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L-parabolic linear Weingarten spacelike submanifolds immersed in an Einstein manifold

Railane Antonia and Henrique Fernandes de Lima

Abstract. In this paper, we deal with complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p. Under some curvature constraints on the ambient space, we establish an L-parabolicity criterion related to a suitable modified Cheng–Yau's operator L and we apply it to obtain sufficient conditions which guarantee that such a spacelike submanifold must be an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple.

1. Introduction

Let L_p^{n+p} denote an $(n+p)$-dimensional semi-Riemannian manifold of index p. A submanifold M^n immersed in L_p^{n+p} is said to be *spacelike* if the metric on M^n induced from that of L_p^{n+p} is positive definite and it is called *linear Weingarten* when its mean curvature function H and its normalized scalar curvature R satisfy a linear relation of the type $R = aH + b$ for some constants $a, b \in \mathbb{R}$.

In this setting, the second author jointly with de Lima [7] obtained characterization results for linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Einstein manifold $\mathcal{E}_1^{n+1}$ of index 1 considering restrictions on the square length of the second fundamental form and some appropriate curvature constraints of the ambient space which were inspired by the works of Nishikawa [11] and Choi et al. [6, 15]. Later, the second author jointly with Araújo, dos Santos and Velásquez [4] extended these results for the context of an n-dimensional spacelike submanifold M^n immersed with a parallel normalized mean curvature vector in a locally symmetric semi-Riemannian manifold L_p^{n+p} of index p. For this, they assumed the existence of real constants c_1, c_2 and c_3 such that the sectional curvature $\overline{K}$ and curvature tensor

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$\overline{R}$ of L_p^{n+p} satisfy the following conditions:

$$\overline{K}(u,\eta) = \frac{c_1}{n}, \tag{1}$$

for any $u \in TM$ and $\eta \in TM^{\perp}$;

$$\overline{K}(u,v) \geq c_2, \tag{2}$$

for any $u, v \in TM$;

$$\overline{K}(\eta,\xi) = \frac{c_3}{p}, \tag{3}$$

for $\eta, \xi \in TM^{\perp}$; and

$$\langle \overline{R}(\xi,u)\eta, u\rangle = 0, \tag{4}$$

for $u \in TM$ and $\xi, \eta \in TM^{\perp}$ with $\langle \xi, \eta\rangle = 0$. We note that, when $p = 1$, conditions (3) and (4) are automatically satisfied. Afterwards, also assuming this set of constraints, the second author jointly with Araújo, Barboza and Velásquez [3] applied the techniques developed by Yang and Hou in [16] and by Liu and Zhang in [10] to get sufficient conditions guaranteeing that such a spacelike submanifold M^n is either totally umbilical or isometric to an isoparametric hypersurface of a totally geodesic submanifold $L_1^{n+1} \hookrightarrow L_p^{n+p}$, with two distinct principal curvatures, one of which is simple.

More recently, the authors [8] studied complete linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Einstein manifold obeying curvature constraints (1) and (2). In this setting, they proved a parabolicity criterion related to a suitable Cheng–Yau modified operator and they applied it to obtain sufficient conditions which guarantee that such a spacelike hypersurface must be either totally umbilical or isometric to an isoparametric spacelike hypersurface with two distinct principal curvatures one of which is simple. In [9], Liu and Xie used the Omori–Yau's generalized maximum principle to obtain classification results concerning a complete spacelike hypersurface M^n with constant mean curvature in $\mathcal{E}_1^{n+1}$ without asking that the ambient space is locally symmetric but also assuming the curvature constraints (1) and (2).

Motivated by the works described above, here we deal with complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p obeying curvature constraints (1), (2), (3) and (4). In this setting, we establish an L-parabolicity criterion related to a suitable modified Cheng–Yau's operator L (see Proposition 3.2), which is a consequence of a more general criterion related to divergent-type operators due to Pigola, Rigoli and Setti (see [13, Theorem 2.6]), and we apply it to obtain sufficient conditions which guarantee that such a spacelike submanifold must be an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple (see Theorem 3.4 and Corollary 3.5).

It is worth mentioning that the works [3, 4] contain results similar to ours, under the assumption that the ambient space is locally symmetric. But, in our setting, we can find examples of Einstein manifolds which are not locally symmetric. Indeed, according to [9, Example 1.1], the semi-Riemannian product space $\mathbb{R}_p^p \times M^n$, where M^n is a Ricci flat Riemannian manifold, is an Einstein manifold of index p. Moreover, supposing that the sectional curvature K_M of M^n is such that $K_M(u, v) \geq c_2$ for any $u, v \in TM$ and some constant c_2 and considering the spacelike submanifold given by the inclusion $\iota : M^n \hookrightarrow \mathbb{R}_p^p \times M^n$, we can verify that the curvature constraints (1), (2), (3) and (4) are satisfied. However, if M^n is not locally symmetric, then $\mathbb{R}_p^p \times M^n$ is not a locally symmetric manifold.

2. Preliminaries

Let M^n be a spacelike submanifold immersed in a semi-Riemannian space L_p^{n+p} of index p. In this context, we choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in L_p^{n+p}, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and $e_{n+1}, \ldots, e_{n+p}$ are normal to M^n. Using the following convention of indices:

$$1 \leq A, B, C, \ldots \leq n + p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p,$$

and taking the corresponding dual coframes $\omega_1, \ldots, \omega_{n+1}$, the semi-Riemannian metric of L_p^{n+p} is given by $ds^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$, $1 \leq i \leq n$ and $n + 1 \leq \alpha \leq n + p$. Denoting by $\{\omega_{AB}\}$ the connection forms of L_p^{n+p}, we have that the structure equations of L_p^{n+p} are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{5}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D, \tag{6}$$

where $\overline{R}_{ABCD}$ denote the components of the curvature tensor of L_p^{n+p}. In this setting, denoting by $\overline{R}_{CD}$ and $\overline{R}$ the Ricci tensor and the scalar curvature of L_p^{n+p}, respectively, we have

$$\overline{R}_{CD} = \sum_B \varepsilon_B \overline{R}_{CBDB}, \quad \overline{R} = \sum_A \varepsilon_A \overline{R}_{AA}.$$

Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the curvature tensor of L_p^{n+p} are defined by

$$\sum_E \varepsilon_E \overline{R}_{ABCD;E}\omega_E = d\overline{R}_{ABCD} - \sum_E \varepsilon_E \big(\overline{R}_{EBCD}\omega_{EA} + \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED}\big).$$

Restricting all the tensors to M^n in L_p^{n+p}, since $\omega_\alpha = 0$ on M^n, we get

$$\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0.$$

So, from Cartan's Lemma we obtain

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad \text{with } h_{ij}^\alpha = h_{ji}^\alpha. \tag{7}$$

This gives the second fundamental form of M^n, that is $B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha$, and its square length $S = |B|^2 = \sum_{\alpha,i,j}(h_{ij}^\alpha)^2$. Furthermore, the mean curvature vector h and the mean curvature function H of M^n are defined, respectively, by

$$h = \frac{1}{n}\sum_\alpha \Big(\sum_i h_{ii}^\alpha\Big) e_\alpha \quad \text{and} \quad H = |h| = \frac{1}{n}\sqrt{\sum_\alpha \Big(\sum_i h_{ii}^\alpha\Big)^2}.$$

From (5) and (6), we deduce that the connection forms $\{\omega_{ij}\}$ of M^n are characterized by the following structure equations:

$$\begin{aligned} d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \end{aligned} \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n. From previous structure equations, we obtain the Gauss equation (see [12, Theorem 4.5])

$$R_{ijkl} = \overline{R}_{ijkl} - \sum_\beta (h_{ik}^\beta h_{jl}^\beta - h_{il}^\beta h_{jk}^\beta). \tag{9}$$

From (9) we also get the relation

$$S = n^2H^2 + n(n-1)R - \sum_{i,j} \overline{R}_{ijij}, \tag{10}$$

where R stands for the normalized scalar curvature of M^n. Moreover, the first covariant derivatives h_{ijk}^α of h_{ij} satisfy

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{jk}^\alpha \omega_{ki} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}. \tag{11}$$

Then, by exterior differentiation of (7) we get the Codazzi equation (see [12, Theorem 4.33])

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = \overline{R}_{\alpha ijk}. \tag{12}$$

The second covariant derivatives h^{α}_{ijkl} of h^{α}_{ij} are given by

$$\sum_l h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} - \sum_l h^{\alpha}_{ljk}\omega_{li} - \sum_l h^{\alpha}_{ilk}\omega_{lj} - \sum_l h^{\alpha}_{ijl}\omega_{lk} - \sum_\beta h^{\beta}_{ijk}\omega_{\beta\alpha}.$$

Taking the exterior derivative in (11), we obtain the Ricci formula

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = -\sum_m h^{\alpha}_{im}R_{mjkl} - \sum_m h^{\alpha}_{mj}R_{mikl}. \tag{13}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ to M^n, we get

$$\begin{aligned}\overline{R}_{\alpha ijkl} = \overline{R}_{\alpha ijk;l} &+ \sum_\beta \overline{R}_{\alpha\beta jk}h^{\beta}_{il} + \sum_\beta \overline{R}_{\alpha i\beta k}h^{\beta}_{jl}\\ &+ \sum_\beta \overline{R}_{\alpha ij\beta}h^{\beta}_{kl} + \sum_{m,k} \overline{R}_{mijk}h^{\alpha}_{lm},\end{aligned} \tag{14}$$

where $\overline{R}_{\alpha ijkl}$ denotes the covariant derivative of $\overline{R}_{\alpha ijk}$ as a tensor on M^n. Moreover, since we are supposing that M^n has a flat normal bundle, that is, $R^{\perp} = 0$ (equivalently, $R_{\alpha\beta jk} = 0$), $\overline{R}_{\alpha\beta jk}$ satisfy the Ricci equation

$$\overline{R}_{\alpha\beta ij} = \sum_k (h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{kj}h^{\beta}_{ik}). \tag{15}$$

3. Main results

From now on, we will deal with a spacelike submanifold M^n immersed with parallel normalized mean curvature vector in $\mathcal{E}^{n+p}_p$, which means that the mean curvature function H is positive and h is parallel as a section of the normal bundle. In particular, we can choose a orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ of $T\mathcal{E}^{n+p}_p$ such that $e_{n+1} = \frac{h}{H}$. So, we get

$$H^{n+1} := \frac{1}{n}\operatorname{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} := \frac{1}{n}\operatorname{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2, \tag{16}$$

where h^{α} denotes the matrix (h^{α}_{ij}).

Considering this previous setting, we obtain the following Simons type formula which is derived from the proofs of [4, Lemma 2] and [9, Lemma 3.1].

Lemma 3.1. *Let M^n be a spacelike submanifold immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all operators B_η with $\eta \in TM^\perp$, where $\langle B_\eta u, v\rangle := \langle B(u,v), \eta\rangle$ for any u, $v \in TM$. Then*

$$\begin{aligned}\frac{1}{2}\Delta S = {} & |\nabla B|^2 + 2\Big(\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik}\Big) \\ & + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha i \beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha k \beta i} \\ & + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \overline{R}_{\alpha k \beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{kk}^\beta \overline{R}_{\alpha i \beta j} + n \sum_{i,j} h_{ij}^{n+1} H_{ij} \\ & - nH \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} + \sum_{\alpha,\beta} [\mathrm{tr}(h^\alpha h^\beta)]^2 \\ & + \frac{3}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha), \end{aligned} \tag{17}$$

where $N(A) = \mathrm{tr}(AA^t)$, for any matrix $A = (a_{ij})$.

Proof. The Laplacian Δh_{ij}^α of the components h_{ij}^α of the second fundamental form is defined by $\Delta h_{ij}^\alpha := \sum_k h_{ijkk}^\alpha$. Consequently, from the Codazzi equation (12) we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{\alpha,i,j} h_{ij}^\alpha \Big(\sum_k h_{ijkk}^\alpha\Big) + \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 \\ &= \sum_{i,j,k,\alpha} h_{ij}^\alpha \overline{R}_{\alpha ijkk} + \sum_{\alpha,i,j,k} h_{ij}^\alpha h_{kijk}^\alpha + |\nabla B|^2. \end{aligned} \tag{18}$$

On the other hand, since $(\mathcal{E}_p^{n+p}, \overline{g})$ is an Einstein manifold, the components of its Ricci tensor satisfy $\overline{R}_{AB} = \lambda \overline{g}_{AB}$, for some constant $\lambda \in \mathbb{R}$. Moreover, since we are supposing that there exists an orthogonal basis for TM that diagonalizes simultaneously all B_η with $\eta \in TM^\perp$, we can consider $\{e_1, \dots, e_n\}$ a local orthonormal frame on M^n such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ for all $\alpha \in \{n+1, \dots, n+p\}$. So, proceeding as in [9], from the differential Bianchi identity and the fact that $\overline{g}_{AB;C} \equiv 0$ we get

$$\begin{aligned}\sum_{i,k,\alpha} \lambda_i^\alpha \overline{R}_{\alpha iik;k} &= -\sum_{i,k,\alpha} \lambda_i^\alpha \big(\overline{R}_{ikik;\alpha} + \overline{R}_{k\alpha ik;i}\big) \\ &= -\sum_{i,\alpha} \lambda_i^\alpha \big(\overline{R}_{ii;\alpha} - \overline{R}_{\alpha i;i}\big) \\ &= -\sum_{i,\alpha} \lambda_i^\alpha \big(\lambda \overline{g}_{ii;\alpha} - \lambda \overline{g}_{\alpha i;i}\big) = 0 \end{aligned} \tag{19}$$

and

$$\sum_{i,k,\alpha} \lambda_i^\alpha \overline{R}_{\alpha kik;i} = \sum_{i,\alpha} \lambda_i^\alpha \overline{R}_{\alpha i;i} = \sum_{i,\alpha} \lambda_i^\alpha \lambda \overline{g}_{\alpha i;i} = 0, \tag{20}$$

where $\overline{R}_{ijkl;m}$ are the covariant derivatives of $\overline{R}_{ijkl}$ on $\mathcal{E}_p^{n+p}$. Consequently, from (19) and (20) we obtain

$$\sum_{i,j,k,\alpha} (\overline{R}_{\alpha ijk;k} + \overline{R}_{\alpha iki;j}) h_{ij}^\alpha = 0. \tag{21}$$

Therefore, using (9), (13)–(18) jointly with (21), we can reason as in the proof of [4, Lemma 2] to deduce formula (17). ■

According to [5], the Cheng–Yau's operator $\square$ acting on a smooth function $f : M^n \to \mathbb{R}$ is given by

$$\square f = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij} = nH\Delta f - \sum_{i,j} h_{ij}^{n+1} f_{ij}, \tag{22}$$

where f_{ij} denote the components of the Hessian of f and the normal vector field e_{n+1} is taken in the direction of the mean curvature vector, that is $e_{n+1} = \frac{h}{H}$. From (22), we also have

$$\square f = \mathrm{tr}(P_1 \circ \nabla^2 f), \tag{23}$$

where

$$P_1 = nHI - h^{n+1}, \tag{24}$$

I being the identity in the algebra of smooth vector fields on M^n, $h^{n+1} = (h_{ij}^{n+1})$ denotes the second fundamental form of M^n in the direction e_{n+1} and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of f.

In order to study linear Weingarten submanifolds of $\mathcal{E}_p^{n+p}$ whose mean curvature function H and normalized scalar curvature R satisfy $R = aH + b$ for some $a, b \in \mathbb{R}$, we will consider the following modified Cheng–Yau's operator

$$L = \square + \frac{n-1}{2} a\Delta. \tag{25}$$

Equivalently, for all smooth function $f : M^n \to \mathbb{R}$, the definition (25) can be rewritten as follows:

$$Lf = \mathrm{tr}(P \circ \nabla^2 f),$$

where

$$P = \left(nH + \frac{n-1}{2} a\right) I - h^{n+1}. \tag{26}$$

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a smooth function $f : M^n \to \mathbb{R}$ such that

$$\Delta f \geq 0 \quad \text{and} \quad \sup_M f < +\infty \quad \text{implies} \quad f = \text{constant}.$$

Extending this previous concept to the operator L defined in (25), M^n is said to be L-parabolic (or parabolic with respect to the operator L) if the constant functions are the only smooth functions $f : M^n \to \mathbb{R}$ which are bounded from above and satisfy $Lf \geq 0$. In other words, for a smooth function $f : M^n \to \mathbb{R}$ such that

$$Lf \geq 0 \quad \text{and} \quad \sup_M f < +\infty \quad \text{implies} \quad f = \text{constant}.$$

At this point, we also observe that, denoting by $\overline{R}_{CD}$ the components of the Ricci tensor of $\mathcal{E}_p^{n+p}$, the scalar curvature $\overline{R}$ of $\mathcal{E}_p^{n+p}$ is given by

$$\overline{R} = \sum_{A}^{n+p} \varepsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} - 2\sum_{i,\alpha} \overline{R}_{i\alpha i\alpha} + \sum_{\alpha,\beta} \overline{R}_{\alpha\beta\alpha\beta}.$$

Consequently, assuming that $\mathcal{E}_p^{n+p}$ satisfies conditions (1) and (3), we get

$$\overline{R} = n(n-1)\overline{\mathcal{R}} - 2pc_1 + (p-1)c_3, \tag{27}$$

where $\overline{\mathcal{R}} := \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$. Hence, since the scalar curvature of an Einstein manifold is constant, from (27) we conclude that $\overline{\mathcal{R}}$ is a constant naturally attached to an Einstein manifold $\mathcal{E}_p^{n+p}$ satisfying (1) and (3).

The next result provides sufficient conditions which guarantee the L-parabolicity of a linear Weingarten spacelike submanifold immersed in $\mathcal{E}_p^{n+p}$. This L-parabolicity criterion is obtained as an application of [13, Theorem 2.6].

Proposition 3.2. *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p, such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \leq \overline{\mathcal{R}}$. If H is bounded on M^n and, for some reference point $o \in M^n$ and some $\delta > 0$,*

$$\int_{\delta}^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{28}$$

where B_t denotes the geodesic ball of radius t in M^n centered at o, then M^n is L-parabolic.

Proof. Let us consider on M^n the symmetric $(0, 2)$-tensor field ξ given by

$$\xi(X, Y) := \langle PX, Y \rangle,$$

for all $X, Y \in TM$ or, equivalently,

$$\xi(\nabla f, \cdot)^{\sharp} = P(\nabla f),$$

where $\sharp : T^*M \to TM$ denotes the musical isomorphism, for all smooth function $f : M^n \to \mathbb{R}$, and P is defined in (26).

Since $\mathcal{E}_p^{n+p}$ is an Einstein manifold, denoting by $\overline{\mathrm{Ric}}$ the Ricci tensor of $\mathcal{E}_p^{n+p}$, we have $\overline{\mathrm{Ric}} = \lambda\langle\, ,\rangle$ for some constant $\lambda \in \mathbb{R}$. Thus, from [1, Lemma 3.1] we get

$$\begin{aligned}\langle \operatorname{div} P_1, \nabla f\rangle &= \sum_i \langle \overline{\mathrm{R}}(e_{n+1}, e_i)e_i, \nabla f\rangle = \overline{\mathrm{Ric}}(e_{n+1}, \nabla f)\\ &= \lambda\langle e_{n+1}, \nabla f\rangle = 0,\end{aligned} \tag{29}$$

where P_1 is defined in (24) and $\overline{\mathrm{R}}$ denotes the curvature tensor of $\mathcal{E}_p^{n+p}$. Choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n, we have

$$\begin{aligned}\operatorname{div}(P_1(\nabla f)) &= \sum_i \langle (\nabla_{e_i} P_1)(\nabla f), e_i\rangle + \langle P_1(\nabla_{e_i} \nabla f), e_i\rangle\\ &= \langle \operatorname{div} P_1, \nabla f\rangle + \square f.\end{aligned} \tag{30}$$

Thus, from (29) and (30) we obtain $\square f = \operatorname{div}(P_1(\nabla f))$. Consequently, we get

$$L(f) = \operatorname{div}(P(\nabla f)). \tag{31}$$

Hence, from (31) we have

$$Lf = \operatorname{div}\big(\xi(\nabla f, \cdot)^{\sharp}\big).$$

Furthermore, since $R = aH + b$ with $b \leq \overline{\mathcal{R}}$, [3, Lemma 3.4] guarantees that L is semi-elliptic or, equivalently, P is positive semi-definite. Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$, we obtain

$$\sum_{i,j}\left(h_{ij}^{n+1}\right)^2 \leq \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 = S.$$

Consequently, from (10) we get

$$n^2H^2 \geq (\lambda_i^{n+1})^2 - n(n-1)aH,$$

for all $i = 1, \ldots, n$. Moreover, since

$$(\lambda_i^{n+1})^2 \leq n^2H^2 + n(n-1)aH \leq \left(nH + \frac{n-1}{2}a\right)^2$$

and taking into account that the normalized mean curvature vector is parallel, we have

$$-nH - \frac{n-1}{2}a \leq \lambda_i^{n+1} \leq nH + \frac{n-1}{2}a,$$

for all $i = 1, \dots, n$. Hence, for all $i \in \{1, \dots, n\}$, we obtain

$$0 \leq \sigma_i \leq 2nH + (n-1)a,$$

where $\sigma_i := nH + \frac{n-1}{2}a - \lambda_i^{n+1}$ are the eigenvalues of the operator P (see [2, Lemma 3]). Consequently, we can define a positive continuous function ξ_+ on $[0, +\infty)$ by

$$\xi_+(t) := 2n \sup_{\partial B_t} H + (n-1)a.$$

From the assumption that H is bounded on M^n, we have

$$\xi_+(t) \leq 2n \sup_M H + (n-1)a < +\infty.$$

Hence, we reach the following estimate:

$$\int_\delta^{+\infty} \frac{dt}{\xi_+(t)\mathrm{vol}(\partial B_t)} \geq \left(2n \sup_M H + (n-1)a\right)^{-1} \int_\delta^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)}.$$

Consequently, from hypothesis (28) we obtain

$$\int_\delta^{+\infty} \frac{dt}{\xi_+(t)\mathrm{vol}(\partial B_t)} = +\infty.$$

Therefore, we are in position to apply [13, Theorem 2.6] to conclude that M^n is L-parabolic. ■

Remark 3.3. It is worth to note that we can reason as in the proof of Proposition 3.2 to infer that an isometric immersion satisfying (28) is $\mathcal{L}$-parabolic for $\mathcal{L} := \mathrm{div}(\mathcal{P}(\nabla \cdot))$, where $\mathcal{P}$ is a positive semi-definite tensor such that $\sup \mathcal{P} < +\infty$ and $\mathrm{div}\mathcal{P} \equiv 0$.

We will also consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha,$$

where $\Phi_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij}$ and H^α is defined by (16). Consequently, we have that

$$\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij} \quad \text{and} \quad \Phi_{ij}^\alpha = h_{ij}^\alpha, \quad n+2 \leq \alpha \leq n+p. \tag{32}$$

Let $|\Phi|^2 = \sum_{\alpha,i,j} (\Phi_{ij}^\alpha)^2$ be the square of the length of Φ. It is not difficult to check that Φ is traceless with

$$|\Phi|^2 = S - nH^2 = nH^2(n-1) + n(n-1)(R - \overline{\mathcal{R}}). \tag{33}$$

From Proposition 3.2, we obtain the following characterization result for complete linear Weingarten spacelike submanifolds.

Theorem 3.4. *Let M^n be a complete linear Weingarten spacelike submanifold immersed with a parallel normalized mean curvature vector and a flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p satisfying conditions (1), (2), (3) and (4), such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \leq \overline{\mathcal{R}}$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all operators B_η with $\eta \in TM^\perp$, where $\langle B_\eta u, v\rangle := \langle B(u, v), \eta\rangle$ for any u, $v \in TM$. When $c := \frac{c_1}{n} + 2c_2 > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{Q(p)}$, where*

$$Q(p) := p(n-2)^2 + 4(n-1).$$

If H is bounded on M^n, $|\Phi| \geq C(n, p, H)$, where

$$C(n, p, H) := \frac{\sqrt{n}}{2\sqrt{n-1}}\Big(p(n-2)H + \sqrt{pQ(p)H^2 - 4p(n-1)c}\Big),$$

and hypothesis (28) is satisfied, then $p = 1$ and M^n is an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple.

Proof. We note that, since the normalized mean curvature vector of M^n is parallel, from the Ricci equation (15) it follows that $h^\alpha h^{n+1} = h^{n+1} h^\alpha$ for all α, that is, h^{n+1} commutes with all the matrices h^α. So, from (32) we have that Φ^{n+1} commutes with all the matrices Φ^α. Since the matrices Φ^α are symmetric and traceless, we can use [14, Lemma 2.6] with Φ^α and Φ^{n+1} in order to obtain

$$\left|\operatorname{tr}((\Phi^\alpha)^2\Phi^{n+1})\right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}. \tag{34}$$

Moreover, using Cauchy–Schwarz inequality we also have that

$$\begin{aligned} p\sum_{\alpha,\beta}[\operatorname{tr}(\Phi^\alpha\Phi^\beta)]^2 &\geq p\sum_\alpha[\operatorname{tr}(\Phi^\alpha)^2]^2 = p\sum_\alpha[N(\Phi^\alpha)]^2 \\ &\geq \Big(\sum_\alpha N(\Phi^\alpha)\Big)^2 = |\Phi|^4. \end{aligned} \tag{35}$$

On the other hand, taking into account our set of constraints on $M^n \hookrightarrow \mathcal{E}_p^{n+p}$ jointly with Lemma 3.1, we can reason as in the proof of [4, Proposition 1] to obtain

$$L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|), \tag{36}$$

where

$$P_{H,p,c}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2 - c).$$

When $c > 0$, if $H^2 \geq \frac{4(n-1)c}{Q(p)}$, then the polynomial $P_{H,p,c}$ has (at least) a positive real root given by $C(n, p, H)$. Thus, since $|\Phi| \geq C(n, p, H)$, we get $P_{H,p,c}(|\Phi|) \geq 0$, with $P_{H,p,c}(|\Phi|) = 0$ if and only if $|\Phi| = C(n, p, H)$. In the case $c \leq 0$, we have that $P_{H,p,c}(|\Phi|) \geq 0$ without any restriction on the values of the mean curvature function H. Consequently, in both cases, from (36) we get that $L(nH) \geq 0$.

But Proposition 3.2 assures that M^n is L-parabolic. So, from the boundedness of H, we get that it is constant on M^n implying, in particular, that $L(nH) = 0$ on M^n. Since $|\Phi| > 0$, we obtain that $P_{H,p,c}(|\Phi|) = 0$. Thus, inequalities (34) and (35) are, in fact, equalities. In particular, we have that $N(\Phi^{n+1}) = \mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2$. Thus, from (33) we obtain

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2 = S - nH^2. \tag{37}$$

Since M^n has parallel normalized mean curvature vector, from (16) we also have

$$\mathrm{tr}(\Phi^{n+1})^2 = S - \sum_{\alpha > n+1} \sum_{i,j} (h_{ij}^{\alpha})^2 - nH^2. \tag{38}$$

Thus, from (37) and (38) we conclude that

$$\sum_{\alpha > n+1} \sum_{i,j} (h_{ij}^{\alpha})^2 = 0.$$

Now, returning to (35) we get

$$p|\Phi|^4 = pN(\Phi^{n+1})^2 = p \sum_{\alpha \geq n+1} [N(\Phi^{\alpha})]^2 = |\Phi|^4.$$

Hence, we conclude that $p = 1$. Moreover, since we are supposing that $b \leq \overline{\mathcal{R}}$, from [4, Lemma 1] and the fact that H is constant on M^n, we obtain that

$$\sum_{i,j,k} (h_{ijk}^{n+1})^2 = n^2 |\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j, k. Therefore, we have that M^n must be an isoparametric spacelike hypersurface of $\mathcal{E}_1^{n+1}$. ■

We close our paper quoting the following consequence of Theorem 3.4.

Corollary 3.5. *Let M^n be a complete linear Weingarten spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}_1^{n+1}$ of index 1 satisfying conditions (1) and (2), such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \leq \overline{\mathcal{R}}$. When $c := \frac{c_1}{n} + 2c_2 > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{(n-2)^2 + 4(n-1)}$. If H is bounded on M^n,*

$$|\Phi| \geq \frac{\sqrt{n}}{2\sqrt{n-1}} \left((n-2)H + \sqrt{n^2H^2 - 4(n-1)c} \right)$$

and hypothesis (28) is satisfied, then M^n is an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple.

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Railane Antonia
Departamento de Matemática, Universidade Federal da Paraíba, João Pessoa 58.051-900, Paraíba, Brazil; railane.silva@academico.ufpb.br

Henrique Fernandes de Lima
Departamento de Matemática, Universidade Federal de Campina Grande, Campina Grande 58.429-970, Paraíba, Brazil; henriquedelima74@gmail.com