_j^{(i)}} \partial_{j+\delta_j^{(i)}}$$

for $1 \leq j \leq n$. Then

$$\begin{aligned} &NF(\omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}, \mathcal{G}_{i,\beta}) \\ &\quad = (\omega_i - a_{1,i}^{(i)}\omega_1 - \cdots - a_{i-1,1}^{(i)}\omega_{i-1} - a_{i,i}^{(i)}\omega_{i+1} - \cdots - a_{n,i}^{(i)}\omega_{n+1}) x_i \partial_i \\ &\qquad + \beta_1^{(i)}\omega_1 + \cdots + \beta_{i-1}^{(i)}\omega_{i-1} + \beta_i^{(i)}\omega_{i+1} + \cdots + \beta_n^{(i)}\omega_{n+1}, \end{aligned}$$

and hence

$$\alpha_i(\omega) = \omega_i - a_{1,i}^{(i)}\omega_1 - \cdots - a_{i-1,1}^{(i)}\omega_{i-1} - a_{i,i}^{(i)}\omega_{i+1} - \cdots - a_{n,i}^{(i)}\omega_{n+1}$$

and

$$\gamma_i(\omega, \beta) = \beta_1^{(i)}\omega_1 + \cdots + \beta_{i-1}^{(i)}\omega_{i-1} + \beta_i^{(i)}\omega_{i+1} + \cdots + \beta_n^{(i)}\omega_{n+1}.$$

Since the kernel of A is invariant under invertible $\mathbb{Q}$-linear transformations, we have that $A^{(i)} v = \bar{0}$ and therefore

$$\begin{aligned} v_1 + a_{1,i}^{(i)} v_i &= 0, \\ &\vdots \\ v_{i-1} + a_{i-1,i}^{(i)} v_i &= 0, \\ v_{i+1} + a_{i,i}^{(i)} v_i &= 0, \\ &\vdots \\ v_{n+1} + a_{n,i}^{(i)} v_i &= 0. \end{aligned}$$

Since $v_i \neq 0$ (because $i \in \mathcal{C}^+ \cup \mathcal{C}^-$), we can write the i-th column of $A^{(i)}$ as

$$\begin{aligned} a^{(i)}_{1,i} &= -\frac{v_1}{v_i}, \\ &\vdots \\ a^{(i)}_{i-1,i} &= -\frac{v_{i-1}}{v_i}, \\ a^{(i)}_{i,i} &= -\frac{v_{i+1}}{v_i}, \\ &\vdots \\ a^{(i)}_{n,i} &= -\frac{v_{n+1}}{v_i}. \end{aligned} \tag{3.3}$$

Then the claim for α_i follows by the equalities in (3.3) while the claim for γ_i follows by definition of $\omega^{(i)}$ and the fact that $\beta^{(i)} = A_i^{-1}\beta$.

Finally, for $i, j \in \mathcal{C}^+ \cup \mathcal{C}^-$ with $i < j$, we have that $\delta_i^{(j)} = 0$ and therefore

$$E_i^{(j)} - \beta_i^{(j)} = a^{(j)}_{i,j} x_j \partial_j + x_i \partial_i - \beta_i^{(j)} = -\frac{v_i}{v_j} x_j \partial_j + x_i \partial_i - \beta_i^{(j)}.$$

Hence

$$\begin{aligned} s_i - s_j &= \frac{1}{v_i}\langle \omega, v\rangle x_i \partial_i - \frac{1}{v_j}\langle \omega, v\rangle x_j \partial_j + \gamma_i - \gamma_j \\ &= \frac{1}{v_i}\langle \omega, v\rangle (E_i^{(j)} - \beta_i^{(j)}) + \frac{\beta_i^{(j)}}{v_i}\langle \omega, v\rangle + \gamma_i - \gamma_j. \end{aligned}$$

Since $s_i - s_j$ belongs to $D_{n+1}(E_1 - \beta_1, \dots, E_n - \beta_n)$, the rest of dividing by any Gröbner bases must be zero, and the last claim of the statement follows. ■

Notation 3.14. In particular, if $\langle \omega, v\rangle = 0$ then $s_i(\omega, \beta)$ is independent of i. It will be denoted by

$$\gamma(\omega, \beta) = NF\big(\omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}, \mathcal{G}_{i,\beta}\big).$$

Example 3.15. We revisit Example 3.3 to illustrate the content of Lemma 3.13. First notice that in this case $\mathcal{C}^+ = \{1, 3\}$ and $\mathcal{C}^- = \{2\}$. The matrices A_j for $j \in \mathcal{C}^+ \cup \mathcal{C}^-$ are

$$A_1 = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix},$$

and the matrices $A^{(j)}$ are

$$A^{(1)} = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
E_1^{(1)} - \beta_1^{(1)} &= \frac{1}{2}x\partial_x + y\partial_y - \frac{2\beta_1 - \beta_2}{6},\\
E_2^{(1)} - \beta_2^{(1)} &= -\frac{1}{4}x\partial_x + z\partial_z - \frac{1}{4}\beta_2,\\
E_1^{(2)} - \beta_1^{(2)} &= x\partial_x + 2y\partial_y - \frac{2\beta_1 - \beta_2}{3},\\
E_2^{(2)} - \beta_2^{(2)} &= \frac{1}{2}y\partial_y + z\partial_z - \frac{\beta_1 + \beta_2}{6},\\
E_1^{(3)} - \beta_1^{(3)} &= x\partial_x - 4z\partial_z + \beta_2,\\
E_2^{(3)} - \beta_2^{(3)} &= y\partial_y + 2z\partial_z - \frac{\beta_1 + \beta_2}{3}.
\end{aligned}
$$

Dividing $s = \omega_1 x\partial_x + \omega_2 y\partial_y + \omega_3 z\partial_z$ by the elements $E_1^{(i)} - \beta_1^{(i)}$ and $E_2^{(i)} - \beta_2^{(i)}$ with respect to the order $<_i$, it is straightforward to check that

$$
\begin{aligned}
s_1(\omega, \beta) &= \left(\omega_1 - \frac{\omega_2}{2} + \frac{\omega_3}{4}\right)x\partial_x + \frac{2\beta_1 - \beta_2}{6}\omega_2 + \frac{\beta_2}{4}\omega_3,\\
s_2(\omega, \beta) &= \left(\omega_2 - 2\omega_1 - \frac{\omega_3}{2}\right)y\partial_y + \frac{2\beta_1 - \beta_2}{3}\omega_1 + \frac{\beta_1 + \beta_2}{6}\omega_3,\\
s_3(\omega, \beta) &= (\omega_3 + 4\omega_1 - 2\omega_2)z\partial_z - \beta_2\omega_1 + \frac{\beta_1 + \beta_2}{3}\omega_2.
\end{aligned}
$$

Once several notations are introduced together with some technical results, we proceed with computation of the initial ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$. One way to compute initial ideals is, by [13, Theorem 1.1.6], to compute a Gröbner basis for $H_A(\beta)$ with respect to $<_{(-\omega,\omega)}$, for any term order $<$.

In the first non-trivial case, where the matrix A is of size 1×2, i.e., $A = \begin{pmatrix} p & q \end{pmatrix}$, the hypergeometric ideal is

$$H_A(\beta) = D_2(\partial_x^q - \partial_y^p, px\partial_x + qy\partial_y - \beta).$$

In this simple case it is not difficult to prove that $\mathcal{G} = \{\partial_x^q - \partial_y^p, px\partial_x + qy\partial_y - \beta\}$ is such a Gröbner basis of $H_A(\beta)$, by computing the S-polynomial in the homogenized Weyl algebra, as explained in [13].

However, the complexity of the computations for a general $n \times (n+1)$ matrix grows to a point that makes it more reasonable trying another way. We will use [13, Theorem 4.3.5] instead, as we explain next.

Since the toric ideal I_A is principal and generated by a binomial, it is direct to see that, for any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\mathrm{in}_\omega(I_A) = \begin{cases} \partial^{v^+} & \text{if } \langle \omega, v \rangle > 0, \\ \partial^{v^-} & \text{if } \langle \omega, v \rangle < 0, \\ \partial^{v^+} - \partial^{v^-} & \text{if } \langle \omega, v \rangle = 0. \end{cases}$$

By [13, Theorem 3.1.3], the ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ equals the so-called *fake initial ideal*

$$\mathrm{fin}_{(-\omega,\omega)}(H_A(\beta)) := D_{n+1}\mathrm{in}_\omega(I_A) + D_{n+1}(E_1 - \beta_1, \dots, E_n - \beta_n)$$

when the parameter vector β is generic. But in the particular case we are treating of hypergeometric ideals of codimension one, we can prove that the equality holds for every $\beta \in \mathbb{C}^n$.

Proposition 3.16. Let A be a full-rank matrix of size $n \times (n+1)$. Given any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\beta \in \mathbb{C}^n$, we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = \begin{cases} D_{n+1}(\partial^{v^+}, E_1 - \beta_1, \dots, E_n - \beta_n) & \text{if } \langle \omega, v \rangle > 0, \\ D_{n+1}(\partial^{v^-}, E_1 - \beta_1, \dots, E_n - \beta_n) & \text{if } \langle \omega, v \rangle < 0, \\ H_A(\beta) & \text{if } \langle \omega, v \rangle = 0. \end{cases}$$

Proof. By [13, Theorem 4.3.5] (which holds true for non-necessarily homogeneous matrices) we have that if the first homology $H_1(K_\bullet^\beta(\mathrm{gr}^{(-\omega,\omega)}(D/DI_A)))$ vanishes, then the initial ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ equals the fake initial ideal $\mathrm{fin}_{(-\omega,\omega)}(H_A(\beta))$.

The vanishing of the first homology is equivalent to the injectivity of the morphism

$$D_{n+1}/D_{n+1}\mathrm{in}_\omega(I_A) \xrightarrow{\cdot(E-\beta_E)} D_{n+1}/D_{n+1}\mathrm{in}_\omega(I_A)$$

for every $E - \beta_E$ such that E belongs to the Euler space $\mathcal{E}_A$ of A. Recall that the number β_E depends linearly on the components of the parameter vector β. We denote the Euler operator

$$E = c_1 x_1 \partial_1 + \cdots + c_{n+1} x_{n+1} \partial_{n+1} \in \mathcal{E}_A$$

and $c = (c_1, \dots, c_{n+1})$. By $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ we denote the canonical basis of $\mathbb{R}^{n+1}$. Then, given a monomial $\mathbf{x}^u \partial^w$, we have

$$\mathbf{x}^u \partial^w (E - \beta_E) = \big(\langle c, w \rangle - \beta_E\big)\mathbf{x}^u \partial^w + \sum_{j=1}^{n+1} c_j \mathbf{x}^{u+\mathbf{e}_j} \partial^{w+\mathbf{e}_j}. \tag{3.4}$$

Let us suppose that $\langle \omega, v\rangle > 0$. The morphism

$$\cdot(E-\beta_E):\ D_{n+1}/D_{n+1}\partial^{v^+} \longrightarrow D_{n+1}/D_{n+1}\partial^{v^+}$$

is well defined since

$$\partial^{v^+}(E-\beta_E) = \Big(E-\beta_E + \sum_{i\in\mathcal{C}+} c_i v_i\Big)\partial^{v^+}.$$

Let $P=\sum_\alpha c_\alpha \mathbf{x}^{u_\alpha}\partial^{w_\alpha}\in D_{n+1}$ be an operator such that $P\notin D_{n+1}\partial^{v^+}$. The operator $P(E-\beta_E)$ belongs to $D_{n+1}\partial^{v^+}$ if and only if every monomial of $P(E-\beta_E)$ is of the form $\mathbf{x}^u\partial^w$ with $v^+\le w$, where $\le$ stands for the partial order

$$u\le w \quad \text{if and only if} \quad u_j\le w_j \quad \text{for } j=1,\dots,n+1.$$

For every $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}$ in P we have that the monomials in the expansion of $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}(E-\beta_E)$ are (see equation (3.4))

$$\begin{array}{ll} \mathbf{x}^{u_\alpha}\partial^{w_\alpha} & \text{if } \beta_E\neq\langle c, w_\alpha\rangle, \\ \mathbf{x}^{u_\alpha+\mathbf{e}_j}\partial^{w_\alpha+\mathbf{e}_j} & \text{for } 1\le j\le n+1 \text{ such that } c_j\neq 0. \end{array}$$

We claim that if $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}\notin D_{n+1}\partial^{v^+}$ then $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}(E-\beta_E)\notin D_{n+1}\partial^{v^+}$. It is obvious when $\beta_E\neq\langle c, w_\alpha\rangle$. Otherwise, notice that there is at most one $j\in\{1,\dots,n+1\}$ such that

$$\partial^{w_\alpha+\mathbf{e}_j}\in D_{n+1}\partial^{v^+}.$$

Then for any $k\in\{1,\dots,n+1\}\setminus\{j\}$ the monomial $\mathbf{x}^{u_\alpha+\mathbf{e}_k}\partial^{w_\alpha+\mathbf{e}_k}$ does not belong to $D_{n+1}\partial^{v^+}$ and appears in $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}(E-\beta_E)$ unless $c_k=0$. Therefore $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}(E-\beta_E)\in D_{n+1}\partial^{v^+}$ if and only if $\beta=\langle c, w_\alpha\rangle$, $c=\mathbf{e}_j$ and $\partial^{w_\alpha+\mathbf{e}_j}\in D_{n+1}\partial^{v^+}$. But these conditions are incompatible with $v^+\not\le w_\alpha$. Indeed, the fact that $c=\mathbf{e}_j$ belongs to the row-span of A implies that $v_j=0$, and hence $v_j^+=0$. Therefore, we have $v^+\not\le w_\alpha$ and $v^+\le w_\alpha+\mathbf{e}_j$, which leads to contradiction, and the claim is proved.

On the other hand, we have

$$P(E-\beta_E)=\sum_\alpha c_\alpha\big(\langle c, w_\alpha\rangle-\beta_E\big)\mathbf{x}^{u_\alpha}\partial^{w_\alpha}+\sum_\alpha\sum_{j=1}^{n+1} c_\alpha c_j \mathbf{x}^{u_\alpha+\mathbf{e}_j}\partial^{w_\alpha+\mathbf{e}_j}.$$

Let $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}\in D_{n+1}$ be a monomial in P such that $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}\notin D_{n+1}\partial^{v^+}$, i.e.,

$$v^+\not\le w_\alpha.$$

We have seen that, for every α with $c_\alpha\neq 0$, there exists $j\in\{1,\dots,n+1\}$ such that $c_j\neq 0$ and

$$v^+\not\le w_\alpha+\mathbf{e}_j.$$

Considering the monomials in P having this property for j fixed, we see that the monomials

$$\mathbf{x}^{u_\alpha+\mathbf{e}_j}\partial^{w_\alpha+\mathbf{e}_j},$$

such that the j-th coordinate of $w_{\alpha,j}$ is maximal, cannot cancel in $P(E-\beta_E)$ and hence $P(E-\beta_E)\notin D_{n+1}\partial^{v^+}$.

If $\langle\omega,v\rangle<0$ the proof is completely analogous.

Finally, let us suppose that $\langle\omega,v\rangle=0$. The morphism is well defined since

$$\begin{aligned}(\partial^{v^+}-\partial^{v^-})(E-\beta_E)&=(E-\beta_E)(\partial^{v^+}-\partial^{v^-})\\&\quad+\Big(\sum_{i\in\mathcal{C}+}c_iv_i^+\Big)\partial^{v^+}-\Big(\sum_{i\in\mathcal{C}-}c_iv_i^-\Big)\partial^{v^-}\\&=(E-\beta_E)(\partial^{v^+}-\partial^{v^-})+\Big(\sum_{i\in\mathcal{C}+}c_iv_i^+\Big)(\partial^{v^+}-\partial^{v^-})\end{aligned}$$

because $\langle c,v\rangle=0$ implies $\langle c,v^+\rangle=\langle c,v^-\rangle$.

To prove the injectivity of the morphism

$$\cdot(E-\beta_E):\ D_{n+1}/D_{n+1}(\partial^{v^+}-\partial^{v^-})\to\ D_{n+1}/D_{n+1}(\partial^{v^+}-\partial^{v^-}),$$

let $P=\sum_\alpha c_\alpha\mathbf{x}^{u_\alpha}\partial^{w_\alpha}\in D_{n+1}$ be an operator such that $P\notin D_{n+1}(\partial^{v^+}-\partial^{v^-})$, and let $<$ be any term order on D_{n+1} such that $LT_<(\partial^{v^+}-\partial^{v^-})=\partial^{v^+}$. We can suppose that no monomial $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}$ in P is divisible by ∂^{v^+}. Now, the difference to the previous case is that, if there exists j such that $v^+\leq w_\alpha+\mathbf{e}_j$, we can reduce by the binomial. Hence, in $\mathbf{x}^{u_\alpha}\partial^{w_\alpha}(E-\beta_E)$, instead of the monomial $\mathbf{x}^{u_\alpha+\mathbf{e}_j}\partial^{w_\alpha+\mathbf{e}_j}$ we have $\mathbf{x}^{u_\alpha+\mathbf{e}_j}\partial^{w_\alpha+\mathbf{e}_j-v^++v^-}$. Since $\operatorname{supp}(v^+)\cap\operatorname{supp}(v^-)=\emptyset$, this monomial is not reducible by $\partial^{v^+}-\partial^{v^-}$, and we can use the same argument as before to deduce that

$$NF\big(P(E-\beta_E),\{\partial^{v^+}-\partial^{v^-}\}\big)\neq 0.\qquad\blacksquare$$

Notation 3.17. For simplicity, we will drop the dependence on ω and β in $s_i(\omega,\beta)$, $\alpha_i(\omega)$ and $\gamma_i(\omega,\beta)$ (as defined in Lemma 3.13), and simply write s_i, α_i and γ_i.

For $i\in\mathcal{C}^+\cup\mathcal{C}^-$ we set the polynomial

$$b_i(s)=\prod_{j=0}^{|v_i|-1}(s-\gamma_i-j\alpha_i).$$

Remark 3.18. Recall that we may have (see Remark 3.8) a toric ideal of the form

$$I_A=(\partial^v-1).$$

In this case, if $\omega\in\mathbb{R}^{n+1}\setminus\{0\}$ is such that $\langle\omega,v\rangle<0$, then the initial ideal is

$$\operatorname{in}_{(-\omega,\omega)}(H_A(\beta))=D_{n+1}$$

and the b-function is not defined, since it is only defined for proper holonomic ideals. But for $\langle \omega, v \rangle \geq 0$ it is defined and we describe it below, with the notations introduced in Notation 3.14 and Notation 3.17.

Theorem 3.19. *Let $A \in \mathcal{M}_{n\times(n+1)}(\mathbb{Z})$ be a matrix with maximal rank. For any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ and for any $\beta \in \mathbb{C}^n$,*

$$b_{\omega,\beta}(s) = \begin{cases} \prod_{i\in\mathcal{C}^+} b_i(s) & \text{if } \langle \omega, v \rangle > 0, \\ \prod_{i\in\mathcal{C}^-} b_i(s) & \text{if } \langle \omega, v \rangle < 0, \\ s - \gamma(\omega, \beta) & \text{if } \langle \omega, v \rangle = 0 \end{cases}$$

is the b-function with respect to the weight ω of the holonomic ideal $H_A(\beta)$.

Proof. As we have seen in Section 2, there are two steps to compute the b-function of a holonomic ideal. The first one is achieved in Proposition 3.16. For the second step, both via elimination or using Noro's algorithm, we need a Gröbner basis of the initial ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$. The advantage of using the second procedure is that we can use any term order in the Weyl algebra. However, despite the simple form of the initial ideal described in Proposition 3.16, the computation of a Gröbner basis of $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ is too long and complex. Therefore, we will not compute the b-function directly. Instead we will compute a multiple of the b-function (see Remark 2.8) and some divisors of the b-function, by dealing with ideals simpler than $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$.

Let us denote by $b(s)$ the right-hand side of the equality in the statement. First we will prove that $b(s)$ is a multiple of the b-function $b_{\omega,\beta}(s)$. For this it is enough to show that for any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$

$$b(\omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}) \in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)).$$

For any operator E in the Euler space there exists $\beta_E \in \mathbb{C}$ uniquely defined such that $E - \beta_E \in H_A(\beta)$. Then it is obvious that for any $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$

$$E - \beta_E \in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)).$$

Moreover,

$$\begin{aligned} \partial^{v^+} &\in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) && \text{if } \langle \omega, v \rangle > 0, \\ \partial^{v^-} &\in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) && \text{if } \langle \omega, v \rangle < 0, \\ \partial^{v^+} - \partial^{v^-} &\in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) && \text{if } \langle \omega, v \rangle = 0. \end{aligned}$$

Let s denote the operator

$$s = \omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}.$$

First, suppose that $\langle \omega, v\rangle = 0$. Then

$$\omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1} \in \mathcal{E}_A$$

and, see Notation 3.14,

$$s = \omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1} = \sum_i c_i (E_i - \beta_i) + \gamma(\omega, \beta).$$

Then $b(s) = s - \gamma(\omega, \beta)$ is a multiple of the b-function.

Suppose that $\langle \omega, v\rangle > 0$. By Remark 3.18 we assume that $\mathcal{C}^+ \neq \emptyset$. Notice that for every $i \in \mathcal{C}^+ \cup \mathcal{C}^-$, we can write s as follows:

$$s = s_i + \sum_j \lambda_j^{(i)} (E_j - \beta_j), \tag{3.5}$$

where $\lambda_j^{(i)}$ are linear functions on ω. Then, by definition of $b(s)$ and by (3.5), we have

$$b(s) = \prod_{i \in \mathcal{C}^+} b_i \big(s_i + \sum_j \lambda_j^{(i)} (E_j - \beta_j)\big)$$

for a certain $\lambda_j^{(i)} \in \mathbb{Q}$.

If $i \in \mathcal{C}^+$, then $v_i \neq 0$ and by Lemma 3.13 we have $s_i = \alpha_i x_i \partial_i + \gamma_i$ with $\alpha_i = \alpha_i(\omega) \neq 0$. Hence

$$\begin{aligned} b_i \big(s_i + \sum_j \lambda_j^{(i)} (E_j - \beta_j)\big) &= \prod_{k=0}^{v_i - 1} \Big(\alpha_i (x_i \partial_i - k) + \sum_{j=1}^n \lambda_j^{(i)} (E_j - \beta_j)\Big) \\ &= \alpha_i^{v_i} \prod_{k=0}^{v_i - 1} (x_i \partial_i - k) + Q_i \sum_{j=1}^n \lambda_j^{(i)} (E_j - \beta_j) \end{aligned}$$

for a certain operator $Q_i \in D_{n+1}$. Then we have

$$b_i(s) = b_i \big(s_i + \sum_j \lambda_j^{(i)} (E_j - \beta_j)\big) = \alpha_i^{v_i} x_i^{v_i} \partial_i^{v_i} + Q_i \sum_{j=1}^n \lambda_j^{(i)} (E_j - \beta_j),$$

where we are using the well-known relation

$$x_i \partial_i (x_i \partial_i - 1)(x_i \partial_i - 2) \cdots (x_i \partial_i - \ell) = x_i^{\ell+1} \partial_i^{\ell+1},$$

for any positive integer ℓ and $1 \leq i \leq n+1$. Now we deduce

$$b(s) = \prod_{i \in \mathcal{C}^+} \alpha_i^{v_i} x_i^{v_i} \partial_i^{v_i} + \sum_{j=1}^n R_j (E_j - \beta_j)$$

for certain operators $R_j \in D_{n+1}$. And since

$$\prod_{i\in\mathcal{C}^+} \alpha_i^{v_i} x_i^{v_i} \partial_i^{v_i} = \Big(\prod_{i\in\mathcal{C}^+} \alpha_i^{v_i}\Big)\mathbf{x}^{v^+}\partial^{v^+}$$

we deduce that

$$b(s) \in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)),$$

as we wanted to prove.

When $\langle \omega, v\rangle < 0$, the proof is completely analogous.

Now we prove that for generic parameters $\beta \in \mathcal{C}^n$ the polynomial $b(s)$ divides the b-function $b_{\omega,\beta}(s)$, and hence it follows that $b(s) = b_{\omega,\beta}(s)$.

If $\langle \omega, v\rangle = 0$ there is nothing to prove since we know by Theorem 2.7 that the b-function is non-zero. Otherwise, suppose $\langle \omega, v\rangle > 0$ (the case $\langle \omega, v\rangle < 0$ is analogous). For any $i \in \mathcal{C}^+$ consider any term order $<_i$ satisfying (3.1). If we define the ideal

$$J_{i,\beta} := D_{n+1}(\partial_i^{v_i}, E_1 - \beta_1, \dots, E_n - \beta_n),$$

by Proposition 3.16 it is clear that we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \subseteq J_{i,\beta}.$$

We claim that

$$\bar{\mathcal{G}}_{i,\beta} := \{\partial_i^{v_i}\} \cup \mathcal{G}_{i,\beta}$$

is a Gröbner basis of $J_{i,\beta}$ with respect to $<_i$ (see Definition 3.10 for $\mathcal{G}_{i,\beta}$). Indeed, since $\mathcal{G}_{i,\beta}$ is a Gröbner basis of the ideal it generates, with respect to $<_i$, we only have to compute the S-polynomials $S(\partial_i^{v_i}, E_k^{(i)} - \beta_k^{(i)})$ for $1 \le k \le n$. First recall that the Euler operators can be written as

$$E_k^{(i)} - \beta_k^{(i)} = a_{k,i}^{(i)} x_i \partial_i + x_{k+\delta_k^{(i)}} \partial_{k+\delta_k^{(i)}} - \beta_k^{(i)},$$

where

$$\delta_k^{(i)} = \begin{cases} 0 & \text{if } k < i, \\ 1 & \text{if } k \ge i. \end{cases}$$

Notice that for $1 \le k \le n$, we have $k + \delta_k^{(i)} \ne i$. Then, since

$$LT_{<_i}(E_k^{(i)} - \beta_k^{(i)}) = x_{k+\delta_k^{(i)}} \partial_{k+\delta_k^{(i)}}$$

we have

$$\begin{aligned} S(\partial_i^{v_i}, E_k^{(i)} - \beta_k^{(i)}) &= x_{k+\delta_k^{(i)}} \partial_{k+\delta_k^{(i)}} \partial_i^{v_i} - \partial_i^{v_i}(E_k^{(i)} - \beta_k^{(i)}) \\ &= (\beta_k^{(i)} - a_{k,i}^{(i)} v_i - a_{k,i}^{(i)} x_i \partial_i)\partial_i^{v_i} \longrightarrow_{\bar{\mathcal{G}}_i} \{0\}. \end{aligned}$$

Now we claim that the ideal $J_{i,\beta}$ is holonomic. Indeed, since $\bar{\mathcal{G}}_{i,\beta}$ is a Gröbner basis of $J_{i,\beta}$ we deduce that (see Definition 2.3)

$$\mathrm{in}_{(0,e)}(J_{i,\beta}) = D_{n+1}(\partial_i^{v_i}, E_1^{(i)}, \dots, E_n^{(i)}).$$

The components of the characteristic variety are of the form

$$V(w_i, p_j)_{j\in\{1,\dots,n+1\}\setminus\{i\}} \subseteq \mathbb{C}^{2(n+1)} = \mathrm{Spec}(\mathbb{C}[u_1, \dots, u_{n+1}, w_1, \dots, w_{n+1}]),$$

where p_j is either u_j or w_j. Hence the dimension of the characteristic variety is $n+1$ and the ideal $J_{i,\beta}$ is holonomic as claimed. Let us compute its b-function $b_{J_{i,\beta},\omega}(s)$.

If $v_i = 1$ then

$$NF(s, \bar{\mathcal{G}}_{i,\beta}) = \gamma_i$$

and hence $b_{J_{i,\beta},\omega}(s) = s - \gamma_i$. Otherwise

$$NF(s, \bar{\mathcal{G}}_{i,\beta}) = \alpha_i x_i \partial_i + \gamma_i$$

and now we can only reduce with $\partial_i^{v_i}$. We cannot look for a solution to

$$NF(s^\ell + a_{\ell-1}s^{\ell-1} + \cdots + a_0, \bar{\mathcal{G}}_{i,\beta}) = 0 \tag{3.6}$$

as long as $\ell < v_i$. For $\ell = v_i$ we find a solution as follows. By Lemma 2.9 we have

$$NF(s^{v_i} + a_{v_i-1}s^{v_i-1} + \cdots + a_0, \bar{\mathcal{G}}_{i,\beta}) = NF(s_i^{v_i} + a_{v_i-1}s_i^{v_i-1} + \cdots + a_0, \bar{\mathcal{G}}_{i,\beta}),$$

and by equation (1.1)

$$s_i^\ell = \sum_{j=0}^{\ell}\sum_{k=0}^{j}\binom{\ell}{j}\alpha_i^j\gamma_i^{\ell-j}S(j,k)x_i^k\partial_i^k.$$

Hence, setting $a_{v_i} := 1$,

$$\begin{aligned} s_i^{v_i} + \cdots + a_0 &= \sum_{r=0}^{v_i} a_r\left(\sum_{j=0}^{r}\sum_{k=0}^{j}\binom{r}{j}\alpha_i^j\gamma_i^{r-j}S(j,k)x_i^k\partial_i^k\right) \\ &= \sum_{k=0}^{v_i}\left(\sum_{j=k}^{v_i}\sum_{r=j}^{v_i} a_r\binom{r}{j}\alpha_i^j\gamma_i^{r-j}S(j,k)\right)x_i^k\partial_i^k. \end{aligned}$$

And then we deduce

$$\begin{aligned} &NF(s^{v_i} + a_{v_i-1}s^{v_i-1} + \cdots + a_0, \bar{\mathcal{G}}_{i,\beta}) \\ &\quad= \sum_{k=0}^{v_i-1}\left(\sum_{j=k}^{v_i-1}\sum_{r=j}^{v_i-1} a_r\binom{r}{j}\alpha_i^j\gamma_i^{r-j}S(j,k)\right)x_i^k\partial_i^k. \end{aligned}$$

This polynomial vanishes if and only if every coefficient vanishes, i.e. for every $0 \leq k \leq v_i - 1$,

$$\sum_{j=k}^{v_i-1} \sum_{r=j}^{v_i-1} a_r \binom{r}{j} \alpha_i^j \gamma_i^{r-j} S(j,k) = 0, \tag{3.7}$$

which is a linear system of equations in $a_{v_i-1}, \ldots, a_0$.

This system is of the type studied in [4]. Let us briefly recall some definitions of this paper. Given $(k_1, k_2) \in \mathbb{Z}^2_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$, we define the polynomial

$$C(k_1, k_2, \ell) = \sum_{\substack{i_1 \geq k_1, \\ i_2 \geq k_2, \\ i_1 + i_2 \leq \ell}} \binom{\ell}{i_1} \binom{\ell - i_1}{i_2} S(i_1, k_1) S(i_2, k_2) x^{i_1} y^{i_2} z^{\ell - i_1 - i_2}.$$

Further, given a finite set of points $R \subseteq \mathbb{Z}^2_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$, we define the matrix

$$M_{R,m} = \big(C(k_1, k_2, \ell)\big)_{(k_1,k_2) \in R,\ 0 \leq \ell \leq m-1}.$$

Then the system in (3.7) can be written as

$$M_{R_i, v_i} (\mathbf{a}, 1)^t = \bar{0},$$

where $(\mathbf{a}, 1) = (a_0, \ldots, a_{v_i-1}, 1)$, the set R_i is

$$R_i = \{(0,0), (1,0), \ldots, (v_i - 1, 0)\},$$

and the polynomials $C(k_1, k_2, \ell) \in \mathbb{Z}[x, y, z]$ are evaluated at $x = \alpha_i$ and $z = \gamma_i$ (notice that for points of the form $(j, 0)$, the corresponding polynomial $C(j, 0, \ell)$ does not depend on the variable y).

By [4, Theorem 3.6] this system is solvable and its solution $(a_0, \ldots, a_{v_i-1})$ gives rise to the polynomial $b_i(s)$. Then, we have proved that

$$b_{J_{i,\beta},\omega}(s) = b_i(s)$$

and therefore, for any $1 \leq i \leq n+1$ such that $i \in \mathcal{C}^+$, the polynomial $b_i(s)$ is a divisor of the b-function.

Hence, $(\prod_{i \in \mathcal{C}^+} b_i(s))_{\text{red}}$ is a multiple of the b-function and therefore, if we have $(\prod_{i \in \mathcal{C}^+} b_i(s))_{\text{red}} = b(s)$ we are done. But this is not the case in general. For any $i \in \mathcal{C}^+$ the polynomial $b_i(s)$ is reduced, but for $i, j \in \mathcal{C}^+$ the polynomials $b_i(s)$ and $b_j(s)$ may have roots in common. Let us characterize when this occurs. There are two common roots $\gamma_i + k\alpha_i = \gamma_j + r\alpha_j$ if and only if

$$\gamma_i - \gamma_j = -k\alpha_i + r\alpha_j = \left(\frac{r}{v_j} - \frac{k}{v_i}\right)\langle \omega, v \rangle.$$

By Lemma 3.13, if $i > j$,

$$\gamma_i - \gamma_j = \frac{\beta_j^{(i)}}{v_j}\langle \omega, v\rangle$$

and since $\langle \omega, v\rangle \neq 0$ we have proved that there is a root of $b_i(s)$ equal to a root of $b_j(s)$ for different $i, j \in \mathcal{C}^+$ if and only if there exists $r \in \{0, \dots, v_i - 1\}$ and $k \in \{0, \dots, v_j - 1\}$ such that

$$\frac{\beta_j^{(i)}}{v_j} = \frac{r}{v_j} - \frac{k}{v_i},$$

which is a condition independent of ω.

Then, since $\beta_j^{(i)}$ is a non-zero linear combination of $\beta_1, \dots, \beta_n$, the set

$$Z_{ij} := \bigcup_{k,r}\Big\{\frac{\beta_j^{(i)}}{v_j} = \frac{r}{v_j} - \frac{k}{v_i}\Big\}$$

defines a Zariski closed set in $\mathcal{C}^n$, such that the polynomial $b_i(s)b_j(s)$ is reduced if and only if $\beta \notin Z_{ij}$. The condition for the polynomial $b(s)$ to be non-reduced, when $\langle \omega, v\rangle > 0$, is then

$$\beta \in \bigcup_{i,j \in \mathcal{C}^+} Z_{ij},$$

which is a closed condition on β.

So far we have proved that generically in β the b-function has $|v^+|$ roots (recall that we are assuming that $\langle \omega, v\rangle > 0$), while for β in a Zariski closed set of $\mathbb{C}^n$ the b-function has degree less than or equal to $|v^+|$.

To finish, we show that for β in a Zariski closed set of $\mathbb{C}^n$ we cannot have less roots than for generic β. Suppose that, instead of using Noro's algorithm to compute the b-function, we look for the minimal generator of $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[s]$ by elimination. Let $\mathcal{G}$ be a Gröbner basis of $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ with respect to a term order of the type $<_i$. Since this order eliminates the variables $x_j\partial_j$ for $j \neq i$ in $s = \omega_1 x_1 \partial_1 + \cdots + \omega_{n+1} x_{n+1} \partial_{n+1}$, we get

$$NF(s, \mathcal{G}) = s_i = \alpha_i x_i \partial_i + \gamma_i.$$

Here we are implicitly assuming that $x_i\partial_i \notin \mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$, since otherwise $b(s) = s - \gamma_i$ and there is nothing to prove. By Lemma 2.9

$$NF(s^\ell + a_{\ell-1}s^{\ell-1} + \cdots + a_0, \mathcal{G}) = NF(s_i^\ell + a_{\ell-1}s_i^{\ell-1} + \cdots + a_0, \mathcal{G})$$

and by equation (1.1), expressions of the form $s_i^\ell + a_{\ell-1}s_i^{\ell-1} + \cdots + a_0$ live in $\mathbb{C}[x_i, \partial_i]$. Hence, to reduce $s_i^\ell + a_{\ell-1}s_i^{\ell-1} + \cdots + a_0$ we look for an operator

$P_i \in \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[x_i, \partial_i]$ with minimal degree. Such an operator can be found by applying the Buchberger algorithm to the set

$$\left\{\partial^{v^+}, E_j^{(i)} - \beta_j^{(i)} \mid 1 \le j \le n,\ j + \delta_j^{(i)} \in \mathcal{C}^+\right\}.$$

Recall that

$$\begin{aligned} E_j^{(i)} - \beta_j^{(i)} &= a_{j,i}^{(i)} x_i \partial_i + x_{j+\delta_j^{(i)}} \partial_{j+\delta_j^{(i)}} - \beta_j^{(i)}, \\ LT_{<_i}(E_j^{(i)} - \beta_j^{(i)}) &= x_{j+\delta_j^{(i)}} \partial_{j+\delta_j^{(i)}}, \end{aligned}$$

where

$$\delta_j^{(i)} = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j \ge i. \end{cases}$$

Here it is crucial that

$$a_{j,i}^{(i)} = -\frac{v_{j+\delta_j^{(i)}}}{v_i} \neq 0$$

to deduce that the operator P_i has its leading term $LT_{<_i}(P_i)$ independent of the parameters β.

It follows, that we cannot have generically more roots than for β in a Zariski closed set. ■

Example 3.20. Let us consider the hypergeometric system of Example 3.3,

$$H_A(\beta) = D_3\left(\partial_x^4 \partial_z - \partial_y^2, x\partial_x + 3y\partial_y + 2z\partial_z - \beta_1, -x\partial_x + 4z\partial_z - \beta_2\right).$$

Let $\omega \in \mathbb{R}^3 \setminus \{0\}$ be such that $4\omega_1 + \omega_3 < 2\omega_2$. Then, by Proposition 3.16 we have that

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = D_3\left(\partial_y^2, x\partial_x + 3y\partial_y + 2z\partial_z - \beta_1, -x\partial_x + 4z\partial_z - \beta_2\right).$$

For the moment we do not care whether $\mathcal{G} = \{\partial_y^2, x\partial_x + 3y\partial_y + 2z\partial_z - \beta_1, -x\partial_x + 4z\partial_z - \beta_2\}$ is a Gröbner basis of $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ or not, but we reduce s^k by its elements. In this way we will find a multiple of the b-function as follows. First, we define

$$\bar{s} = \alpha_2 y \partial_y + \gamma_2$$

which is nothing but $s_2(\omega, \beta)$ (see Lemma 3.13), with $\alpha_2 = \omega_2 - 2\omega_1 - \frac{\omega_3}{2}$ and $\gamma_2 = \frac{2\beta_1 - \beta_2}{3}\omega_1 + \frac{\beta_1 + \beta_2}{6}\omega_3$ as computed in Example 3.15. Since we have

$$\begin{aligned} \bar{s}^2 &= \alpha_2^2 (y\partial_y)^2 + 2\alpha_2\gamma_2 y\partial_y + \gamma_2^2 \\ &= \alpha_2^2 y^2 \partial_y^2 + (\alpha_2^2 + 2\alpha_2\gamma_2) y\partial_y + \gamma_2^2 \\ &\equiv (\alpha_2^2 + 2\alpha_2\gamma_2) y\partial_y + \gamma_2^2 \quad \mathrm{mod}\ \mathcal{G}, \end{aligned}$$

we can look for a_0 and a_1 such that

$$\bar{s}^2 + a_1\bar{s} + a_0 \equiv 0 \mod \mathcal{G}.$$

In other words,

$$(\alpha_2^2 + 2\alpha_2\gamma_2)y\partial_y + \gamma_2^2 + a_1(\alpha_2 y\partial_y + \gamma_2) + a_0 = 0,$$

or equivalently

$$\begin{aligned} \alpha_2^2 + 2\alpha_2\gamma_2 + \alpha_2 a_1 &= 0, \\ \gamma_2^2 + a_1\gamma_2 + a_0 &= 0. \end{aligned} \tag{3.8}$$

To solve this system it is enough to notice that the hypothesis $4\omega_1 + \omega_3 < 2\omega_2$ is equivalent to $\alpha_2 > 0$. Then we get

$$\begin{aligned} a_1 &= -\alpha_2 - 2\gamma_2, \\ a_0 &= \gamma_2^2 + \gamma_2\alpha_2, \end{aligned}$$

and therefore

$$b(s) = s^2 + a_1 s + a_0 = (s - \gamma_2)(s - \gamma_2 - \alpha_2)$$

is a multiple of the b-function. In Theorem 3.19 it is proved that it is indeed the b-function, since the polynomial $b(s)$ just found is, by definition, the polynomial $b_2(s)$ (see Notation 3.17) and $\mathcal{C}^- = \{2\}$.

The linear system (3.8) is a very particular case of the kind of systems solved in [4].

To complete the example we give here the full description of $b_{\omega,\beta}(s)$. By Theorem 3.19, the b-function of $H_A(\beta)$ with respect to $\omega \in \mathbb{R}^3 \setminus \{0\}$ is

$$\begin{cases} (s-\gamma_1)(s-\gamma_1-\alpha_1)(s-\gamma_1-2\alpha_1) & \\ \qquad \cdot (s-\gamma_1-3\alpha_1)(s-\gamma_3) & \text{if } 4\omega_1+\omega_3 > 2\omega_2, \\ (s-\gamma_2)(s-\gamma_2-\alpha_2) & \text{if } 4\omega_1+\omega_3 < 2\omega_2, \\ s-\gamma(\omega,\beta) & \text{if } 4\omega_1+\omega_3 = 2\omega_2, \end{cases}$$

where, when $4\omega_1 + \omega_3 = 2\omega_2$, we have (see Notation 3.14)

$$\gamma(\omega,\beta) = (\beta_1 + \beta_2)\frac{1}{3}\omega_1 - \beta_2\omega_1.$$

Notice that there are multiple roots. When $4\omega_1 + \omega_3 > 2\omega_2$, it is straightforward to check that, if

$$-\beta_2 = j \in \{0,1,2,3\}$$

then the roots $\gamma_1 + j\alpha_1$ and γ_3 coincide for any $\omega \in \mathbb{R}^3 \setminus \{0\}$.

4. Final remarks

Given $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$, we have proved in Theorem 3.19 that the number of roots of the b-function of $H_A(\beta)$ is independent of the parameter $\beta \in \mathbb{C}^n$. This is due to the equality of the fake and the initial ideal proved in Proposition 3.16, and seems to be particular of the case of codimension one.

Remark 4.1. Notice that for $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$, if we consider the weight

$$\omega' := \omega + \sum_{i=1}^{n} \mu_i (a_{i,1}, \dots, a_{i,n+1})$$

with $\mu_i \in \mathbb{R}$, then $\langle \omega', v \rangle = \langle \omega, v \rangle$, and hence, by Proposition 3.16,

$$\mathrm{in}_{(-\omega',\omega')}(H_A(\beta)) = \mathrm{in}_{(-\omega,\omega)}(H_A(\beta)).$$

This equality suggests a relation among the b-functions $b_{\omega,\beta}(s)$ and $b_{\omega',\beta}(s)$, and it is indeed not difficult to check that the following symmetry relation holds:

$$b_{\omega',\beta}(s) = b_{\omega,\beta}(s + \langle \mu, \beta \rangle),$$

where $\mu = (\mu_1, \dots, \mu_n)$.

There are some properties that are natural to ask to the matrix A when dealing with hypergeometric systems. Some of them have already appeared above.

Definition 4.2. Given a matrix $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$, with columns denoted by $a_1, \dots, a_n \in \mathbb{Z}^d$, we can associate to A the semigroup

$$S_A := \mathbb{N}A = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{N}\},$$

the lattice

$$L_A := \mathbb{Z}A = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{Z}\},$$

the cone

$$C_A := \mathbb{R}_{\geq 0}A = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{R}_{\geq 0}\},$$

and the polytope

$$P_A := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid 0 \leq \lambda_i \leq 1\}.$$

We have $S_A \subseteq L_A \cap C_A$.

The vertices of the polytope P_A are in the lattice L_A. We define the volume of A as the normalized volume of P_A, i.e.,

$$\mathrm{vol}(A) = \frac{\mathrm{vol}(P_A)}{\mathrm{vol}(L_A)},$$

where the volume of the lattice is defined by any basis of L_A.

(i) We say that the matrix A is homogeneous if $(1, \dots, 1)$ belongs to the $\mathbb{Q}$-row span of A. This is equivalent to the fact that the ideal $H_A(\beta)$ is regular.

(ii) We say that A is pointed if $\mathbb{N}A \cap (-\mathbb{N}A) = \{0\}$, where $-\mathbb{N}A = \{-w \mid w \in \mathbb{N}A\}$. Or equivalently, if there exists a vector in the $\mathbb{Q}$-row span of A with all coordinates positive. Another characterization for a matrix A to be pointed is that C_A is a strictly convex cone.

(iii) We say that the matrix A is saturated if its columns generate a saturated semigroup, i.e., if
$$\mathbb{N}A = (\mathbb{Z}A) \cap \mathbb{R}^n_{\geq 0}.$$

Definition 4.3. We say that $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ is generic with respect to the ideal I_A, or simply generic, if the initial ideal $\text{in}_\omega(I_A)$ is monomial.

In the case under consideration, it follows by Proposition 3.16 that $\omega \in \mathbb{R}^{n+1}$ is generic if and only if $\langle \omega, v \rangle \neq 0$.

Corollary 4.4. Let $A \in \mathcal{M}_{n\times(n+1)}(\mathbb{Z})$ be a full-rank matrix and let $\beta \in \mathbb{C}^n$ a vector of parameters.

(i) The b-function of $H_A(\beta)$ has a constant number of roots for ω generic if and only if A is homogeneous.

(ii) The b-function is defined in the whole $\mathbb{R}^{n+1}$ if and only if A is pointed and has not a zero column.

(iii) The b-function of $H_A(\beta)$ is reduced for any ω and any β if and only if $\text{Ker}_{\mathbb{Z}}(A)$ is generated by a vector of the form
$$v = v_i \mathbf{e}_i - v_j \mathbf{e}_j$$
with $v_i, v_j \in \mathbb{Z}_{>0}$ and $1 \leq i < j \leq n+1$.

(iv) There exists $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ generic such that the b-function has degree equal to $\text{vol}(A)$ and it is reduced for every $\beta \in \mathbb{C}^n$ if and only if the matrix A is saturated. In particular, if there exists $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ generic such that $b_{\omega,\beta}(s)$ has degree one then the matrix A is saturated.

Proof. Recall that we are denoting by $a_i \in \mathbb{Z}^n$, for $1 \leq i \leq n+1$, the columns of the matrix A, and by $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ the standard basis of $\mathbb{R}^{n+1}$.

Referring to (i) there is nothing to prove, since it follows directly from Theorem 3.19.

We prove (ii). Suppose that the matrix A has a zero column,
$$a_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

for certain $1 \leq i \leq n+1$. Then A_i has to be an invertible matrix and the solution to $Av = 0$ is generated by $\mathbf{e}_i$. Therefore, the toric ideal I_A is generated by the operator $\partial_i - 1$ and the b-function is not defined for $\langle \omega, \mathbf{e}_i \rangle < 0$ (see Remark 3.18).

Suppose now that A is not pointed, then there exists a non-zero $u \in \mathbb{N}A \cap (-\mathbb{N}A)$. Then,

$$u = \lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1}, \quad \text{with } \lambda_i \geq 0$$

and

$$-u = \mu_1 a_1 + \cdots + \mu_{n+1} a_{n+1}, \quad \text{with } \mu_i \geq 0.$$

Therefore,

$$0 = (\lambda_1 + \mu_1) a_1 + \cdots + (\lambda_{n+1} + \mu_{n+1}) a_{n+1}$$

and hence $(\lambda_1 + \mu_1, \ldots, \lambda_{n+1} + \mu_{n+1})$ is a multiple of v. This implies that $v \in \mathbb{R}^{n+1}_{\geq 0}$, or in other words, $I_A = (\partial^v - 1)$. And, again, the b-function is not defined for $\langle \omega, v \rangle < 0$.

For the other implication, we have that $\mathrm{Ker}_{\mathbb{Z}}(A)$ is generated by v where

$$v_1 a_1 + \cdots + v_{n+1} a_{n+1} = \bar{0} \in \mathbb{R}^n.$$

By hypothesis the vector v is not of the form $\mathbf{e}_i$ with $1 \leq i \leq n+1$. If A is pointed there exists a homomorphism $\phi : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ such that $\phi(a_i) > 0$ for every i. Hence

$$v_1 \phi(a_1) + \cdots + v_{n+1} \phi(a_{n+1}) = 0$$

and not every v_i can be non-negative. Hence $\mathcal{C}^+ \neq \emptyset$ and $\mathcal{C}^- \neq \emptyset$, and the b-function is defined for every $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$.

Let us prove (iii). The condition of the statement is equivalent to $|\mathcal{C}^+| = |\mathcal{C}^-| = 1$. It follows directly from Theorem 3.19 that when $|\mathcal{C}^+| = |\mathcal{C}^-| = 1$ the b-function is reduced.

Suppose now that $\langle \omega, v \rangle > 0$ and $|\mathcal{C}^+| > 1$. In the proof of Theorem 3.19 we show that for any $i, j \in \mathcal{C}^+$ there exists a non-empty Zariski closed set $Z_{ij} \subseteq \mathbb{C}^n$ such that $b_i(s) b_j(s)$ is reduced if and only if $\beta \notin Z_{ij}$.

To finish, let us prove (iv). The existence of a generic weight $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$ such that for every $\beta \in \mathbb{C}^n$ the polynomial $b_{\omega,\beta}(s)$ is reduced and has degree equal to $\mathrm{vol}(A)$ is equivalent, by Theorem 3.19, to the following situation:

$$\langle \omega, v \rangle > 0, \quad \mathcal{C}^+ = \{j\} \quad \text{and} \quad v^+ = \mathrm{vol}(A) \mathbf{e}_j$$

for certain $j \in \{1, \ldots, n+1\}$. Notice that the condition $Av = 0$ is then

$$\mathrm{vol}(A) a_j = \sum_{k \neq j} v_k a_k,$$

where $v_k \geq 0$. Since $\mathrm{vol}(A)$ equals the number of lattice points in the so-called fundamental parallelogram of $C_A \cap L_A$, the equation above implies that $S_A = L_A \cap \mathbb{Z}^n_{\geq 0}$, and therefore A is saturated. ■

Remark 4.5. A possible generalization of Corollary 4.4 for matrices of any size $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ might be straightforward for the points (ii) and (iv), together with the following.

(i) The b-function of $H_A(\beta)$ has constant number of roots for both ω and β generic if and only if A is homogeneous.

(iii) The b-function is reduced for any ω and any β if and only if $\mathrm{Ker}_{\mathbb{Z}}(A)$ can be generated by vectors of the form

$$a_i \mathbf{e}_i - b_j \mathbf{e}_j$$

with $1 \leq i < j \leq n$ and $a_i, b_j > 0$.

This agrees with [13, Proposition 5.1.9], where the b-function is described for generic weights and parameters, for homogeneous matrices of any size. It also agrees with the main result in [3], where the b-function is described for generic β in the case of a particular family of matrices which are non-homogeneous, pointed and saturated.

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