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Approximation to the classical fractals by using non-affine contraction mappings

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Abstract. In the literature, there are various methods to obtain the fractal sets such as escape time algorithm, L-systems and iterated function system (IFS), etc. In this study, we aim to approximate to the classical fractals by using non-affine contraction mappings. In order to get these non-affine mappings, we utilize from the sequences of suitable Lipschitz continuous functions. Then, we obtain some approximations to the fractals which can be constructed as the attractor of an IFS. Finally, we give some illustrations for some specific cases.

1. Introduction

In recent years, fractals have been one of the popular subjects with many applications in different areas such as mathematics, meteorology, engineering, physics, biology, etc. (see [7–10, 12–15, 17, 19–21]). It is clearly seen that the history of these self-similar sets dates back to older times. As an example of a crucial development about fractals, one can take into account the Hutchinson's theory given in 1981, to construct the fractals by using the iterated function system (IFS) in [16]. Many well-known fractal sets such as the Cantor set defined by Georg Cantor in 1883, the Sierpinski triangle defined by Vaclav Sierpinski in 1915, the Koch curve defined by Helge von Koch in 1904 can be easily obtained by their related IFSs (see [8, 14, 18]). On the other hand, fractals can be constructed by using different methods and transformations. For instance, considering the contraction mappings, Barnsley defined a function and obtained the right Sierpinski triangle via the escape time algorithm in [8]. Differently from Barnsley's method, some classical fractals are obtained via the escape time algorithm by using expanding, folding, translation and rotation mappings, which are defined independently from their related IFSs in [6]. Büyükyılmaz, Yaylı and Gök also give a different construction for Sierpinski triangles with Galilean transformations in [11].

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In the present paper, our purpose is to get approximations to the fractals which can be obtained as a fixed point of an IFS. For this approximation, inspired by the papers [1, 3–5], we construct non-affine contraction mappings. As is known, most of the fractals such as Sierpinski triangle, Sierpinski carpet, Sierpinski tetrahedron, Vicsek fractal and Koch curve can be obtained as the attractor of an iterated function systems. It is seen that the contraction mappings used in obtaining these fractals (see Examples 3.1–3.8) are generally affine transformations ($y = ax + b$). In this study, it is aimed to approach to these types of fractals by using new non-affine transformations. Our method is also valid for the fractals which can be obtained by non-affine mappings. For this purpose, firstly, the sequences of non-affine contraction mappings will be constructed by applying (sequences of) nonlinear transformations to known contraction mappings. We recall that these transformations are also used in turning a linear operator into nonlinear one [1–5]. Thanks to these sequences of non-affine contraction mappings, it is possible to obtain (a sequence of) new iterated function systems. Thus, we get approximations to fractals by using the sequence of self-similar sets, which are obtained as the attractor of the sequence of new IFSs. That means, we will have new fractal sequences, which converge to the initial fractal. In the last section, in order to verify our study, we define some non-affine contraction mappings and get approximations to the Sierpinski triangles, Sierpinski carpet, Sierpinski tetrahedron, Vicsek (box) fractal and Koch curve. Using some algorithms in Maple, we also display our approximations in Figures 1–7.

Now, we give some required definitions and theorems.

Definition 1.1 ([8]). Let (X, d) be a metric space. Then, the function $f : X \to X$ is called a contraction mapping, if there exists a constant $0 \leq k < 1$ satisfying that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$. Here, k is called by the contractivity factor of f.

Theorem 1.2 ([8]). *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping. Then, there exists a unique fixed point x_0, (namely $f(x_0) = x_0$) such that*

$$\lim_{n\to\infty} f^n(x) = f(x_0) = x_0$$

holds for all $x \in X$.

Here and throughout the paper, we denote n-times composition of a function f by $f^n(x)$, i.e., $f^n(x) := (f \circ f \circ \cdots \circ f)(x)$.

Definition 1.3 ([8]). Let (X, d) be a complete metric space and let $f_i : X \to X$ $(i = 1, 2, \ldots, N)$ be contraction mappings. Then, the set $\{X; f_1, f_2, \ldots, f_N\}$ is called an iterated function system (IFS).

We give the following fundamental theorem for fractal geometry which is introduced by Hutchinson:

Theorem 1.4 ([16]). *Let (X, d) be a complete metric space and $\{X; f_1, f_2, \ldots, f_N\}$ be an IFS. Then $F : \mathcal{H}(X) \to \mathcal{H}(X)$,*

$$F(S) := \bigcup_{i=1}^{N} f_i(S)$$

is a contraction mapping for every $S \in \mathcal{H}(X) := \{S \subseteq X | S \text{ is compact and } S \neq \emptyset\}$ and it has a unique fixed point $A \in \mathcal{H}(X)$ such that $F(A) = A$. For each $S \in \mathcal{H}(X)$, the composition sequence $(F^n(S))_{n=0}^{\infty}$ is convergent to the fixed point A and this fixed point is called the attractor of the IFS. In addition, F has an α-contractivity factor, where

$$\alpha = \max\big\{\alpha_i \mid \alpha_i \text{ is the contractivity factor of } f_i \text{ for all } i = 1, \ldots, N\big\}.$$

2. Approximation to fractals by non-affine contraction mappings

In this section, we give our main approximation theorem. To this end, we need to define the following sequence of nonlinear transformations $(H_k)_{k\in\mathbb{N}}$.

We define $H_k : X \to X$ such that H_k is a Lipschitz continuous function with Lipschitz constant K_H, namely, there exists a constant K_H satisfying

$$d(H_k(x), H_k(y)) \leq K_H d(x, y)$$

for all $x, y \in X$ and $k \in \mathbb{N}$.

In addition, we need the following assumption: Let $S \in \mathcal{H}(X)$ be given. Then

$$\lim_{k\to\infty} H_k(u) = u \text{ (uniformly)} \tag{1}$$

holds for all $u \in S$, i.e., for all $\varepsilon > 0$, there exists a number $k_0 \in \mathbb{N}$ such that for all $u \in S$ and $k \geq k_0$

$$d(H_k(u), u) < \varepsilon.$$

Now, we obtain the following lemmas.

Lemma 2.1. *If (1) is satisfied, then for a given $S \in \mathcal{H}(X)$*

$$\lim_{k\to\infty} H_k(S) = S$$

holds.

Proof. First of all, let us recall the definition of the Hausdorff metric. Let $A, B \subset X$ be given, then the Hausdorff metric $\tilde{d}$ is defined by

$$\tilde{d}(A, B) = \max\Big\{\sup_{x\in A} \bar{d}(x, B), \sup_{y\in B} \bar{d}(A, y)\Big\},$$

where $\bar{d}(u_0, U) = \inf_{u\in U} d(u_0, u)$.

Let $S \in \mathscr{H}(X)$ be fixed and let $y \in H_k(S)$ be arbitrarily given. Then there exists an element $u \in S$ such that $y = H_k(u)$. Now from (1), for any $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $d(H_k(u), u) < \varepsilon/2$ for all $k \geq k_0$ and $u \in S$. So we can clearly see that

$$\bar{d}(y, S) = \bar{d}(H_k(u), S) = \inf_{s\in S} d(H_k(u), s) \leq d(H_k(u), u) < \frac{\varepsilon}{2}$$

for all $k \geq k_0$. Then, since the convergence in (1) is uniform, from the previous inequality

$$\begin{aligned}\sup_{y\in H_k(S)} \bar{d}(y, S) &= \sup_{H_k(u)\in H_k(S)} \bar{d}(H_k(u), S)\\ &\leq \sup_{H_k(u)\in H_k(S)} d(H_k(u), u) \leq \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for all $k \geq k_0$. On the other hand, let $x \in S$ be arbitrarily given. Then from (1) there exists an $H_k(x) \in H_k(S)$ such that for all $\varepsilon > 0$, there exists a $k_1 \in \mathbb{N}$ such that $d(H_k(x), x) < \varepsilon/2$ for all $x \in S$ and $k \geq k_1$. Therefore, for all $k \geq k_0$,

$$\begin{aligned}\bar{d}(H_k(S), x) &= \inf_{H_k(u)\in H_k(S)} d(H_k(u), x)\\ &\leq d(H_k(x), x) < \frac{\varepsilon}{2}.\end{aligned}$$

It follows from the previous inequality and (1) that

$$\sup_{x\in S} \bar{d}(H_k(S), x) \leq \sup_{x\in S} d(H_k(x), x) \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $k \geq k_1$. Therefore, we finally conclude

$$\tilde{d}(H_k(S), S) < \varepsilon$$

for all $k \geq \max\{k_0, k_1\}$, which completes the proof. ■

Lemma 2.2. *Let $\{X; f_1, f_2, \ldots, f_N\}$ be an IFS and K_F be the contractivity factor of $F(S) := \bigcup_{i=1}^{N} f_i(S)$. Then for each $k \in \mathbb{N}$, $\{X; H_k(f_1), H_k(f_2), \ldots, H_k(f_N)\}$ is also an IFS, provided that*

$$K_F K_H < 1 \tag{2}$$

where K_H is the Lipschitz constant of H_k.

Proof. From the definition of H_k, for every $1 \le i \le N$, one can easily see that the inequalities

$$\begin{aligned} d(H_k(f_i(x)), H_k(f_i(y))) &\le K_H d(f_i(x), f_i(y)) \\ &\le K_H K_F d(x, y) < d(x, y) \end{aligned}$$

hold, which implies $H_k(f_i)$ is a contraction mapping for all $1 \le i \le N$ and $k \in \mathbb{N}$. Thus, $\{X; H_k(f_1), H_k(f_2), \dots, H_k(f_N)\}$ is an IFS for each $k \in \mathbb{N}$. ∎

Our main approximation theorem is given below.

Theorem 2.3. *Assume that* $\{X; f_1, f_2, \dots, f_N\}$ *be an IFS and* (1) *holds. Then, for any* $S \in \mathcal{H}(X)$

$$\lim_{k\to\infty} \lim_{n\to\infty} F_k^n(S) = \lim_{n\to\infty} F^n(S) = A$$

holds, where A *is the attractor of* $\{X; f_1, f_2, \dots, f_N\}$, $F_k(S) := \bigcup_{i=1}^{N}(H_k \circ f_i)(S)$ *and* $F(S) := \bigcup_{i=1}^{N} f_i(S)$.

Proof. By Lemma 2.1, $\{X; H_k(f_1), H_k(f_2), \dots, H_k(f_N)\}$ is an IFS for each $k \in \mathbb{N}$. Therefore, from Theorem 1.4 there exists a unique fixed point A_k such that

$$\lim_{n\to\infty} F_k^n(S) = A_k$$

with the contractivity factor $K_H K_F \in [0, 1)$, where K_F is the contractivity factor of $F(S)$. Now, our aim is to show that

$$\lim_{k\to\infty} A_k = A.$$

Changing the order of limits, we get

$$\begin{aligned} \lim_{k\to\infty} A_k &= \lim_{k\to\infty} \lim_{n\to\infty} F_k^n(S) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} F_k^n(S) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} (F_k \circ F_k \circ \dots \circ F_k)(S) \\ &= \lim_{n\to\infty} \lim_{k_1,\dots,k_n\to\infty} (F_{k_1} \circ F_{k_2} \circ \dots \circ F_{k_n})(S) \\ &= \lim_{n\to\infty} \lim_{k_n\to\infty} \cdots \lim_{k_1\to\infty} (F_{k_1} \circ F_{k_2} \circ \dots \circ F_{k_n})(S). \end{aligned} \tag{3}$$

On the other hand, from Lemma 2.1 we observe the equalities

$$\begin{aligned} \lim_{k\to\infty} F_k(S) = \lim_{k\to\infty} \bigcup_{i=1}^{N} H_k(f_i(S)) &= \lim_{k\to\infty} H_k\left(\bigcup_{i=1}^{N} f_i(S)\right) \\ = \bigcup_{i=1}^{N} f_i(S) = F(S). \end{aligned}$$

Using this expression in (3) for $k_1 \to \infty$ yields

$$\begin{aligned}\lim_{k\to\infty} A_k &= \lim_{n\to\infty}\lim_{k_n\to\infty}\cdots\lim_{k_1\to\infty} F_{k_1}(F_{k_2}\circ\cdots\circ F_{k_n}(S))\\ &= \lim_{n\to\infty}\lim_{k_n\to\infty}\cdots\lim_{k_2\to\infty} F(F_{k_2}\circ F_{k_3}\circ\cdots\circ F_{k_n}(S)).\end{aligned}$$

Since F is continuous, by a similar process we easily get

$$\begin{aligned}\lim_{k\to\infty} A_k &= \lim_{n\to\infty}(F\circ F\circ\cdots\circ F)(S)\\ &= \lim_{n\to\infty} F^n(S) = A,\end{aligned}$$

which completes the proof. ■

Corollary 2.4. *Theorem 2.3 states that, for every $k \in \mathbb{N}$, we get a sequence of attractors which correspond to new fractals and these attractors approach to the fractal obtained as the attractor of $\{X; f_1, f_2, \ldots, f_N\}$ when $k \to \infty$.*

3. Applications

In this section, we give some sequence of nonlinear transformations (H_k) for some well-known fractals and then we illustrate our approximations using non-affine contraction mappings. Let S be any fixed compact subset of $\mathbb{R}^N$ for some suitable $N \in \mathbb{N}$. Consider the following examples.

Example 3.1 (Approximation to the right Sierpinski gasket). Let $H_k : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $H_k(u_1, u_2) := (H_k^1(u_1, u_2), H_k^2(u_1, u_2))$, where

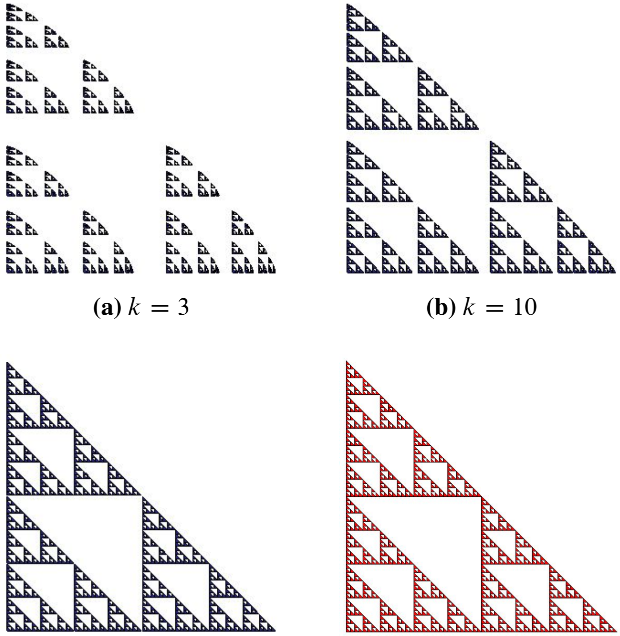
$$H_k^i(u_1,u_2) := \begin{cases} k\ln\left(1+\dfrac{u_i}{k}\right), & \text{if } u_i \in [0,1),\\ ku_i\ln\left(1+\dfrac{1}{k}\right), & \text{if } u_i \in [1,\infty),\end{cases}$$

for $1 \le i \le 2$. It is possible to extend H_k in the odd way. We see from [1] that H_k satisfies the Lipschitz condition under Euclidean metric with Lipschitz constant $K_H = 1$. In addition, since $H_k(u) \le H_{k+1}(u)$ for all $u \in S$, by Dini's theorem

$$\lim_{k\to\infty} H_k(u_1,u_2) = (u_1,u_2) \text{ (uniformly)}$$

holds and hence, condition (1) is satisfied. Although our estimation is true for every compact subset S of $\mathbb{R}^2$, we consider the set

$$\begin{aligned} S =& \{(u_1,0) : 0 \le u_1 \le 1\} \cup \{(0,u_2) : 0 \le u_2 \le 1\}\\ &\cup \{(u_1,u_2) : 0 \le u_1, u_2 \le 1, u_1+u_2 = 1\}\end{aligned}$$



(a) $k = 3$ **(b)** $k = 10$

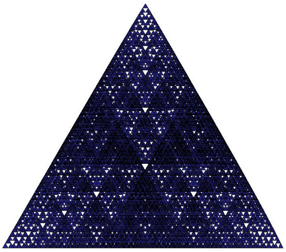
(c) $k = 100$ **(d)** Right Sierpinski gasket

Figure 1. Approximation to the right Sierpinski gasket.

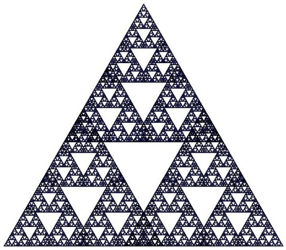
and the IFS $\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3)\}$, where $H_k(f_i)$ are clearly non-affine and the f_i's are the affine contraction mappings,

$$f_1(u_1, u_2) = \left(\frac{u_1}{2}, \frac{u_2}{2}\right),$$
$$f_2(u_1, u_2) = \left(\frac{u_1 + 1}{2}, \frac{u_2}{2}\right),$$
$$f_3(u_1, u_2) = \left(\frac{u_1}{2}, \frac{u_2 + 1}{2}\right),$$

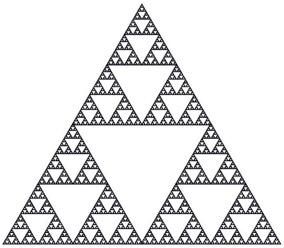
for $1 \leq i \leq 3$. Using non-affine contraction mappings, from Theorem 2.3 we may estimate to the right Sierpinski gasket. For this approximation, see Figure 1, here (d) represents the right Sierpinski gasket, and (a), (b) and (c) represent our approximation for certain values of $k \in \mathbb{N}$.



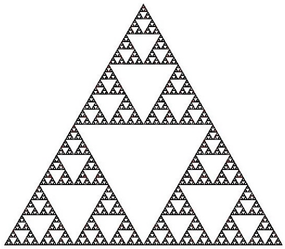
(a) $k = 3$



(b) $k = 7$



(c) $k = 100$



(d) Sierpinski Gasket

Figure 2. Approximation to the Sierpinski gasket using H_k as defined in (4).

Example 3.2 (Approximation to the Sierpinski gasket). In this example, we consider the kernel

$$H_k(u_1, u_2) = \left(u_1 + \sin\left(\frac{u_1}{k}\right), u_2 + \sin\left(\frac{u_2}{k}\right)\right). \tag{4}$$

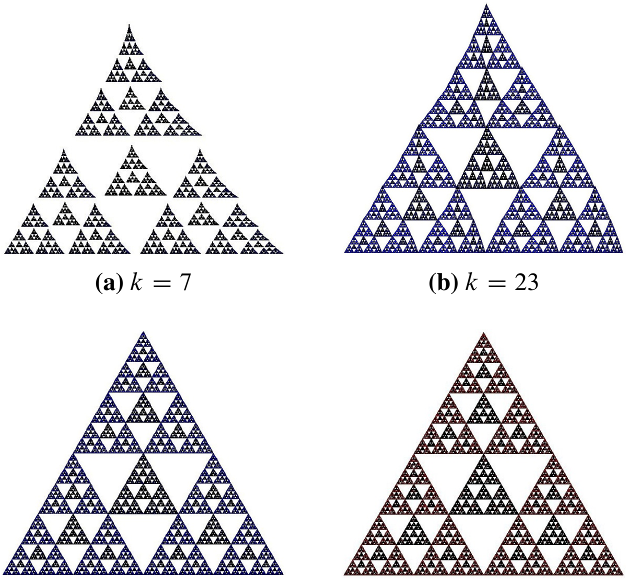
By the definition of H_k, it is clear that

$$|H_k(u_1, u_2) - H_k(v_1, v_2)| \le \frac{3}{2}|(u_1, u_2) - (v_1, v_2)|$$

for all $k > 1$. Since the contractivity factor of the following contraction mappings of Sierpinski gasket is $1/2$, the new contractivity factor will be $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} < 1$:

$$\begin{aligned} f_1(u_1, u_2) &= \left(\frac{u_1}{2}, \frac{u_2}{2}\right), \\ f_2(u_1, u_2) &= \left(\frac{u_1 + 1}{2}, \frac{u_2}{2}\right), \\ f_3(u_1, u_2) &= \left(\frac{2u_1 + 1}{4}, \frac{2u_2 + \sqrt{3}}{4}\right). \end{aligned}$$

Furthermore, since there exists a constant $M > 0$ such that $|\sin\frac{u}{k}| \le \frac{u}{k} \le \frac{M}{k}$ on a compact set S, condition (1) holds. Now, let us consider the new IFS $\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3)\}$, which satisfies all the assumptions of our main theorem. Then it is possible to approach the Sierpinski gasket. See Figure 2 for this estimation.



(a) $k = 7$ **(b)** $k = 23$ **(c)** $k = 450$ **(d)** added Sierpinski Gasket

Figure 3. Approximation to the added Sierpinski gasket using H_k as defined in (5).

Example 3.3 (Approximation to the added Sierpinski gasket). Let us consider the kernel

$$H_k(u_1, u_2) = \left(u_1 + \frac{\sin\left(\frac{k}{u_1^2+1}\right)}{4k}, u_2 + \frac{\sin\left(\frac{k}{u_2^2+1}\right)}{4k} \right). \tag{5}$$

Then, one can clearly observe that

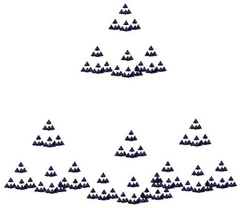
$$|H_k(u_1, u_2) - H_k(v_1, v_2)| \leq \frac{3}{2} |(u_1, u_2) - (v_1, v_2)|$$

for all $k \in \mathbb{N}$. Also, condition (1) is clearly satisfied. Since the contractivity factor of the set $\{\mathbb{R}^2, f_1, f_2, f_3, f_4\}$ is $1/2$, the new contractivity factor will be equal to $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} < 1$. Then, from the above lemmas, $\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3), H_k(f_4)\}$ is an IFS. Using the contraction mappings of the added Sierpinski triangle

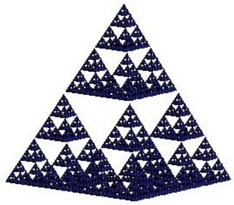
$$f_1(x, y) = \left(\frac{x}{4} + \frac{3}{8}, \frac{y}{4} + \frac{\sqrt{3}}{8} \right), \qquad f_2(x, y) = \left(\frac{x}{2}, \frac{y}{2} \right),$$

$$f_3(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2} \right), \qquad f_4(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4} \right),$$

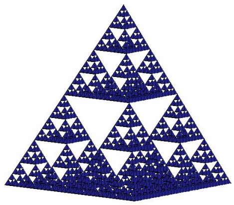
then we get the approximation in Figure 3.



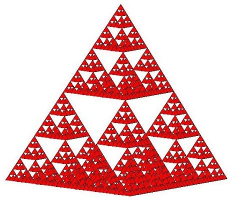
(a) $k = 1$



(b) $k = 4$



(c) $k = 30$



(d) Sierpinski Tetrahedron

Figure 4. Approximation to the Sierpinski tetrahedron using H_k as defined in (6).

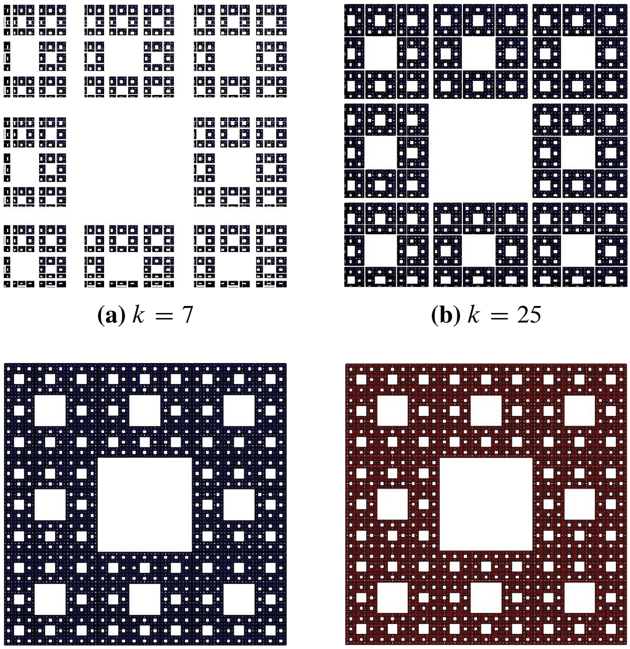
Example 3.4 (Approximation to the Sierpinski tetrahedron). Now, we will investigate the approximation to the Sierpinski tetrahedron. For this reason, we consider the transformation

$$H_k(u_1, u_2, u_3) = \left(\frac{e^k u_1}{e^k + 1}, \frac{e^k u_2}{e^k + 1}, \frac{e^k u_3}{e^k + 1}\right) \tag{6}$$

for every $k \in \mathbb{N}$. As given in [2], it is clear that H_k is Lipschitz continuous with Lipschitz constant $K_H = \frac{1}{2}$, and satisfies condition (1). Therefore, by Lemma 2.1 and Lemma 2.2 the set $\{\mathbb{R}^3, H_k(f_1), H_k(f_2), H_k(f_3), H_k(f_4)\}$ is an IFS. Here, the contraction mappings of Sierpinski tetrahedron are given by:

$$\begin{aligned}
f_1(u_1, u_2, u_3) &= \left(\frac{u_1}{2}, \frac{u_2}{2}, \frac{u_3}{2}\right), \\
f_2(u_1, u_2, u_3) &= \left(\frac{u_1 + 1}{2}, \frac{u_2}{2}, \frac{u_3}{2}\right), \\
f_3(u_1, u_2, u_3) &= \left(\frac{2u_1 + 1}{4}, \frac{2u_2 + \sqrt{3}}{4}, \frac{u_3}{2}\right), \\
f_4(u_1, u_2, u_3) &= \left(\frac{2u_1 + 1}{4}, \frac{6u_2 + \sqrt{3}}{12}, \frac{3u_3 + \sqrt{6}}{6}\right).
\end{aligned}$$

By Theorem 2.3, we may obtain an approximation to the Sierpinski tetrahedron, and this approximation is illustrated in Figure 4.



(a) $k = 7$ (b) $k = 25$

(c) $k = 295$ (d) Sierpinski Carpet

Figure 5. Approximation to the Sierpinski carpet using H_k as defined in (7).

Example 3.5 (Approximation to the Sierpinski carpet). Let a Lipschitz continuous function H_k with $K_H = 1$ be defined by

$$H_k(u_1, u_2) = \Big(\frac{ku_1^2}{1 + ku_1}, \frac{ku_2^2}{1 + ku_2}\Big) \tag{7}$$

as given in [1]. Since H_k satisfies the Lipschitz continuity, we see by Lemma 2.1 and Lemma 2.2 that

$$\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3), H_k(f_4), H_k(f_5), H_k(f_6), H_k(f_7), H_k(f_8)\}$$

is an IFS with non-affine mappings $H_k(f_i)$, where f_i $(1 \le i \le 8)$ are the usual contraction mappings of Sierpinski carpet defined by

$$f_1(u_1, u_2) = \Big(\frac{u_1}{3}, \frac{u_2}{3}\Big), \qquad f_2(u_1, u_2) = \Big(\frac{u_1 + 1}{3}, \frac{u_2}{3}\Big),$$

$$f_3(u_1, u_2) = \Big(\frac{u_1 + 2}{3}, \frac{u_2}{3}\Big), \qquad f_4(u_1, u_2) = \Big(\frac{u_1}{3}, \frac{u_2 + 1}{3}\Big),$$

$$f_5(u_1, u_2) = \Big(\frac{u_1}{3}, \frac{u_2 + 2}{3}\Big), \qquad f_6(u_1, u_2) = \Big(\frac{u_1 + 2}{3}, \frac{u_2 + 1}{3}\Big),$$

$$f_7(u_1, u_2) = \Big(\frac{u_1 + 1}{3}, \frac{u_2 + 2}{3}\Big), \qquad f_8(u_1, u_2) = \Big(\frac{u_1 + 2}{3}, \frac{u_2 + 2}{3}\Big).$$

Since $H_k(u) \leq H_{k+1}(u)$ for all $u \in S$, by Dini's theorem the condition (1) is satisfied and therefore, by Theorem 2.3, we may obtain the Sierpinski carpet whenever $k \to \infty$ (see Figure 5).

Example 3.6 (Approximation to the box fractal). For this example, we define

$$H_k(u_1, u_2) = \left(u_1 + \frac{\arctan(k/(u_1^2+1))}{k}, u_2 + \frac{\arctan(k/(u_2^2+1))}{k}\right). \qquad (8)$$

Since

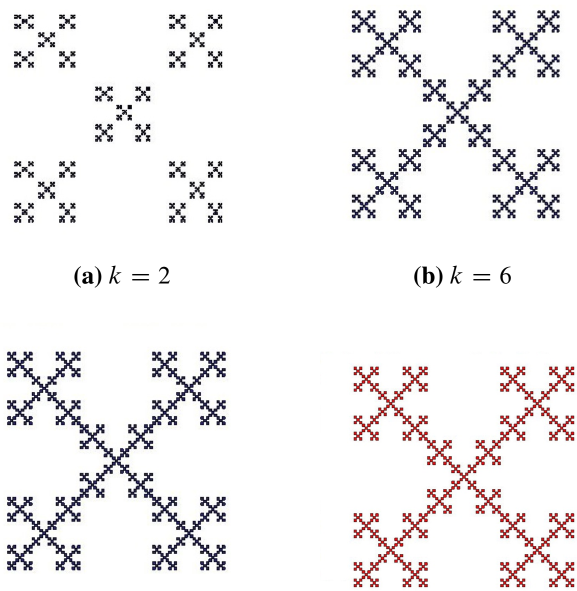
$$\left|\frac{d}{dx}\frac{\arctan(k/(x^2+1))}{k}\right| = \left|\frac{2x}{k^2+x^4+2x^2+1}\right| < 2,$$

we have

$$d(H_k(x), H_k(y)) < 3d(x, y).$$

It is known, that the contractivity factor of the box fractal is $1/3$, and therefore condition (2) is clearly satisfied. In addition, since (1) holds, we get a new IFS

$$\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3), H_k(f_4), H_k(f_5)\}$$



(a) $k = 2$ **(b)** $k = 6$ **(c)** $k = 29$ **(d)** Box fractal

Figure 6. Approximation to the box fractal using H_k as defined in (8).

from the contraction mappings of the box fractal, where

$$f_1(u_1,u_2) = \Big(\frac{u_1}{3}, \frac{u_2}{3}\Big),$$
$$f_2(u_1,u_2) = \Big(\frac{u_1+2}{3}, \frac{u_2}{3}\Big),$$
$$f_3(u_1,u_2) = \Big(\frac{u_1}{3}, \frac{u_2+2}{3}\Big),$$
$$f_4(u_1,u_2) = \Big(\frac{u_1+2}{3}, \frac{u_2+2}{3}\Big),$$
$$f_5(u_1,u_2) = \Big(\frac{u_1+1}{3}, \frac{u_2+1}{3}\Big).$$

By using Theorem 2.3, we have an approximation to the box fractal, which is displayed in Figure 6.

Remark 3.7. So far, we take nonlinear transformations for which $H_k^i = H_k^j$ for all $i \neq j$, however it is possible to take the ones for which $H_k^i \neq H_k^j$ for $i \neq j$. An example for this case is given below.

Example 3.8 (Approximation to the Koch curve). Now, we consider $H_k(u_1,u_2) := (H_k^1(u_1,u_2), H_k^2(u_1,u_2))$, where

$$H_k^1(u_1,u_2) := \begin{cases} k\ln\Big(1+\dfrac{u_1}{k}\Big), & \text{if } u_1 \in [0,1], \\ ku_1\ln\Big(1+\dfrac{1}{k}\Big), & \text{if } u_1 \in (1,\infty), \end{cases}$$

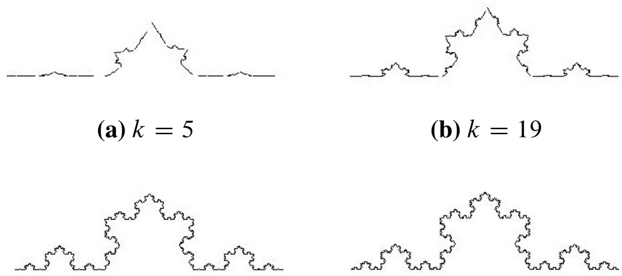
and extended in the odd way, and $H_k^2(u_1,u_2) := \frac{ku_2^2}{1+k_2u_2}$. It is easy to observe that $K_H = 1$ and (1) is satisfied by Dini's theorem. So, we have that

$$\big\{\mathbb{R}^2, H_k(f_1), H_k(f_2), H_k(f_3), H_k(f_4)\big\}$$

is a new IFS with non-affine contraction mappings. By Theorem 2.3, it is possible to approach to the Koch curve considering the following contraction mappings in the above IFS system:

$$f_1(u_1,u_2) = \Big(\frac{u_1}{3}, \frac{u_2}{3}\Big),$$
$$f_2(u_1,u_2) = \Big(\frac{u_1-\sqrt{3}u_2+2}{6}, \frac{\sqrt{3}u_1+u_2}{6}\Big),$$
$$f_3(u_1,u_2) = \Big(\frac{u_1+\sqrt{3}u_2+3}{6}, \frac{-\sqrt{3}u_1+u_2+\sqrt{3}}{6}\Big),$$
$$f_4(u_1,u_2) = \Big(\frac{u_1+2}{3}, \frac{u_2}{3}\Big).$$

This approximation can be seen in Figure 7.



(a) $k = 5$ **(b)** $k = 19$

(c) $k = 192$ **(d)** Koch curve

Figure 7. Approximation to the Koch curve.

Remark 3.9. If we take $H_k(u) = u$, then all the non-affine contraction mappings turn into usual affine contraction mappings. Hence, our estimation both preserve the classical case and extend it to the non-affine setting.

4. Conclusion

In this study, we aim to approximate fractals by using non-affine transformations. For this purpose, we use the IFS and suitable nonlinear mappings. Then, we obtain the sequence of new fractal sets which approach to our main fractal, which is the attractor of the initial IFS. For further study, the geometrical and topological properties of these new fractal sets could be another research problem. On the other hand, this approximation method may also be useful in image processing by fractals.

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