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A fixed point result for mappings on the ℓ_∞-sum of a closed and convex set based on the degree of nondensifiability

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Abstract. Let C a non-empty, bounded, closed and convex subset of a Banach space X, and denote by $\ell_\infty(C)$ the ℓ_∞-sum of C. In the present paper, by using the degree of nondensifiability (DND), we introduce the class of r-Δ-DND-contraction maps $f : \ell_\infty(C) \to X$ and prove that if $f(\ell_\infty(C)) \subset C$ then there is some $x^* \in C$ with $f(x^*, x^*, \ldots, x^*, \ldots) = x^*$. Our result, in the specified framework, generalizes other fixed point results for the so called generalized r-contraction and even other existing fixed point result based on the DND. Also, we derive a new Krasnosel'skiĭ-type fixed point result.

1. Introduction

In 1922 the great Polish mathematician Stefan Banach published in the paper [5] his celebrated fixed point theorem which, for completeness, we recall here.

Theorem 1.1 (Banach Fixed Point Theorem). *Let (X, d) be a complete metric space and $f : X \to X$ an r-contraction, for some $r \in [0, 1)$, i.e. $d(f(x), f(y)) \le rd(x, y)$ for all $x, y \in X$. Then, f has a unique fixed point $x^* \in X$ and, for any $x_0 \in X$, the iteration $x_n := f(x_{n-1})$ for all $n \ge 1$ converges to x^*.*

Due to its many applications (see, for instance, the monograph [33]), the above fixed point result has been generalized in many directions, see, for instance, [1–3,7,16,17,21,22,28–31] and references therein. In particular, for a given integer $m \ge 1$, if we consider X^m, the Cartesian product of m copies of the metric space (X, d) endowed with the metric $d_s(\mathbf{x}, \mathbf{y}) := \max\{d(x_i, y_i) : i = 1, \ldots, m\}$ for all $\mathbf{x} := (x_1, \ldots, x_m)$, $\mathbf{y} := (y_1, \ldots, y_m) \in X^m$, Miculescu [22] and Mihail [21] proved the following result:

Theorem 1.2. *Let (X, d) be a complete metric space and $f : X^m \to X$ such that $d(f(\mathbf{x}), f(\mathbf{y})) \le rd_s(\mathbf{x}, \mathbf{y})$, for some $r \in [0, 1)$ and all $\mathbf{x}, \mathbf{y} \in X^m$ (i.e., f is a generalized r-contraction). Then, f has a unique generalized fixed point, that is, there exists*

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a unique $x^* \in X$ *such that* $f(x^*, \ldots, x^*) = x^*$. *Moreover, for every* $x_0, \ldots, x_{m-1} \in X$ *the sequence*

$$x_{m+k} := f(x_{k+m-1}, \ldots, x_k), \quad \textit{for all } k \geq 0,$$

converges to x^*.

Later, Secelean [28] generalized the above result (see also [20]). Before showing such result, we need to introduce the notation and recall some known facts. Following [15], we denote by $\ell_\infty(X)$ the ℓ_∞-sum of the metric space (X, d), that is, the set of all bounded sequences of X endowed with the supremum metric $d_s := \sup\{d(x_n, y_n) : n \geq 1\}$ for all $\mathbf{x} := (x_n)_{n\geq 1}, \mathbf{y} := (y_n)_{n\geq 1} \in \ell_\infty(X)$.

The notion of ℓ_∞-sum originates from Functional Analysis (see, for instance, [6,8] or [19, p. xii]). Let us note that if X is not bounded, then $\ell_\infty(X)$ is a proper subspace of $\prod_{n\geq 1} X$, but if X is bounded then $\ell_\infty(X) = \prod_{n\geq 1} X$. Some interesting results related with the Tychonoff product topology (see, for instance, [32]) on $\ell_\infty(X)$ and the topology induced by certain metrics in $\ell_\infty(X)$ (assuming X is bounded and not a singleton) were proved in [15].

The above mentioned Secelean fixed point result is the following:

Theorem 1.3. *Let* (X, d) *be a complete metric space and* $f : \ell_\infty(X) \to X$ *such that* $d(f(\mathbf{x}), f(\mathbf{y})) \leq r d_s(\mathbf{x}, \mathbf{y})$, *for some* $r \in [0, 1)$ *and all* $\mathbf{x}, \mathbf{y} \in \ell_\infty(X)$ *(i.e.,* f *is a generalized* r*-contraction). Then,* f *has a unique generalized fixed point, that is, there exists a unique* $x^* \in X$ *such that* $f(x^*, \ldots, x^*, \ldots) = x^*$. *Moreover, for every* $(x_n)_{n\geq 1} \in \ell_\infty(X)$ *the sequence*

$$y_k := f\big(f^k(x_1, \ldots, x_1, \ldots), f^k(x_2, \ldots, x_2, \ldots), \ldots, f^k(x_n, \ldots, x_n, \ldots), \ldots\big),$$

for all $k \geq 0$, *converges to* x^* *where* f^k *means the composition of* f *with itself* k*-times (*f^0 *being the identity mapping).*

Remark 1.4. Actually, the above result holds for more general contractiveness conditions on f, but here this fact is not relevant.

On the other hand, the main goal of this paper is to generalize Theorem 1.3 when X is a non-empty, bounded, closed and convex subset of a Banach space. For this, our main tool will be the so called degree of nondensifiability (DND), explained in detail in Section 2. Also, our main result (see Theorem 3.6) generalizes an existing fixed point result based on the DND. To show that our results are real generalizations of the indicated ones, we provide some examples. As consequence of Theorem 3.6, we derive a new Krasnosel'skiĭ-type fixed point result, see Corollary 3.7.

2. The degree of nondensifiability

Before introducing the main concept and result of this section, it is convenient to recall the concepts of α-dense curve and densifiable set, introduced in [24]. Following the notation of the previous section, and by the moment, (X, d) will be a metric space. Also, we denote $I := [0, 1]$ and, as usual, for a non-empty subset B of X or E, $\overline{B}$ is the closure of B and if X is a Banach space, $\mathrm{Conv}(B)$ is the convex hull of B. The class of non-empty and bounded subsets of X is denoted by $\mathcal{B}(X)$.

Definition 2.1. Let $B \in \mathcal{B}(X)$ and $\alpha \geq 0$. A continuous mapping $\gamma : I \to (X, d)$ is said to be an α-dense curve in B if the following conditions hold:

(i) $\gamma(I) \subset B$.

(ii) For each $x \in B$ there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

If for each $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Some comments are necessary:

(a) For a given $B \in \mathcal{B}(X)$, fixed $x_0 \in B$ the mapping $\gamma : I \to (X, d)$ defined as $\gamma(t) := x_0$ for all $t \in I$ is, trivially, an α-dense curve in B for each $\alpha \geq \mathrm{Diam}(B)$, the diameter of B.

(b) If $B := I^n$ is the n-dimensional closed unit cube of $\mathbb{R}^n$, and $\gamma : I \to \mathbb{R}^n$ is a *space-filling curve* (see [27]), that is to say, γ is continuous and $\gamma(I) = B$, then γ is, precisely, a 0-dense curve in B. So, the α-dense curves are a generalization of the space-filling curves. Moreover, as Mora proved in [23], the space-filling curves can be characterized in terms of the α-dense curves.

(c) It is not very hard to prove (see [26, Proposition 2]) that an arc-connected $B \in \mathcal{B}(X)$ is densifiable if and only if it is precompact.

For a detailed exposition of the α-dense curves, as well as its applications, see [9, 23, 25, 26] and references therein.

Next, we can give the definition of degree of nondensifiability, which was first defined in [25] and analyzed later in [13].

Definition 2.2. For a given $B \in \mathcal{B}(X)$ the degree of nondensifiability, in short DND, of B is defined as

$$\phi(B) := \inf\{\alpha \geq 0 : \Gamma_{B,\alpha} \neq \emptyset\},$$

$\Gamma_{B,\alpha}$ being the class of the α-dense curves in B.

Let us note that the DND is well defined. Indeed, from the above considerations, $\Gamma_{B,\alpha} \neq \emptyset$ for each $\alpha \geq \mathrm{Diam}(B)$ and therefore $\phi(B) \in [0, \mathrm{Diam}(B)]$. Also, by virtue of the Hahn–Mazurkiewicz Theorem (see, for instance, [27, 32]) a set $B \in \mathcal{B}(X)$ is a Peano Continuum (i.e., a compact, connected and locally connected set) if and only if it is the continuous image of I. Therefore, the DND *measures*, in the specified sense, the distance from a given $B \in \mathcal{B}(X)$ to the class of the Peano Continua contained in B. The following example was proved in [25]:

Example 2.3. Let U_X be the closed unit ball of a Banach space X. Then,

$$\phi(U_X) = \begin{cases} 1, & \text{if } X \text{ has infinite dimension,} \\ 0, & \text{if } X \text{ has finite dimension.} \end{cases}$$

The following property of the DND will be useful for our goals (see [13, Proposition 2.6]):

Proposition 2.4. *Let $\mathcal{B}_{\text{arc}}(X)$ be the class of non-empty, bounded and arc-connected sets of X. Then, $\phi(B) = 0$ if and only if B is precompact, for each $B \in \mathcal{B}_{\text{arc}}(X)$.*

It is worth saying that, despite the DND ϕ shares some properties (see [10, 13]) similar to those of the so-called measures of noncompactness (MNC), see, for instance [1, 4], the DND is not a MNC. Moreover, in [10, 11, 14] we have proved several fixed point results based on the DND which works under conditions that similar fixed point results based on the MNCs do not work.

In what follows, if otherwise not specified, we assume that $(X, \|\cdot\|)$ is a Banach space. At this point, it is convenient to introduce the following concept (see also [14]).

Definition 2.5. Let $f : C \to X$ continuous, with $C \in \mathcal{B}(X)$ closed and convex, such that $f(B) \in \mathcal{B}(X)$ for each non-empty $B \subset C$. For a given $r \in [0, 1)$, we will say that f is an r-DND-contraction if

$$\phi(f(B)) \leq r\phi(B),$$

for all non-empty, closed and convex $B \subset C$.

The following fixed point result was proved in [11, Corollary 3.3].

Theorem 2.6. *Let $f : C \to C$ be an r-DND-contraction for some $r \in [0, 1)$, with $C \in \mathcal{B}(X)$ closed and convex. Then, f has some fixed point.*

Of course, the above result is, in forms, similar to the celebrated Darbo Fixed Point Theorem (see, for instance, [1, 4]). However, as we have pointed out above, Theorem 2.6 and other fixed point results based on the MNCs (and, in particular, Darbo Fixed Point Theorem) are essentially different. We conclude this section with some comments to emphasize some common properties of the Darbo Fixed Point Theorem and Theorem 2.6.

(I) If $f : C \subseteq X \to X$ is an r-contraction, for some $r \in [0, 1)$, it is not very hard to check that given a non-empty, closed and convex $B \subset C$ and an α-dense curve in B for some $\alpha > \phi(B)$, then the mapping $f \circ \gamma : I \to E$ is an $r\alpha$-dense curve in $f(B)$ (see also the proof of Proposition 3.2). Therefore, f is an r-DND-contraction. So, in the frame of Theorem 2.6, the above fixed point result is a generalization of the Banach Fixed Point Theorem.

(II) According to [4, Definition I.2.5], a continuous mapping f between two Banach spaces is said to be compact if it maps bounded subsets into pre-compact ones. Assume $f : C \to C$ is compact, with $C \in \mathcal{B}(X)$ compact and convex. By Proposition 2.4, under these conditions, $\phi(f(B)) = 0$. Thus, f is a 0-DND-contraction and therefore Theorem 2.6 generalizes the well-known Schauder Fixed Point Theorem (see, for instance, [4, Theorem I.2.1]).

3. The main result

In what follows, for a given $C \in \mathcal{B}(X)$, we consider $\ell_\infty(C)$ endowed with the supremum metric d_s defined in Section 1. As we are assuming X is a Banach space endowed with a norm $\|\cdot\|$, then

$$d_s(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_s := \sup\left\{\|x_n - y_n\| : n \geq 1\right\},$$

for all $\mathbf{x} := (x_n)_{n\geq 1}$, $\mathbf{y} := (y_n)_{n\geq 1} \in \ell_\infty(C)$. Let us note that, as we have pointed out in Section 1, in this context $\ell_\infty(C) = \prod_{n\geq 1} C$ but we maintain the notation of Section 1. Of course, C is bounded in $(\ell_\infty(X), d_s)$. Therefore, we can consider the DND of C. To emphasize that we refer to the DND in the ℓ_∞-sum of C, $\ell_\infty(C)$, the DND of C will be denoted by $\phi_\infty(C)$.

Recently, in [12] we have generalized Theorem 2.6. As we have pointed out in Section 1, our main goal is to generalize Theorems 1.3 and 2.6 for mappings defined in $\ell_\infty(C)$, with $C \in \mathcal{B}(X)$ convex and closed. To simplify writing and notation, we introduce the following.

Definition 3.1. Given $B \in \mathcal{B}(X)$, we define

$$\ell_\infty(B)_\Delta := \left\{(x_n)_{n\geq 1} \in \ell_\infty(B) : x_n := x \text{ for all } n \geq 1 \text{ and some } x \in B\right\}.$$

Roughly speaking, $\ell_\infty(B)_\Delta$ is the "diagonal section" of $\ell_\infty(B)$. Also, let $f : \ell_\infty(C) \to X$ be continuous, with $C \in \mathcal{B}(X)$ closed and convex, such that $f(B) \in \mathcal{B}(X)$ for each non-empty $B \subset \ell_\infty(C)$. If there is $r \in [0, 1)$, such that

$$\phi(f(\ell_\infty(B)_\Delta)) \leq r\phi_\infty(\ell_\infty(B)_\Delta),$$

for all non-empty, closed and convex $B \subset C$, we will say that f is an r-Δ-DND contraction.

Next, we show that the above concept generalizes that of r-generalized contraction (see Section 1) for mappings defined in $\ell_\infty(C)$ with $C \in \mathcal{B}(X)$ closed and convex.

Proposition 3.2. *Let $C \in \mathcal{B}(X)$ closed and convex and $f : \ell_\infty(C) \to X$ an r-generalized contraction, for some $r \in [0,1)$. Then, f is an r-Δ-DND contraction.*

Proof. Let any non-empty, closed and convex $B \subset C$ and $\gamma := (\gamma_n)_{n\geq 1}$ an α-dense curve in $\ell_\infty(B)_\Delta$, for some $\alpha > \phi(\ell_\infty(B)_\Delta)$. Then, given any $\mathbf{x} \in \ell_\infty(B)_\Delta$ there is $t \in I$ such that $\|\mathbf{x} - \gamma_n(t)\|_s \leq \alpha$.

It is clear that $\tilde{\gamma} := f \circ (\gamma_n)_{n\geq 1} : I \to X$ is continuous and $\tilde{\gamma}(I) \subset f(\ell_\infty(B)_\Delta)$. So, from the above considerations and noticing that f is an r-generalized contraction, we have

$$\|f(\mathbf{x}) - \tilde{\gamma}(t)\| \leq r\|\mathbf{x} - (\gamma_n(t))_{n\geq 1}\|_s \leq r\alpha$$

and therefore $\tilde{\gamma}$ is an α-dense curve in $f(\ell_\infty(B)_\Delta)$. From the arbitrariness of α, we conclude that

$$\phi\big(f(\ell_\infty(B)_\Delta)\big) \leq r\phi_\infty\big(\ell_\infty(B)_\Delta\big),$$

and the proof is complete. ∎

We show by the following examples that the class of the r-generalized contraction defined in $\ell_\infty(C)$, with $C \in \mathcal{B}(X)$ closed and convex, is strictly contained in the class of the r-Δ-DND contractions.

Example 3.3. Let c_0 be the Banach space of the (real) null sequences endowed with its usual supremum norm and U_{c_0} its closed unit ball. Define $f : \ell_\infty(U_{c_0}) \to U_{c_0}$ as

$$f\big((x_n^1)_{n\geq 1}, (x_n^2)_{n\geq 1}, \ldots, (x_n^k)_{n\geq 1}, \ldots\big) := \Big(x_1^1, \frac{1}{2}x_2^2, \ldots, \frac{1}{k}x_k^k, \ldots\Big),$$

for all $(x_n^k)_{n\geq 1} \in U_{c_0}$, $k \geq 1$. Then f is continuous and it is easy to check that is compact (see also [4, Theorem II.4.1]). So, as $f(\ell_\infty(B)_\Delta)$ is precompact and arc-connected for each non-empty, closed and convex $B \subset C$, by Proposition 2.4 $\phi(f(\ell_\infty(B)_\Delta)) = 0$ and therefore f is a 0-Δ-contraction.

If for each $k \geq 1$ we denote by e_k the k-th basic vector of c_0 (recall, $e_k = (0, \ldots, 0, 1, 0, \ldots)$ with the one in the k-th position), and $\mathbf{0}$ the null vector of c_0 we find that

$$\begin{aligned}&\|f(e_1, e_2, \ldots, e_k, \ldots) - f(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \ldots)\| \\ &\qquad = 1 = \|(e_1, e_2, \ldots, e_k, \ldots) - (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \ldots)\|_s,\end{aligned}$$

and consequently f can not be an r-generalized contraction for any $r \in [0,1)$.

Example 3.4. Let $\mathcal{C}(I)$ be the Banach space of the continuous functions $x : I \to \mathbb{R}$ endowed with the supremum norm and $C := \{x \in \mathcal{C}(I) : 0 \leq x(t) \leq 1, \text{ for all } t \in I\}$. Consider $f : \ell_\infty(C) \to C$ given by

$$f\big((x_n(t))_{n\geq 1}\big) := \frac{1}{2}x_1(t) + \sum_{k\geq 2} \frac{|\sin\big(\frac{\pi}{2}(x_k(t) - x_{k-1}(t))\big)|}{2^k}$$

for all $(x_n(t))_{n\geq 1} \in \ell_\infty(C)$. Then, given $x, y \in C$

$$\begin{aligned}&\|f(x(t), x(t), \ldots, x(t), \ldots) - f(y(t), y(t), \ldots, y(t), \ldots)\|\\ &\quad = \frac{1}{2}\|x - y\| = \frac{1}{2}\|x - y\|_s,\end{aligned}$$

and therefore f is a $\frac{1}{2}$-Δ-contraction. Indeed, from the above inequality and reasoning as in the proof of Proposition 3.2, if γ is an α-dense curve in $\ell_\infty(B)_\Delta$ for a given non-empty, closed and convex $B \subset C$, then $f \circ \gamma$ is a $\frac{1}{2}\alpha$-dense curve in $f(\ell_\infty(B))_\Delta$.

Next, take the sequence

$$x_n(t) := \begin{cases} \mathbf{1}, & \text{for } n \text{ odd,}\\ \mathbf{0}, & \text{for } n \text{ even,}\end{cases}$$

where $\mathbf{1}$ and $\mathbf{0}$ stand for the functions identically equal to 1 and 0, respectively. Then

$$\|f((x_n(t))_{n\geq 1}) - f((\mathbf{0})_{n\geq 1})\| = 1 = \|(x_n(t))_{n\geq 1} - (\mathbf{0})_{n\geq 1}\|_s,$$

and therefore f is not an r-generalized contraction for any $r \in [0, 1)$.

The following lemma will be useful for our goal:

Lemma 3.5. *Let $B \in \mathcal{B}(X)$ convex. Then,*

$$\phi_\infty\big(\ell_\infty(B)_\Delta\big) = \phi(B).$$

Proof. Let any $\alpha > \phi_\infty(\ell_\infty(B)_\Delta)$ and γ an α-dense curve in $\ell_\infty(B)_\Delta$. Then,

$$\ell_\infty(B)_\Delta \subset \gamma(I) + \alpha U_{\ell_\infty(X)}, \tag{3.1}$$

$U_{\ell_\infty(X)}$ being the closed unit ball of $\ell_\infty(X)$. Also, as $\gamma(I)$ is compact, given $\varepsilon > 0$ there exists a finite set $\{(y_n^1)_{n\geq 1}, \ldots, (y_n^m)_{n\geq 1}\} \subset \gamma(I)$ such that

$$\gamma(I) \subset \big\{(y_n^1)_{n\geq 1}, \ldots, (y_n^m)_{n\geq 1}\big\} + \varepsilon U_{\ell_\infty(X)}. \tag{3.2}$$

Then, from (3.1) and (3.2) we have

$$\ell_\infty(B)_\Delta \subset \big\{(y_n^1)_{n\geq 1}, \ldots, (y_n^m)_{n\geq 1}\big\} + (\alpha + \varepsilon) U_{\ell_\infty(X)}. \tag{3.3}$$

Because of $\gamma(I) \subset \ell_\infty(B)_\Delta$, for each $j = 1, \ldots, m$, there is $y_j \in B$ such that $y_n^j = y_j$ for all $n \geq 1$. Define the (continuous) mapping $\widetilde{\gamma} : I \to X$ as a polygonal line joining the vectors $y_1, \ldots, y_n$. By the convexity of B, $\widetilde{\gamma}(I) \subset B$ and given $x \in B$, noticing (3.3), there is $1 \leq j \leq n$ such that

$$\|(x, \ldots, x, \ldots) - (y_j, \ldots, y_j, \ldots)\|_s \leq \alpha + \varepsilon.$$

So, taking $t \in I$ such that $\widetilde{\gamma}(t) = y_j$ we have

$$\|x - \widetilde{\gamma}(t)\| = \|x - y_j\| = \|(x, \ldots, x, \ldots) - (y_j, \ldots, y_j, \ldots)\|_s \leq \alpha + \varepsilon$$

and consequently $\widetilde{\gamma}$ is an $(\alpha + \varepsilon)$-dense curve in B. By letting $\alpha \to \phi_\infty(\ell_\infty(B)_\Delta)$ and from the arbitrariness of $\varepsilon > 0$, we infer

$$\phi(B) \leq \phi_\infty(\ell_\infty(B))_\Delta. \tag{3.4}$$

On the other hand, if γ is an α-dense curve in B, the mapping

$$\widehat{\gamma}(t) := (\gamma(t), \gamma(t), \ldots, \gamma(t), \ldots), \quad \text{for all } t \in I,$$

is continuous and it is easy to check that it is an α-dense curve in $\ell_\infty(B)_\Delta$. So, by the arbitrariness of α, we have

$$\phi_\infty(\ell_\infty(B)_\Delta) \leq \phi(B), \tag{3.5}$$

and the result follows from inequalities (3.4) and (3.5). ∎

Now, we can state and prove our main result:

Theorem 3.6. *Let $C \in \mathcal{B}(X)$ be closed and convex and $f : \ell_\infty(C) \to C$ an r-Δ-DND-contraction, for some $r \in [0, 1)$. Then, f has some generalized fixed point, that is, there is $x^* \in C$ such that $f(x^*, \ldots, x^*, \ldots) = x^*$.*

Proof. Define $f_s : C \to C$ as $f_s(x) := f(x, x, \ldots, x, \ldots)$ for all $x \in C$. We will prove that f_s is an r-DND contraction (see Definition 2.5).

It is clear that f_s is continuous. Let any $B \subset C$ non-empty, closed and convex. Then, noticing that f is an r-Δ-DND-contraction and Lemma 3.5, we have

$$\phi(f_s(B)) = \phi(f(\ell_\infty(B)_\Delta)) \leq r\phi_\infty(\ell_\infty(B)_\Delta) = r\phi(B),$$

and so, f_s is an r-DND contraction as claimed. Therefore, by Theorem 2.6 f_s has some fixed point which is a generalized fixed point of f. ∎

So, by Proposition 3.2 and noticing Examples 3.3 and 3.4, the above result generalizes Theorem 1.3 in the case that the considered metric spaces be a convex and closed set $C \in \mathcal{B}(X)$. Also, it is clear that Theorem 3.6 generalizes Theorem 2.6.

On the other hand, the well-known Krasnosel'skiĭ fixed point theorem [18] states that if $f, K : C \to C$ are continuous, with $C \in \mathcal{B}(X)$ closed and convex, $f(C) + K(C) \subset C$, f is an r-contraction and K compact, then $f + K$ has some fixed point. This result has been extended in many directions, see [12, 14] and references therein. Here, and as consequence of Theorem 3.6, we propose a new Krasnosel'skiĭ-type fixed point theorem. We will need the inequality (see [12, Proposition 2.2])

$$\phi(\ell_\infty(A)) \leq 2\big[\phi(\ell_\infty(B)) + \phi(\ell_\infty(C))\big], \tag{3.6}$$

for each arc-connected $A, B, C \in \mathcal{B}(X)$ with $A \subset B + C$. Next, we have the following result:

Corollary 3.7. *Let $C \in \mathcal{B}(X)$ be closed and convex and $T := f + K : \ell_\infty(C) \to C$ be the sum of the continuous mappings f and K such that*

(1) *f is an r-Δ-DND-contraction, for some $r \in [0, 1/2)$,*

(2) *the mapping $x \in C \mapsto K(x, x, \ldots, x, \ldots)$ is compact (that is to say, K restricted to $\ell_\infty(C)_\Delta$ is compact).*

Then, T has some generalized fixed point.

Proof. We will prove that T is an $2r$-Δ-DND-contraction, and then the result holds by Theorem 3.6. Let any $B \subset C$ non-empty, closed and convex. As

$$\begin{aligned} T(\ell_\infty(B)_\Delta) &= \big\{f(x, x, \ldots, x, \ldots) + K(x, x, \ldots, x, \ldots) : (x_n)_{n\geq 1} \in \ell_\infty(B)_\Delta\big\} \\ &\subset f(\ell_\infty(B)_\Delta) + K(\ell_\infty(B)_\Delta) \end{aligned}$$

from (3.6), and noticing (2) of the statement and Proposition 2.4, we find that

$$\phi(T(\ell_\infty(B)_\Delta)) \leq 2\big[\phi(f(\ell_\infty(B)_\Delta)) + \phi(K(\ell_\infty(B)_\Delta))\big] = 2\phi(f(\ell_\infty(B)_\Delta)).$$

Therefore, from the above inequality and condition (1) of the statement, we have

$$\phi(T(\ell_\infty(B)_\Delta)) \leq 2r\phi(\ell_\infty(B)_\Delta).$$

As $2r \in [0, 1)$, we conclude that T is a $2r$-Δ-DND-contraction and the result follows from Theorem 3.6. ∎

Of course, if K is compact the condition (2) of the above result holds trivially. However, the reciprocal is not true: a mapping K can obey (2) but not be compact. To conclude our exposition and evidence this fact, we show an example.

Example 3.8. Let $\mathcal{C}(I)$ and C as in Example 3.4. Define $f : \ell_\infty(C) \to C$ as

$$f((x_n(t))_{n\geq 1}) := \frac{1}{3}x_1(t), \quad \text{for all } (x_n(t))_{n\geq 1} \in \ell_\infty(C).$$

Then, it is clear that f is a $1/3$-generalized contraction and therefore, noticing Proposition 3.2, is a $1/3$-Δ-DND-contraction. Now, consider $K : \ell_\infty(C) \to C$ given by

$$K((x_n(t))_{n\geq 1}) := \frac{2}{3}\Big[\int_0^t x_2(s)\,ds + \sum_{n\geq 1} \frac{|x_{n+2}(t) - x_{n+1}(t)|}{2^n}\Big],$$

for all $(x_n(t))_{n\geq 1} \in \ell_\infty(C)$. Let us note that K is not a compact mapping. Indeed, taking, for instance, the sequence $x_k(t) := t^k \in C$ for each $k \geq 1$, the sequence

$$K(\mathbf{0}, x_k(t), \mathbf{0}, \ldots, \mathbf{0}, \ldots) = \frac{2}{3}\Big[\frac{t^{k+1}}{(k+1)} + t^k\Big],$$

where $\mathbf{0}$ is the identically null function, does not have any convergent sub-sequence and therefore K is not a compact mapping.

Next, define $T := f + K$. It is clear that $T(\ell_\infty(C)) \subset C$ and is continuous. Let any non-empty, convex and closed $B \subset C$. As we have pointed out above f is a $1/3$-Δ-DND-contraction and then condition (1) of Corollary 3.7 is fulfilled. Now,

$$K((x_n(t))_{n\geq 1}) = \frac{2}{3}\int_0^t x_2(s)\,ds, \quad \text{for all } (x_n(t))_{n\geq 1} \in \ell_\infty(B)_\Delta,$$

and consequently (by a direct application of the Arzelá–Ascoli theorem, see also [4, Example I.3]) $K(\ell_\infty(B)_\Delta)$ is precompact. So, condition (2) of Corollary 3.7 holds.

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