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On the optional and orthogonal decompositions of a class of semimartingales

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Abstract. We consider a set $\mathcal{Q}$ of probability measures, which are absolutely continuous with respect to the physical probability measure $\mathbb{P}$ and at least one is equivalent to $\mathbb{P}$. We investigate necessary and sufficient conditions on $\mathcal{Q}$, under which any $\mathcal{Q}$-supermartingale can be decomposed into the sum of a local $\mathcal{Q}$-martingale and a decreasing process. We also provide an orthogonal decomposition of square integrable semimartingale as the orthogonal sum of a local $\mathcal{Q}$-martingale and a square integrable semimartingale. As one application, we state the orthogonal decomposition in an appropriate sense of the polar set of $\mathcal{Q}$. We generalise then the results of a previous article (2021), from finite probability space and discrete time case to general continuous time case.

1. Introduction

One of the key elements in solving problems of pricing and hedging claims in incomplete markets is the well-known optional decomposition theorem. It states that an adapted process V, which is dominated from below, can be written under the form $V = V_0 + \alpha \bullet Y - C$, where $\alpha \bullet Y$ is the stochastic integral with respect to a vector valued semimartingale Y and C is an increasing process if and only if V is a supermartingale under all local equivalent martingale measures $\mathcal{M}_{\text{loc}}(Y)$ of Y. The first version of such theorem was established in the case where Y has continuous paths in El Karoui and Quenez [9], and further extended to the càdlàg paths case in Kramkov [18], Föllmer and Kabanov [10], Delbaen and Schachermayer [8], Föllmer and Kramkov [11]. More generalisations of these results can be found in [1,14,17,22] and [5].

The financial application of the formula is to consider Y as the discounted price process of a finite set of risky financial assets, then the process α represents the investment strategy of an agent in the market, where α^i stands for the units of asset i held

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in the portfolio and C measures the agent's aggregate consumption, while V corresponds to the generated wealth-consumption process starting with initial capital V_0. Inversely for a claim H, we define the process V by $V_t = \text{essup}_{\mathbb{Q}\in\mathcal{M}_{\text{loc}}(Y)} \mathbb{E}^{\mathbb{Q}}(H|\mathcal{F}_t)$, where the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is generated by Y. Thanks to the m-stability property of $\mathcal{M}_{\text{loc}}(Y)$ (see Definition 2.1), the process V is a supermartingale with respect to all elements of $\mathcal{M}_{\text{loc}}(Y)$ and therefore by the optional decomposition theorem, the claim $H = V_T$ can be superhedged via the portfolio V. Unfortunately, this method has a very drastic drawback. The superhedging price V_0, taken as the supremum over all possible scenarios, is very high and far from being the practical one to use. That is why a number of other pricing and hedging alternative methods were investigated. We name for example utility maximization, risk minimization and mean-variance methods. Some of these methods are based on the projection idea, well used for minimizing a certain metric. In our context, the idea is to project the claim H into the vector space vect(Y) of all elements $\alpha \bullet Y_T$ and to write H under sufficient conditions as $H = H_0 + \alpha \bullet Y_T + N$ for some random variable N. By taking the appropriate Hilbert space $\mathcal{L}^2$, N is orthogonal to each element in vect(Y).

A more sophisticated orthogonal decomposition suggests that for a square integrable H, there exists a square integrable martingale L, which is orthogonal to Y, such that $H = H_0 + \alpha \bullet Y_T + L_T$. Two well-known decompositions are stated in this context, namely in Kunita and Watanabe [19] and in Föllmer and Schweizer [12]. We cite other interesting references, which dealt with the same problem under different set of conditions, for example in [6, 16, 20]

In this paper, we consider the framework of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $(\mathcal{F}_t)_{t\in I}$, satisfying the conditions of completeness and right continuity, and a set $\mathcal{Q}$ of $\mathbb{P}$-absolutely continuous probability measures, containing at least one equivalent to $\mathbb{P}$. Our main goal is twofold: first we state necessary and sufficient conditions on $\mathcal{Q}$, under which it is possible to apply the optional decomposition to a given $\mathcal{Q}$-supermartingale. The idea is based on some tools introduced in [3]. Second, we state the orthogonal decomposition of any square integrable semimartingale X, by decomposing it into a $\mathcal{Q}$-martingale Y and a square integrable semimartingale U, which is orthogonal to any $\mathcal{Q}$-martingale. The proof idea is based on the method of projection in $\mathcal{L}^2$ by using the Doob–Meyer decomposition of X and by choosing the appropriate vector space to project into. As one application of the last result, we state the orthogonal decomposition in an appropriate sense of the polar set of $\mathcal{Q}$.

We point out that similar results were stated in [4] for the case of a finite probability space and a finite discrete time axis.

2. Notations and preliminaries

In this section, we introduce the main notations and concepts which will be used later in this paper. We define

- the vector space $\mathcal{L}^p(\mathbb{R}^d)$ of p-integrable random variables with values in $\mathbb{R}^d$ and $\mathcal{L}^p := \mathcal{L}^p(\mathbb{R})$;
- for a semimartingale S, the vector space $\mathcal{L}^p(S)$, of predictable processes α such that the random variable $|\alpha|^p \bullet S_T \in \mathcal{L}^1$, where $\beta \bullet S$ denotes the stochastic integral process of β with respect to S;
- the set $\mathrm{abs}(\mathbb{P})$, to be the set of all $\mathbb{P}$-absolutely continuous probability measures;
- the set $\mathrm{equiv}(\mathbb{P}) := \{\mathbb{Q} \in \mathrm{abs}(\mathbb{P}) : \mathbb{P} \in \mathrm{abs}(\mathbb{Q})\}$;
- $\mathcal{P}$ is the predictable σ-algebra, generated by the class of left continuous processes with limits at right;
- the set Π to be the set of all subsets of $\mathrm{abs}(\mathbb{P})$ and
$$\Pi^{c,e} = \{\mathcal{Q} \in \Pi : \mathcal{Q} \text{is a closed convex set in } \mathcal{L}^1 \text{ and } \mathcal{Q} \cap \mathrm{equiv}(\mathbb{P}) \neq \emptyset\};$$
- for $\mathcal{Q} \in \Pi^{c,e}$, we define
  - the set of densities $\mathcal{Z}^{\mathcal{Q}} := \{Z^{\mathbb{Q}} := d\mathbb{Q}/d\mathbb{P} : \mathbb{Q} \in \mathcal{Q}\}$ and denote $Z_t := \mathbb{E}(Z|\mathcal{F}_t)$ for $Z \in \mathcal{Z}^{\mathcal{Q}}$,
  - the polar set $\mathcal{A}^{\mathcal{Q}} := \{h \in \mathcal{L}^\infty : \mathbb{E}^{\mathbb{Q}}(h) \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\}$;
- for $\mathcal{Q} \in \Pi^{c,e}$, we say that a process X is a $\mathcal{Q}$-supermartingale (resp. $\mathcal{Q}$-martingale, local $\mathcal{Q}$-martingale) if it is a $\mathbb{Q}$-supermartingale (resp. $\mathbb{Q}$-martingale, local $\mathbb{Q}$-martingale) for all $\mathbb{Q} \in \mathcal{Q}$. We denote by $\mathrm{spm}(\mathcal{Q})$ (resp. $m(\mathcal{Q})$, $m_{\mathrm{loc}}(\mathcal{Q})$), the set of all $\mathcal{Q}$-supermartingales (resp. $\mathcal{Q}$-martingales, local $\mathcal{Q}$-martingales);
- the two sets $\mathcal{M}(X) := \{\mathbb{Q} \in \mathrm{abs}(\mathbb{P}) : X \in m(\{\mathbb{Q}\})\}$ and $\mathcal{M}_{\mathrm{loc}}(X) := \{\mathbb{Q} \in \mathrm{abs}(\mathbb{P}) : X \in m_{\mathrm{loc}}(\{\mathbb{Q}\})\}$ for a vector process X;
- for two processes V and W, we say that V is dominated by W and write $V \preceq W$ if $W - V$ is an increasing process;
- $\mathbf{1}_F$ is the indicator function of a set F.

We also recall the definition of the m-stability property of a set $\mathcal{Q} \in \Pi^{c,e}$. We refer to Delbaen [7] for more details on this property.

Definition 2.1. Let $\mathcal{Q} \in \Pi^{c,e}$ with $\mathcal{A} := \mathcal{A}^{\mathcal{Q}}$.

- We say that $\mathcal{Q}$ is m-stable if for any $Z^1, Z^2 \in \mathcal{Z}^{\mathcal{Q}}$ with $Z^2 > 0$ almost surely and a stopping time τ, we have $Z := Z^1_\tau Z^2 / Z^2_\tau \in \mathcal{Z}^{\mathcal{Q}}$.
- We denote by $\mathcal{Q}^{\mathrm{st}}$, the smallest m-stable element in $\Pi^{c,e}$, which contains $\mathcal{Q}$.
- We define the set $\mathcal{Q}' = \{\mathbb{Q} \in \mathrm{abs}(\mathbb{P}) : \mathbb{E}^{\mathbb{Q}}(h) = 0 \text{ for all } h \in \mathcal{A} \cap -\mathcal{A}\}$.

In [3] and for a set $\mathcal{Q} \in \Pi^{c,e}$, a strong link is drawn between the two sets $\mathcal{Q}^{\mathrm{st}}$ and $\mathrm{spm}(\mathcal{Q})$. In a parallel way, the same link is drawn between the two sets $[\mathcal{Q}] := (\mathcal{Q}^{\mathrm{st}})'$ and $m(\mathcal{Q})$. It was shown in particular that $\mathrm{spm}(\mathcal{Q}) = \mathrm{spm}(\mathcal{Q}^{\mathrm{st}})$ and that $[\mathcal{Q}]$ is the greatest set of martingale measures, containing $\mathcal{Q}$ and satisfying $m(\mathcal{Q}) = m([\mathcal{Q}])$.

3. Optional decomposition

We state the main result of this section.

Theorem 3.1. *Let $\mathcal{Q} \in \Pi^{c,e}$. Then the following assertions are equivalent:*

(1) *any positive $\mathcal{Q}$-supermartingale is dominated by a local $\mathcal{Q}$-martingale,*

(2) $[\mathcal{Q}] = \mathcal{Q}^{\mathrm{st}}$,

(3) $\mathcal{Q}' \subseteq \mathcal{Q}^{\mathrm{st}}$.

For the proof of Theorem 3.1, the equivalence (1) $\Leftrightarrow$ (2) and the implication (2) $\Rightarrow$ (3) are not hard to show. But for the implication (3) $\Rightarrow$ (2) to be proved, we need to show that $[\mathcal{H}] = (\mathcal{H}')^{\mathrm{st}}$ for all $\mathcal{H} \in \Pi^{c,e}$. For that we show the equality first for a finite discrete time axis I. We denote respectively by $\mathrm{spm}(\mathcal{H}, I)$, $m(\mathcal{H}, I)$ and $\mathcal{H}^{\mathrm{st},I}$, the corresponding sets $\mathrm{spm}(\mathcal{H})$, $m(\mathcal{H})$ and $\mathcal{H}^{\mathrm{st}}$ on the interval I. To simplify notation we fix $I := \{0 = t_0, t_1, \dots, t_{N-1}, t_N = T\}$ and $J := \{0, \dots, N-1\}$.

Lemma 3.2. *Let us suppose that $\mathcal{H} \in \Pi^{c,e}$ is m-stable on I. Then for any bounded process $X \in \mathrm{spm}(\mathcal{H}, I)$, there exists $\widehat{\mathbb{Q}} \in \mathcal{H}$ with $\widehat{\mathbb{Q}} \sim \mathbb{P}$ and $Y \in \mathrm{spm}(\mathcal{H}, I) \cap m(\{\widehat{\mathbb{Q}}\}, I)$ such that $X \preceq Y$ on I.*

Proof. We define the dynamic sublinear operator ρ, associated to $\mathcal{H}$, by

$$\rho_t(h) := \operatorname{essup}_{\mathbb{Q} \in \mathcal{H}} \mathbb{E}^{\mathbb{Q}}(h|\mathcal{F}_t),$$

for $t \in I$ and $h \in \mathcal{L}^\infty$. We fix $k \in J$ and define $h_k := X_{t_{k+1}} - X_{t_k}$, then $h_k = h_k^1 + h_k^2$ where $h_k^1 = X_{t_{k+1}} - \rho_{t_k}(X_{t_{k+1}})$ and $h_k^2 = \rho_{t_k}(X_{t_{k+1}}) - X_{t_k}$. We have then

$$0 = \rho_{t_k}(h_k^1) = \mathbb{E}^{\mathbb{Q}_k}(h_k^1|\mathcal{F}_{t_k})$$

for some $\mathbb{Q}_k \in \mathcal{H}$, therefore $Y \in \mathrm{spm}(\mathcal{H}, I) \cap m(\mathbb{Q}, I)$, where the process Y is defined by $Y_0 = 0$ and $Y_{t_{k+1}} = Y_{t_k} + h_k^1$ for $k \in J$ and the probability measure $\mathbb{Q}$ is defined by $\mathbb{E}^{\mathbb{Q}}(h|\mathcal{F}_{t_k}) = \mathbb{E}^{\mathbb{Q}_k}(h|\mathcal{F}_{t_k})$ for $h \in \mathcal{L}^\infty(\mathcal{F}_{t_{k+1}})$ and $k \in J$. By following the same methodology as in [15], there exists $\widehat{\mathbb{Q}} \in \mathcal{H}$ with $\widehat{\mathbb{Q}} \sim \mathbb{P}$ such that $Y \in \mathrm{spm}(\mathcal{H}, I) \cap m(\widehat{\mathbb{Q}}, I)$. Moreover, for all $k \in J$, we have

$$X_{t_{k+1}} - X_{t_k} = Y_{t_{k+1}} - Y_{t_k} + h_k^2 \leq Y_{t_{k+1}} - Y_{t_k}$$

since $h_k^2 := \rho_{t_k}(X_{t_{k+1}}) - X_{t_k} \leq 0$, so $X \preceq Y$ on I. ■

Lemma 3.3. *Let $\mathcal{H} \in \Pi^{c,e}$. Then $(\mathcal{H}')^{\mathrm{st},I} = [\mathcal{H}]_I := (\mathcal{H}^{\mathrm{st},I})'$.*

Proof. For the direct inclusion, we have $\mathcal{H} \subseteq \mathcal{H}^{\mathrm{st},I}$, so $\mathcal{H}' \subseteq [\mathcal{H}]_I$ and since $[\mathcal{H}]_I$ is m-stable, $(\mathcal{H}')^{\mathrm{st},I} \subseteq [\mathcal{H}]_I$. For the reverse inclusion, since the two sets $[\mathcal{H}]_I$ and $\widetilde{\mathcal{H}} := (\mathcal{H}')^{\mathrm{st},I}$ are m-stable on I, it suffices to show that $\mathrm{spm}(\widetilde{\mathcal{H}}, I) \subseteq \mathrm{spm}([\mathcal{H}]_I, I)$.

Let a bounded process $X \in \mathrm{spm}(\widetilde{\mathcal{H}}, I)$, so, by Lemma 3.2, there exists $Y \in \mathrm{spm}(\widetilde{\mathcal{H}}, I) \cap m(\{\widehat{\mathbb{Q}}\}, I)$ for some $\widehat{\mathbb{Q}} \in \widetilde{\mathcal{H}}$ with $\widehat{\mathbb{Q}} \sim \mathbb{P}$ such that $X \preceq Y$. So $Y \in \mathrm{spm}(\mathcal{H}', I) \cap m(\{\widehat{\mathbb{Q}}\}, I)$ and since $\mathcal{H}'$ is the set of local martingale measures for a family of adapted processes on the discrete time axis $\{0, T\}$, by a discrete time version of [11, Theorem 3.1 and Example 2.1] and for all $k \in J$, $\lambda \in \mathcal{L}^\infty_+(\mathcal{F}_{t_k})$ and $h_k := Y_{t_{k+1}} - Y_{t_k}$, there exists $M^k \in m_{\mathrm{loc}}(\mathcal{H}', \{0, T\})$ such that

$$\lambda\, h_k \leq M^k_T - M^k_0 =: N_k.$$

There exists also $\widehat{\mathbb{Q}}_k \in \mathcal{H}'$ with $\widehat{\mathbb{Q}}_k \sim \mathbb{P}$, which is equal to $\widehat{\mathbb{Q}}$ on the interval $\{t_k, t_{k+1}\}$, therefore

$$\mathbb{E}^{\widehat{\mathbb{Q}}_k}(\lambda h_k) = \mathbb{E}^{\widehat{\mathbb{Q}}_k}\big[\lambda \mathbb{E}^{\widehat{\mathbb{Q}}_k}(h|\mathcal{F}_{t_k})\big] = 0 \leq \mathbb{E}^{\widehat{\mathbb{Q}}_k}(N_k) \leq 0,$$

and by consequent $\lambda h_k = N_k$. We deduce that $\mathbb{E}^{\mathbb{Q}}(\lambda h_k) = 0$ for all $\mathbb{Q} \in \mathcal{H}'$ and $\lambda \in \mathcal{L}^\infty(\mathcal{F}_{t_k})$ and therefore $\mathbb{E}^{\mathbb{Q}}(h_k|\mathcal{F}_{t_k}) = 0$ for all $\mathbb{Q} \in \mathcal{H}'$. We conclude that $Y \in m(\mathcal{H}', I) \subseteq m(\mathcal{H}, I) = m([\mathcal{H}]_I, I)$ and so $X \in \mathrm{spm}([\mathcal{H}]_I, I)$. ∎

Now, as we show that $[\mathcal{H}]_I = (\mathcal{H}')^{\mathrm{st},I}$ for all finite intervals I, we construct an increasing sequence of finite intervals $(I_n)_{n\geq 1}$, converging to $[0, T]$ and so we need an intermediary result to assure that $([\mathcal{H}]_{I_n}, (\mathcal{H}')^{\mathrm{st},I_n})$ converge to $([\mathcal{H}], (\mathcal{H}')^{\mathrm{st}})$ in an appropriate sense.

Lemma 3.4. *Let a sequence $(\mathcal{Q}_n)_{n\geq 1} \subseteq \Pi^{c,e}$ and define $\mathcal{A}_n$ and $\mathcal{A}'_n$ to be respectively the polar sets of $\mathcal{Q}_n$ and $\mathcal{Q}'_n$ for $n \geq 1$. Then for $\mathcal{B} := \bigcap_n \mathcal{A}'_n$, we have $\mathcal{B} = \mathcal{B}'$.*

Proof. We define $K_n = \mathcal{A}_n \cap -\mathcal{A}_n$ and $L := \mathcal{L}^\infty_-$, then $\mathcal{B} = \mathcal{B}^{**} = (\bigcup_n (\mathcal{A}'_n)^*)^*$. Since $\mathcal{A}'_n = \overline{K_n + L}$ with the closure taken in $\mathcal{L}^\infty$, then $(\mathcal{A}'_n)^* = K^*_n \cap L^*$ and therefore

$$\mathcal{B} = \Big(\bigcup_n (K^*_n \cap L^*)\Big)^* = \Big(\bigcup_n K^*_n \cap L^*\Big)^* = \overline{\bigcap_n K_n + L}.$$

We remark that $\mathcal{B} \cap -\mathcal{B} = \bigcap_n K_n$, so $\mathcal{B} = \mathcal{B}'$. ∎

Next we show the desired result.

Lemma 3.5. *Let $\mathcal{H} \in \Pi^{c,e}$. Then $(\mathcal{H}')^{\mathrm{st}} = [\mathcal{H}]$.*

Proof. For the direct inclusion, we follow the same lines as in Lemma 3.3. For the reverse inclusion, we define, for each integer $n \geq 1$, the interval $I_n = \{k2^{-n}T : k = 0, \dots, 2^n\}$. It was proved in [3, Lemma 2.5] that $\mathcal{R}^{\mathrm{st}} = \overline{\mathrm{co}}(\bigcup_{n\geq 1} \mathcal{R}^{\mathrm{st}, I_n})$ for any $\mathcal{R} \in \Pi^{c,e}$, where $\overline{\mathrm{co}}(C)$ stands for the closed convex hull of a set C in $\mathcal{L}^1$. So

$$[\mathcal{H}] = \Big[\overline{\mathrm{co}}\Big(\bigcup_{n\geq 1} \mathcal{H}^{\mathrm{st},I_n}\Big)\Big]' \subseteq \Big[\overline{\mathrm{co}}\Big(\bigcup_{n\geq 1} (\mathcal{H}^{\mathrm{st},I_n})'\Big)\Big]'.$$

From Lemma 3.4, we have $[\overline{\mathrm{co}}(\bigcup_{n\geq 1} (\mathcal{H}^{\mathrm{st},I_n})')]' = \overline{\mathrm{co}}(\bigcup_{n\geq 1} (\mathcal{H}^{\mathrm{st},I_n})')$, so by Lemma 3.3, we get

$$[\mathcal{H}] \subseteq \overline{\mathrm{co}}\Big(\bigcup_{n\geq 1} (\mathcal{H}^{\mathrm{st},I_n})'\Big) = \overline{\mathrm{co}}\Big(\bigcup_{n\geq 1} (\mathcal{H}')^{\mathrm{st},I_n}\Big) = (\mathcal{H}')^{\mathrm{st}}.$$ ∎

After gathering the necessary results, we now prove the main theorem of this section.

Proof of Theorem 3.1. (1) $\Rightarrow$ (2) The reverse inclusion is trivial, let us prove the direct one. Since the two sets $[\mathcal{Q}]$ and $\mathcal{Q}^{\mathrm{st}}$ are m-stable, then it suffices to show that $\mathrm{spm}(\mathcal{Q}^{\mathrm{st}}) \subseteq \mathrm{spm}([\mathcal{Q}])$. Let $X \in \mathrm{spm}(\mathcal{Q}^{\mathrm{st}}) = \mathrm{spm}(\mathcal{Q})$, so there exists $Y \in m_{\mathrm{loc}}(\mathcal{Q}) = m_{\mathrm{loc}}([\mathcal{Q}]) \subseteq \mathrm{spm}([\mathcal{Q}])$ such that $X \preceq Y$, therefore $X \in \mathrm{spm}([\mathcal{Q}])$.

(2) $\Rightarrow$ (1) Since $\mathrm{spm}(\mathcal{Q}) = \mathrm{spm}(\mathcal{Q}^{\mathrm{st}})$ and $\mathcal{Q}^{\mathrm{st}} = [\mathcal{Q}]$ is the set of local martingale measures for a family of adapted processes, we apply [11, Theorem 3.1 and Example 2.1] and deduce the result.

(2) $\Rightarrow$ (3) We have $\mathcal{Q}' \subseteq [\mathcal{Q}]$ and $[\mathcal{Q}] = \mathcal{Q}^{\mathrm{st}}$, then $\mathcal{Q}' \subseteq \mathcal{Q}^{\mathrm{st}}$.

(3) $\Rightarrow$ (2) We have that $\mathcal{Q} \subseteq \mathcal{Q}' \subseteq \mathcal{Q}^{\mathrm{st}}$, so $\mathcal{Q}^{\mathrm{st}} = (\mathcal{Q}')^{\mathrm{st}}$ and by Lemma 3.5, we obtain the result. ∎

Example 3.6. We consider the setting of a two dimensional Brownian motion $W = (W_t)_{t\in[0,1]}$ and define the set

$$\Lambda = \{\gamma \in \mathbb{R}^2 : \gamma.(1, -1) = 0\}.$$

For $\lambda \in \mathcal{L}^\infty(\mathcal{P}; \mathbb{R}^2)$, we denote by Z^λ, the solution of the equation $dZ^\lambda/Z^\lambda = \lambda.dW$ and $Z_0^\lambda = 1$ and denote by $\mathbb{Q}^\lambda$, the associated probability measure to Z^λ. Now we define the set

$$\mathcal{Q} := \{\mathbb{Q}^{\alpha\lambda} : \alpha \in \mathcal{L}^0(\mathcal{P}; [0, 1]), \lambda \in \Lambda\},$$

then $\mathcal{Q} = \mathcal{Q}'$, $\mathcal{Q}^{\mathrm{st}} = \mathcal{M}((1, 1).W)$ and any positive $\mathcal{Q}$-supermartingale is dominated by a process of the form $\beta(1, 1) \bullet W$ for $\beta \in \mathcal{L}^1_{\mathrm{loc}}$.

4. Orthogonal decomposition

For a semimartingale X, we denote the Doob–Meyer decomposition

$$X = X_0 + M^X + A^X,$$

where M^X is a local martingale and A^X is a predictable process with bounded variation. We define $\mathcal{S}^2$, to be the set of square integrable semimartingales X, which means that

$$\mathbb{E}\big(\langle M^X\rangle_T\big) + \mathbb{E}\big(|A^X|_T^2\big) < \infty,$$

where $\langle M^X\rangle$ is the quadratic variation process of M^X and $|A^X|$ is the total variation process of A^X. We define for $\mathcal{Q} \in \Pi^{c,e}$, the sets $m_2(\mathcal{Q}) := m_{\mathrm{loc}}(\mathcal{Q}) \cap \mathcal{S}^2$, $\mathrm{spm}_2(\mathcal{Q}) := \mathrm{spm}(\mathcal{Q}) \cap \mathcal{S}^2$ and $\Pi^{c,e,2} := \{\mathcal{Q} \in \Pi^{c,e} : d\mathbb{Q}/d\mathbb{P} \in \mathcal{L}^2$ for some $\mathbb{Q} \in \mathcal{Q}\}$.

In this section, we state the orthogonal decomposition of square integrable semimartingales. We say that two processes $X, Y \in \mathcal{S}^2$ are orthogonal if their quadratic covariation process $\langle X, Y\rangle = \langle M^X, M^Y\rangle$ is null and we write $X \perp Y$.

Theorem 4.1. *Let $X \in \mathcal{S}^2$ and $\mathcal{Q} \in \Pi^{c,e,2}$. Then there exists a unique pair of processes $(Y, U) \in m_2(\mathcal{Q}) \times \mathcal{S}^2$ with $Y_0 = U_0 = 0$ such that $X = X_0 + Y + U$ and $U \perp V$ for all $V \in m_2(\mathcal{Q})$.*

An immediate consequence of this theorem is given next.

Theorem 4.2. *Let us consider a vector valued adapted process S such that $\emptyset \neq \mathcal{M}_{\mathrm{loc}}(S) \in \Pi^{c,e,2}$. Then for all $X \in \mathcal{S}^2$, there exists a predictable process β and $U \in \mathcal{S}^2$ such that $X = X_0 + \beta \bullet S + U$ and $U \perp M^S$.*

One application of Theorem 4.2 is pricing claims in incomplete markets. We consider the same setting as in [12, 13, 21]: S is the price process of one risky asset and $C(\varphi)$ is the risk process associated to the trading strategy φ. We suppose that $S \in \mathcal{S}^2$, then by the no-arbitrage assumption, there exists a predictable process α^* such that $A^S = \alpha^* \bullet \langle M^S\rangle$. We define the probability measure $\mathbb{P}^*$, by its derivative Z^*, defined by

$$Z^* = \exp\Big\{-\alpha \bullet M_T^S - \frac{1}{2}(\alpha^*)^2 \bullet \langle M^S\rangle_T\Big\}.$$

We suppose that $Z^* \in \mathcal{L}^2$ and state that the price of a claim $H \in \mathcal{L}^2$, which corresponds to the minimizing strategy of the conditional variance process of the risk process $C(\varphi^*)$, is given by $\mathbb{E}^{\mathbb{P}^*}(H)$. Indeed, we should show (2.25) and (2.26) in [12, Proposition 2.24]. We define the process V by $V_t = \mathbb{E}^{\mathbb{P}^*}(H|\mathcal{F}_t)$, then, by Theorem 4.2, there exists a predictable process β and $U \in \mathcal{S}^2$ such that $V = V_0 + \beta \bullet S + U$ and $U \perp M^S$. We prove now that U is a martingale, we apply Itô's formula and get

$$d(Z^*U) = Z^*\,dU + U\,dZ^*.$$

Since the processes Z^*U and Z^* are martingales, then $Z^* \bullet U$ is a martingale, which means that $Z^* \bullet A^U = 0$ and therefore $A^U = 0$.

Before proving Theorem 4.1, we state a preliminary result. We recall from [2], that for any square integrable process Y, admitting at least one equivalent martingale measure, there exists $\alpha^Y \in \mathcal{L}^2(M^Y)$ such that $Y = Y_0 + M^Y - \alpha^Y \bullet \langle M^Y \rangle$.

Lemma 4.3. *Let $\mathcal{Q} \in \Pi^{c,e,2}$. Then we have the following:*

(1) *the set $\mathcal{K}^{\mathcal{Q}} := \{M_T^Y : Y \in m_2(\mathcal{Q})\}$ is a closed vector space in $\mathcal{L}^2$;*

(2) *let $V, W \in m_2(\mathcal{Q})$ with $V_0 = W_0 = 0$, so $V = W$ iff $M^V = M^W$.*

Proof. (1) From the uniqueness of the Doob–Meyer decomposition, we deduce easily that $\mathcal{K}^{\mathcal{Q}}$ is a vector space. Now we show that it is closed in $\mathcal{L}^2$. Let $(Y^n)_{n\geq 1} \subseteq m_2(\mathcal{Q})$ such that $M_T^n := M_T^{Y^n}$ converges in $\mathcal{L}^2$ to some $G \in \mathcal{L}^2$, and define the martingale M by $M_t := \mathbb{E}(G|\mathcal{F}_t)$ for $t \in [0, T]$. Let $\mathbb{Q} \in \mathcal{Q}$ such that $Z = d\mathbb{Q}/d\mathbb{P} \in \mathcal{L}^2$, then by the Kunita–Watanabe decomposition, there exists $\alpha^n \in \mathcal{L}^2(M^n)$, $R^n \in m_2(\{\mathbb{P}\})$ with $R^n \perp M^n$ such that

$$dD^Z := dZ/Z = \alpha^n \, dM^n + dR^n.$$

So $\langle D^Z, M^n \rangle_T = \alpha^n \bullet \langle M^n \rangle_T$ converges in $\mathcal{L}^1$ to $\langle D^Z, M \rangle_T$. Therefore $Y_T^n = M_T^n - \alpha^n \bullet \langle M^n \rangle_T$ converges in $\mathcal{L}^1$ to $Y_T := G - \langle D^Z, M \rangle_T$ and by consequent Y_t^n converges in $\mathcal{L}^1$ to $Y_t := \mathbb{E}^{\mathbb{Q}}(Y_T|\mathcal{F}_t)$ for all $\mathbb{Q} \in \mathcal{Q}$. We deduce that $Y \in m_2(\mathcal{Q})$ and thus $G = M_T^Y \in \mathcal{K}^{\mathcal{Q}}$.

(2) The direct implication is trivial. Inversely, let us suppose that $M^V = M^W$, then $Y := V - W = M^Y - \alpha^Y \bullet \langle M^Y \rangle$ with $M^Y = M^V - M^W = 0$, so $Y = 0$ which means that $V = W$. ∎

Proof of Theorem 4.1. We suppose without loss of generality that $X_0 = 0$. From Lemma 4.3, the set $\mathcal{K}^{\mathcal{Q}} = \{M_T^Y : Y \in m_2(\mathcal{Q})\}$ is a closed vector space in $\mathcal{L}^2$. So by projection of $M_T^X \in \mathcal{L}^2$ into $\mathcal{K}^{\mathcal{Q}}$, there exists $Y \in m_2(\mathcal{Q})$ and $G \in \mathcal{L}^2$ such that $M_T^X = M_T^Y + G$ and $\mathbb{E}(GG') = 0$ for all $G' \in \mathcal{K}^{\mathcal{Q}}$. We define $U := X - Y \in \mathcal{S}^2$ and show the following:

$$M_T^U = M_T^X - M_T^Y = G,$$

so $\mathbb{E}(\langle M^U, M^V \rangle_T) = \mathbb{E}(M_T^U \, M_T^V) = 0$ for all $V \in m_2(\mathcal{Q})$. Therefore,

$$\mathbb{E}\big(\mathbf{1}_F \bullet \langle M^U, M^V \rangle_T\big) = \mathbb{E}\big(M_T^U \mathbf{1}_F \bullet M_T^V\big) = 0$$

for all predictable set F since $\mathbf{1}_F \bullet M^V = M^{\mathbf{1}_F \bullet V}$, we deduce that $\langle M^U, M^V \rangle = 0$.

For the uniqueness property, we suppose that there exists another pair of processes (Y', U') satisfying the assumptions of Theorem 4.1. So $M^Y = M^{Y'}$ and then $Y = Y'$ from assertion (2) in Lemma 4.3. We deduce that $U - U' = Y' - Y = 0$ and therefore $U = U'$. ∎

Next, we introduce some new concepts, based on the orthogonal decomposition, stated in Theorem 4.1.

Definition 4.4. Let $\mathcal{Q} \in \Pi^{c,e,2}$.

(1) We define the two operators $\pi_0^{\mathcal{Q}}$ and $\pi_1^{\mathcal{Q}}$ by $\pi_0^{\mathcal{Q}}(X) = Y$ and $\pi_1^{\mathcal{Q}}(X) = U$ for $X \in \mathcal{S}^2$, where the pair of processes (Y, U) is given in Theorem 4.1.

(2) A process $X \in \mathcal{S}^2$ with $X_0 = 0$ is said to be a $\pi^{\mathcal{Q}}$-martingale if

$$\pi_0^{\mathcal{Q}}(X) = X.$$

(3) A process $X \in \mathcal{S}^2$ with $X_0 = 0$ is said to be a $\pi^{\mathcal{Q}}$-non martingale if

$$\pi_0^{\mathcal{Q}}(X) = 0.$$

The notion of $\pi^{\mathcal{Q}}$-martingale, introduced above is identical to the notion of $\mathcal{Q}$-martingale for $\mathcal{Q} \in \Pi^{c,e}$. Some properties of the two operators $\pi_0^{\mathcal{Q}}$ and $\pi_1^{\mathcal{Q}}$ are given below.

Proposition 4.5. *Let $\mathcal{Q} \in \Pi^{c,e,2}$. Then*

(1) $\pi_i^{\mathcal{Q}} = \pi_i^{[\mathcal{Q}]}$ *for* $i = 0, 1$;

(2) $\pi_0^{\mathcal{Q}} \circ \pi_0^{\mathcal{Q}} = \pi_0^{\mathcal{Q}}$ *and* $\pi_1^{\mathcal{Q}} \circ \pi_0^{\mathcal{Q}} = 0$;

(3) $\pi_1^{\mathcal{Q}} \circ \pi_1^{\mathcal{Q}} = \pi_1^{\mathcal{Q}}$ *and* $\pi_0^{\mathcal{Q}} \circ \pi_1^{\mathcal{Q}} = 0$;

(4) *for* $\pi = \pi_0^{\mathcal{Q}}, \pi_1^{\mathcal{Q}}$, *we have* $\pi(X + Y) = \pi(X) + \pi(Y)$ *and* $\pi(\alpha \bullet X) = \alpha \bullet \pi(X)$ *for all* $X, Y \in \mathcal{S}^2$ *and* $\alpha \in \mathcal{L}^2(X)$ *such that* $\alpha \bullet X \in \mathcal{S}^2$;

(5) *for* $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$, *we have* $\pi_0^{\mathcal{Q}} = \pi_0^{\tilde{\mathcal{Q}}} + \pi_0^{\mathcal{Q}} \circ \pi_1^{\tilde{\mathcal{Q}}}$ *and* $\pi_1^{\mathcal{Q}} = \pi_1^{\mathcal{Q}} \circ \pi_1^{\tilde{\mathcal{Q}}}$.

Proof. (1) Since $m_2(\mathcal{Q}) = m_2([\mathcal{Q}])$, we get the result.

(2) Let $X \in \mathcal{S}^2$ and define $Y = \pi_0^{\mathcal{Q}}(X)$, $U = \pi_1^{\mathcal{Q}}(X)$, $Z = \pi_0^{\mathcal{Q}}(Y)$ and $V = \pi_1^{\mathcal{Q}}(Y)$. So $X = X_0 + Z + (V + U)$ such that $Z \in m_2(\mathcal{Q})$, $V + U \in \mathcal{S}^2$, $Z_0 = V_0 + U_0 = 0$ and $V + U \perp W$ for all $W \in m_2(\mathcal{Q})$. Therefore, by uniqueness we get $Y = Z$ and $V = 0$.

We proceed in the same way for the remaining assertions. ■

Definition 4.6. Let $\mathcal{Q} \in \Pi^{c,e,2}$ and $X \in \mathcal{S}^2$. The two processes $\pi_0^{\mathcal{Q}}(X)$ and $\pi_1^{\mathcal{Q}}(X)$ are called respectively the $\pi^{\mathcal{Q}}$-martingale and $\pi^{\mathcal{Q}}$-non martingale parts of X.

Another characterization of the notions of $\pi^{\mathcal{Q}}$-martingale and $\pi^{\mathcal{Q}}$-non martingale, is given below.

Corollary 4.7. *Let $\mathcal{Q} \in \Pi^{c,e,2}$ and $X \in \mathcal{S}^2$ with $X_0 = 0$. Then*

(1) *X is a $\pi^{\mathcal{Q}}$-martingale iff $\pi_1^{\mathcal{Q}}(X) = 0$.*

(2) *X is a $\pi^{\mathcal{Q}}$-non martingale iff $\pi_1^{\mathcal{Q}}(X) = X$.*

Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e}$ and let τ be a stopping time, we define $\mathcal{Q} := \mathcal{Q}^1 *_\tau \mathcal{Q}^2 \in \Pi^{c,e}$ to be the $\mathcal{L}^1$-closed convex hull of the set of probability measures $\mathbb{Q} := \mathbb{Q}^1 *_\tau \mathbb{Q}^2$ defined by $\mathbb{Q}(F) = \mathbb{E}^{\mathbb{Q}^1}(\mathbb{E}^{\mathbb{Q}^2}(\mathbf{1}_F|\mathcal{F}_\tau))$, for $\mathbb{Q}^1 \in \mathcal{Q}^1$, $\mathbb{Q}^2 \in \mathcal{Q}^2$ and $F \in \mathcal{F}$. Let respectively $\mathcal{A}$, $\mathcal{A}^1$ and $\mathcal{A}^2$ be the polar sets of $\mathcal{Q}$, $\mathcal{Q}^1$ and $\mathcal{Q}^2$, then thanks to [7], $\mathcal{Q}$ is m-stable on the time axis $\{0, \tau, T\}$ and $\mathcal{A} = \mathcal{A}^1 \cap \mathcal{L}^\infty(\mathcal{F}_\tau) + \mathcal{A}^2_\tau$, where $\mathcal{A}^2_\tau := \{h \in \mathcal{L}^\infty : bh \in \mathcal{A}^2 \text{ for all } b \in \mathcal{L}^\infty_+(\mathcal{F}_\tau)\}$.

Proposition 4.8. *Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e,2}$ and let τ be a stopping time. Then for $\mathcal{Q} := \mathcal{Q}^1 *_\tau \mathcal{Q}^2$ and $X \in \mathcal{S}^2$, we have*

$$\pi_i^{\mathcal{Q}}(X) = \pi_i^{\mathcal{Q}^1}(\mathbf{1}_{[0,\tau)} \bullet X) + \pi_i^{\mathcal{Q}^2}(\mathbf{1}_{[\tau,T)} \bullet X),$$

for $i = 0, 1$.

Proof. Let us define the processes $Y^1 := \pi_0^{\mathcal{Q}^1}(\mathbf{1}_{[0,\tau)} \bullet X)$, $Y^2 := \pi_0^{\mathcal{Q}^2}(\mathbf{1}_{[\tau,T)} \bullet X)$, $U^1 := \pi_1^{\mathcal{Q}^1}(\mathbf{1}_{[0,\tau)} \bullet X)$, $U^2 := \pi_1^{\mathcal{Q}^2}(\mathbf{1}_{[\tau,T)} \bullet X)$, $Y := Y^1 + Y^2$ and $U := U^1 + U^2$, then $U \in \mathcal{S}^2$ and $X = X_0 + Y + U$. In order to apply the uniqueness property in Theorem 4.1, we have to show that $Y \in m_2(\mathcal{Q})$ and that $U \perp V$ for all $V \in m_2(\mathcal{Q})$. We remark that $\mathcal{Q} = \mathcal{Q}^1$ on $\mathcal{F}_\tau$ and $\mathcal{Q} = \mathcal{Q}^2$ conditionally on $\mathcal{F}_\tau$, and that

$$\begin{aligned} \mathbf{1}_{[0,\tau)} \bullet Y &= \mathbf{1}_{[0,\tau)} \bullet Y^1, \qquad & \mathbf{1}_{[\tau,T)} \bullet Y &= \mathbf{1}_{[\tau,T)} \bullet Y^2, \\ \mathbf{1}_{[0,\tau)} \bullet U &= \mathbf{1}_{[0,\tau)} \bullet U^1, \qquad & \mathbf{1}_{[\tau,T)} \bullet U &= \mathbf{1}_{[\tau,T)} \bullet U^2. \end{aligned}$$

So for $\alpha \in \mathcal{L}^\infty$, $\mathbb{Q}^1 \in \mathcal{Q}^1$, $\mathbb{Q}^2 \in \mathcal{Q}^2$ and $\mathbb{Q} := \mathbb{Q}^1 *_\tau \mathbb{Q}^2$, we have

$$\mathbb{E}^{\mathbb{Q}}(\alpha \bullet Y_T) = \mathbb{E}^{\mathbb{Q}^1}(\mathbf{1}_{[0,\tau)}\alpha \bullet Y^1_T) + \mathbb{E}^{\mathbb{Q}^2}(\mathbf{1}_{[\tau,T)}\alpha \bullet Y^2_T) = 0,$$

then $\mathbb{E}^{\mathbb{Q}}(\alpha \bullet Y_T) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$, and so $Y \in m_2(\mathcal{Q})$. Now, for $V \in m_2(\mathcal{Q})$, we have $\langle U, V\rangle = \mathbf{1}_{[0,\tau)} \bullet \langle U^1, V\rangle + \mathbf{1}_{[\tau,T)} \bullet \langle U^2, V\rangle$, with $\mathbf{1}_{[0,\tau)} \bullet V \in m_2(\mathcal{Q}^1)$ and $\mathbf{1}_{[\tau,T)} \bullet V \in m_2(\mathcal{Q}^2)$ and therefore $\langle U, V\rangle = 0$. We conclude the result. ∎

For a family $\mathcal{V}$ of adapted processes, we denote by $\mathcal{M}\text{sp}(\mathcal{V})$, the set of all supermartingales measures of the family $\mathcal{V}$, which means that $\mathbb{Q} \in \mathcal{M}\text{sp}(\mathcal{V})$ if $\mathbb{Q} \in \text{abs}(\mathbb{P})$ and $X \in \text{spm}(\mathbb{Q})$ for all $X \in \mathcal{V}$.

Theorem 4.9. *Let $\mathcal{V} \subseteq \mathcal{S}^2$ such that $\emptyset \neq \mathcal{Q} := \mathcal{M}\text{sp}(\mathcal{V}) \in \Pi^{c,e,2}$ and define the set $\widehat{\mathcal{V}} := \{\pi_0^{\mathcal{Q}}(V) : V \in \mathcal{V}\}$. Then*

(1) $[\mathcal{Q}] = \mathcal{M}(\widehat{\mathcal{V}})$;

(2) *for all $W \in \mathcal{S}^2$, there exists $\widehat{V} \in \widehat{\mathcal{V}}$ and $\beta \in \mathcal{L}^2(\widehat{V})$ such that $\pi_0^{\mathcal{Q}}(W) = \beta \bullet \widehat{V}$;*

(3) *for all $W \in \mathcal{S}^2$, there exists $\widehat{V} \in \widehat{\mathcal{V}}$, $\beta \in \mathcal{L}^2(\widehat{V})$ and $L \in \mathcal{S}^2$ such that $W = W_0 + \beta \bullet \widehat{V} + L$ and $L \perp Y$ for all $Y \in \widehat{\mathcal{V}}$.*

Proof. (1) Since from [3], $[\mathcal{Q}]$ is the greatest set of martingale measures, containing $\mathcal{Q}$ and satisfying $m(\mathcal{Q}) = m([\mathcal{Q}])$, we need to show that

(i) $\mathcal{Q} \subseteq \mathcal{M}(\widehat{\mathcal{V}})$, and

(ii) $m(\mathcal{Q}) \subseteq m(\mathcal{M}(\widehat{\mathcal{V}}))$.

For (i), we have $\widehat{\mathcal{V}} \subseteq m_2(\mathcal{Q})$, so $\mathcal{Q} \subseteq \mathcal{M}(\widehat{\mathcal{V}})$. For (ii), let $X \in m(\mathcal{Q})$ with $X_0 = 0$, then by applying [11, Theorem 3.1 and Example 2.2], we get $X \preceq \beta \bullet V$ for some predictable process β and $V \in \mathcal{V}$. Let $\mathbb{Q} \in \mathcal{Q}$ with $\mathbb{Q} \sim \mathbb{P}$ and $\alpha \in \mathcal{L}^{\infty}_{+}(\mathcal{P})$, then $0 = \mathbb{E}^{\mathbb{Q}}(\alpha \bullet X_T) \leq \mathbb{E}^{\mathbb{Q}}(\alpha\beta \bullet V_T) \leq 0$. Therefore $X = \beta \bullet V$ and so $X = \pi_0^{\mathcal{Q}}(X) = \beta \bullet \pi_0^{\mathcal{Q}}(V) \in m(\mathcal{M}(\widehat{\mathcal{V}}))$.

We deduce that $\mathcal{Q} \subseteq \mathcal{M}(\widehat{\mathcal{V}}) \subseteq [\mathcal{Q}]$, so $[\mathcal{Q}] = [\mathcal{M}(\widehat{\mathcal{V}})] = \mathcal{M}(\widehat{\mathcal{V}})$.

(2) Let $W \in \mathcal{S}^2$, then $\pi_0^{\mathcal{Q}}(W) \in m_2(\mathcal{Q})$ and since $m_2(\mathcal{Q}) = m_2([\mathcal{Q}] = m_2(\mathcal{M}(\widehat{\mathcal{V}}))$, there exists $\widehat{V} \in \widehat{\mathcal{V}}$ and $\beta \in \mathcal{L}^2(\widehat{V})$ such that $\pi_0^{\mathcal{Q}}(W) = \beta \bullet \widehat{V}$.

(3) It is a consequence of assertion (2) and Theorem 4.1. ∎

As an immediate consequence of Theorem 4.9, we state a generalisation of the Föllmer–Schweizer decomposition formula.

Corollary 4.10. *Let $V = (V^1, \ldots, V^d)$ such that $V^i \in \mathcal{S}^2$ for $i = 1, \ldots, d$, $\emptyset \neq \mathcal{Q} := \mathcal{M}\mathrm{sp}(V) \in \Pi^{c,e,2}$ and define $\widehat{V} = (\widehat{V}^1, \ldots, \widehat{V}^d)$ where $\widehat{V}^i = \pi_0^{\mathcal{Q}}(V^i)$ for $i = 1, \ldots, d$. Then for all $W \in \mathcal{S}^2$, there exists $\beta \in \mathcal{L}^2(\widehat{V})$ and $L \in \mathcal{S}^2$ such that $W = W_0 + \beta \bullet \widehat{V} + L$ and $L \perp \widehat{V}$.*

5. Orthogonal decomposition of polar sets

We say that a set $\mathcal{A} \subseteq \mathcal{L}^{\infty}$ is a $\mathcal{Q}$-polar set for some $\mathcal{Q} \in \Pi^{c,e}$ if $\mathcal{A} = \mathcal{A}^{\mathcal{Q}}$. We denote $[\mathcal{Q}] = (\mathcal{Q}^{\mathrm{st}})'$ and $[\mathcal{A}]$ to be the $[\mathcal{Q}]$-polar set. We define $\mathbb{A} := \{\mathcal{A}^{\mathcal{Q}} : \mathcal{Q} \in \Pi^{c,e}\}$ and $\mathbb{A}^{\mathrm{st}} := \{\mathcal{A}^{\mathcal{Q}} : \mathcal{Q} \in \Pi^{c,e}$ is m-stable$\}$.

In this section, we state the orthogonal decomposition of $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$ by showing the existence of a unique element $\mathcal{B} \in \mathbb{A}^{\mathrm{st}}$ such that $\mathcal{A} = [\mathcal{A}] + \mathcal{B}$ and $\mathcal{B}$ is orthogonal to $[\mathcal{A}]$ in a sense to be defined next. We write in this case $\mathcal{A} = [\mathcal{A}] \oplus \mathcal{B}$. We denote by $\mathrm{maxspm}_2(\mathcal{Q})$, the set of maximal elements in $\mathrm{spm}_2(\mathcal{Q})$ with respect to the order $\preceq$.

Definition 5.1. Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e,2}$ such that $\mathcal{A}^i := \mathcal{A}^{\mathcal{Q}^i} \in \mathbb{A}^{\mathrm{st}}$ for $i = 1, 2$. We say that $\mathcal{A}^1$ and $\mathcal{A}^2$ are orthogonal and we write $\mathcal{A}^1 \perp \mathcal{A}^2$, if $\mathrm{maxspm}_2(\mathcal{Q}^1) \perp \mathrm{maxspm}_2(\mathcal{Q}^2)$, which means that $X^1 \perp X^2$ for all $X^i \in \mathrm{maxspm}_2(\mathcal{Q}^i)$ and $i = 1, 2$.

Now we state the orthogonal decomposition of a set $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$.

Theorem 5.2. *Let $\mathcal{Q} \in \Pi^{c,e,2}$ such that $\mathcal{A} = \mathcal{A}^{\mathcal{Q}} \in \mathbb{A}^{\mathrm{st}}$. Then there exists a unique element $\mathcal{A}^{\perp}$ in $\mathbb{A}^{\mathrm{st}}$ such that $\mathcal{A} = [\mathcal{A}] \oplus \mathcal{A}^{\perp}$.*

Proof. We define the family $\mathcal{V} := \{\pi_1^{\mathcal{Q}}(X) : X \in \mathrm{spm}_2(\mathcal{Q})\}$, the set $\mathcal{Q}^\perp := \mathcal{M}\mathrm{sp}(\mathcal{V})$ and its polar set $\mathcal{A}^\perp$.

First we show that $\mathcal{A}^\perp \perp [\mathcal{A}]$. We remark that $\mathrm{maxspm}_2([\mathcal{Q}]) = m_2(\mathcal{Q})$ and for all $X \in \mathrm{maxspm}_2(\mathcal{Q}^\perp)$ and thanks to [11, Theorem 3.1 and Example 2.2], there exists $Y \in \mathrm{spm}(\mathcal{Q})$ such that $X = \pi_1^{\mathcal{Q}}(Y)$ and so $X \perp m_2(\mathcal{Q})$.

Second we show that $\mathcal{A} = [\mathcal{A}] + \mathcal{A}^\perp$ or $\mathcal{Q} = [\mathcal{Q}] \cap \mathcal{Q}^\perp$. The direct inclusion is trivial, for the reverse inclusion we show that $\mathrm{spm}_2(\mathcal{Q}) \subseteq \mathrm{spm}_2([\mathcal{Q}] \cap \mathcal{Q}^\perp)$, so let $X \in \mathrm{spm}_2(\mathcal{Q})$, $\mathbb{Q} \in [\mathcal{Q}] \cap \mathcal{Q}^\perp$ and $\alpha \in \mathcal{L}_+^\infty(\mathcal{P})$, by Theorem 4.1, we get $X = X_0 + \pi_0^{\mathcal{Q}}(X) + \pi_1^{\mathcal{Q}}(X)$ with $\pi_0^{\mathcal{Q}}(X) \in m_2([\mathcal{Q}]) \subseteq \mathrm{spm}_2([\mathcal{Q}])$ and $\pi_1^{\mathcal{Q}}(X) \in \mathrm{spm}_2(\mathcal{Q}^\perp)$. Therefore

$$\mathbb{E}^{\mathbb{Q}}(\alpha \bullet X_T) = \mathbb{E}^{\mathbb{Q}}(\alpha \bullet \pi_0^{\mathcal{Q}}(X)_T) + \mathbb{E}^{\mathbb{Q}}(\alpha \bullet \pi_1^{\mathcal{Q}}(X)_T) \leq 0,$$

and then $X \in \mathrm{spm}_2([\mathcal{Q}] \cap \mathcal{Q}^\perp)$. We deduce by duality that $\mathcal{A} = [\mathcal{A}] + \mathcal{A}^\perp$.

Third we show the uniqueness property. Let us suppose there exists $\mathcal{B} \in \mathbb{A}^{\mathrm{st}}$ satisfying $\mathcal{A} = [\mathcal{A}] \oplus \mathcal{B}$. In order to show that $\mathcal{B} = \mathcal{A}^\perp$, we prove that $\mathrm{spm}_2(\mathcal{Q}^\perp) = \mathrm{spm}_2(\mathcal{Q}^{\mathcal{B}})$ or $\mathrm{maxspm}_2(\mathcal{Q}^\perp) = \mathrm{maxspm}_2(\mathcal{Q}^{\mathcal{B}})$. Let $X \in \mathrm{maxspm}_2(\mathcal{Q}^{\mathcal{B}})$, so $X = X_0 + \pi_0^{\mathcal{Q}}(X) + \pi_1^{\mathcal{Q}}(X)$ and since $X - X_0 - \pi_1^{\mathcal{Q}}(X) \perp \pi_1^{\mathcal{Q}}(X)$, we have $X = X_0 + \pi_1^{\mathcal{Q}}(X) \in \mathrm{spm}_2(\mathcal{Q}^\perp)$. Inversely, let $X \in \mathrm{maxspm}_2(\mathcal{Q}^\perp)$. Since $\mathcal{Q} = [\mathcal{Q}] \cap \mathcal{Q}^{\mathcal{B}}$, by Lemma 5.3 below, we get $\mathrm{spm}_2(\mathcal{Q}) \subseteq \mathrm{spm}_2([\mathcal{Q}]) + \mathrm{spm}_2(\mathcal{Q}^{\mathcal{B}}))$ and therefore $X = X^1 + X^2$ with $X^1 \in \mathrm{spm}_2([\mathcal{Q}])$ and $X^2 \in \mathrm{spm}_2(\mathcal{Q}^{\mathcal{B}})$. From [11], there exists a local $\mathcal{Q}$-martingale M and a decreasing process C such that $X^1 = M + C$. Let τ be a localizing sequence so that the two processes $\mathbf{1}_{[0,\tau)} \bullet M$ and $\mathbf{1}_{[0,\tau)} \bullet C$ are in $\mathcal{S}^2$. We deduce that $\mathbf{1}_{[0,\tau)} \bullet (X - (C + X^2)) = \mathbf{1}_{[0,\tau)} \bullet M$, and since the two processes $\mathbf{1}_{[0,\tau)} \bullet (X - (C + X^2))$ and $\mathbf{1}_{[0,\tau)} \bullet M$ are orthogonal, $\mathbf{1}_{[0,\tau)} \bullet (X - (C + X^2)) = 0$, which means that $\mathbf{1}_{[0,\tau)} \bullet X = \mathbf{1}_{[0,\tau)} \bullet (C + X^2) \in \mathrm{spm}_2(\mathcal{Q}^{\mathcal{B}})$ for all τ and therefore $X \in \mathrm{spm}_2(\mathcal{Q}^{\mathcal{B}})$. ∎

Lemma 5.3. *Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e}$ such that $\mathcal{Q}^1$ and $\mathcal{Q}^2$ are m-stable. Then*

$$\mathrm{spm}_2(\mathcal{Q}^1 \cap \mathcal{Q}^2) \subseteq \mathrm{spm}_2(\mathcal{Q}^1) + \mathrm{spm}_2(\mathcal{Q}^2).$$

Proof. We shall show that $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ where $\mathcal{Q} := \mathcal{Q}^1 \cap \mathcal{Q}^2$ and $\widetilde{\mathcal{Q}} := \mathcal{M}\mathrm{sp}(\mathrm{spm}_2(\mathcal{Q}^1) + \mathrm{spm}_2(\mathcal{Q}^2))$. Let $\mathbb{Q} \in \widetilde{\mathcal{Q}}$ and $h \in \mathcal{A}^{\mathcal{Q}}$, so there exists $h_1 \in \mathcal{A}^{\mathcal{Q}^1}$ and $h_2 \in \mathcal{A}^{\mathcal{Q}^2}$ such that $h = h_1 + h_2$. By applying [11, Theorem 3.1 and Example 2.2], we deduce that there exists $X^1 \in \mathrm{spm}_2(\mathcal{Q}^1)$ and $X^2 \in \mathrm{spm}_2(\mathcal{Q}^2)$ such that $h_1 \leq X_T^1$ and $h_2 \leq X_T^2$. So $\mathbb{E}^{\mathbb{Q}}(h) \leq \mathbb{E}^{\mathbb{Q}}((X^1 + X^2)_T) \leq 0$. ∎

Thanks to the orthogonal decomposition stated in Theorem 5.2, we introduce notions of martingale and non martingale sets.

Definition 5.4. We say that $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$ is a martingale (resp. a non martingale) set if $[\mathcal{A}] = \mathcal{A}$ (resp. $[\mathcal{A}] = \mathcal{L}_-^\infty$).

Next we investigate the properties of the sets $[\mathcal{A}]$ and $\mathcal{A}^{\perp}$.

Proposition 5.5. *Let $\mathcal{Q} \in \Pi^{c,e,2}$ such that $\mathcal{A} = \mathcal{A}^{\mathcal{Q}} \in \mathbb{A}^{\text{st}}$. Then*

(1) *$[\mathcal{A}]$ is the largest martingale subset in $\mathcal{A}$;*

(2) *$\mathcal{A}^{\perp}$ is a non martingale subset in $\mathcal{A}$;*

(3) *$\mathcal{A}^{\perp}$ is a minimal subset in $\mathcal{A}$, which satisfy $\mathcal{A} = [\mathcal{A}] + \mathcal{A}^{\perp}$;*

(4) *$\mathcal{A}^{\perp}$ is a maximal subset in $\mathcal{A}$, which satisfy $\mathcal{A}^{\perp} \perp [\mathcal{A}]$;*

(5) *$[\mathcal{A}]^{\perp} = \mathcal{L}^{\infty}_{-}$;*

(6) *$(\mathcal{A}^{\perp})^{\perp} = \mathcal{A}^{\perp}$.*

Proof. (1) $[[\mathcal{A}]] = [\mathcal{A}]$, so $[\mathcal{A}]$ is a martingale set. Now, let $\mathcal{B} \in \mathbb{A}^{\text{st}}$ with $\mathcal{B} = [\mathcal{B}]$ and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B} = [\mathcal{B}] \subseteq [\mathcal{A}]$.

(2) Let $X \in m_2(\mathcal{Q}^{\perp})$, so $X \in m_2(\mathcal{Q})$ and then $X = X_0 + \pi_0^{\mathcal{Q}}(X)$, but $X - X_0 \perp \pi_0^{\mathcal{Q}}(X)$, therefore $X = X_0$. We deduce that $[\mathcal{A}^{\perp}] = \mathcal{L}^{\infty}_{-}$.

(3) Let $\mathcal{B} \in \mathbb{A}^{\text{st}}$ such that $\mathcal{A} = [\mathcal{A}] + \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}^{\perp}$. So $\text{spm}_2(\mathcal{Q}^{\mathcal{B}}) \subseteq \text{spm}_2(\mathcal{Q}^{\perp})$ and then $\mathcal{B} \perp [\mathcal{A}]$. By uniqueness in Theorem 5.2, we get the result.

(4) Let $\mathcal{B} \in \mathbb{A}^{\text{st}}$ such that $\mathcal{A}^{\perp} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B} \perp [\mathcal{A}]$. Then $\mathcal{A} = [\mathcal{A}] + \mathcal{A}^{\perp} \subseteq [\mathcal{A}] + \mathcal{B} \subseteq \mathcal{A}$, so $\mathcal{A} = [\mathcal{A}] + \mathcal{B}$ and by uniqueness in Theorem 5.2, we get the result.

(5) We have $\text{spm}_2([\mathcal{Q}]^{\perp}) \perp m_2([[\mathcal{Q}]]) = m_2([\mathcal{Q}])$ and $\text{spm}_2([\mathcal{Q}]^{\perp}) \subseteq \text{spm}_2([\mathcal{Q}])$, so $m_2([\mathcal{Q}]^{\perp}) = \{0\}$ and any $X \in \text{spm}_2([\mathcal{Q}]^{\perp})$ is an decreasing process, therefore $[\mathcal{A}]^{\perp} = \mathcal{L}^{\infty}_{-}$.

(6) We apply Theorem 5.2 and get $\mathcal{A}^{\perp} = [\mathcal{A}^{\perp}] + (\mathcal{A}^{\perp})^{\perp} = (\mathcal{A}^{\perp})^{\perp}$. ■

Definition 5.6. Let $\mathcal{Q} \in \Pi^{c,e,2}$ such that $\mathcal{A} = \mathcal{A}^{\mathcal{Q}} \in \mathbb{A}^{\text{st}}$. Then the two sets $[\mathcal{A}]$ and $\mathcal{A}^{\perp}$ are called respectively the martingale and the non martingale parts of $\mathcal{A}$.

Next, we characterize the martingale and non martingale sets in more detail.

Theorem 5.7. *Let $\mathcal{Q} \in \Pi^{c,e,2}$ such that $\mathcal{A} = \mathcal{A}^{\mathcal{Q}} \in \mathbb{A}^{\text{st}}$. Then the following assertions are equivalent:*

(1) *$\mathcal{A}$ is a martingale set.*

(2) *$\mathcal{A}^{\perp} = \mathcal{L}^{\infty}_{-}$.*

(3) *Any $\mathcal{Q}$-supermartingale is dominated by a local $\mathcal{Q}$-martingale.*

Proof. (1) $\Rightarrow$ (2) Let us suppose $\mathcal{A}$ is a martingale set, then by definition $\mathcal{A} = [\mathcal{A}]$ and by assertion (5) in Proposition 5.5, we get $\mathcal{A}^{\perp} = [\mathcal{A}]^{\perp} = \mathcal{L}^{\infty}_{-}$.

(2) $\Rightarrow$ (1) By Theorem 5.2, we get $\mathcal{A} = [\mathcal{A}] + \mathcal{A}^{\perp} = [\mathcal{A}]$.

(1) $\Rightarrow$ (3) since $\mathcal{Q} = [\mathcal{Q}]$, we apply [11, Theorem 3.1 and Example 2.1] and get the result.

(3) $\Rightarrow$ (1) we apply Theorem 3.1 and get that $\mathcal{Q}^{\mathrm{st}} = [\mathcal{Q}]$ and since $\mathcal{Q} = \mathcal{Q}^{\mathrm{st}}$, we have $\mathcal{Q} = [\mathcal{Q}]$ and therefore $\mathcal{A} = [\mathcal{A}]$. ∎

Theorem 5.8. *Let $\mathcal{Q} \in \Pi^{c,e,2}$ such that $\mathcal{A} = \mathcal{A}^{\mathcal{Q}} \in \mathbb{A}^{\mathrm{st}}$. Then the following assertions are equivalent:*

(1) *$\mathcal{A}$ is a non martingale set.*

(2) *$\mathcal{A}^{\perp} = \mathcal{A}$.*

(3) *Any $\mathcal{Q}$-supermartingale is a $\pi^{\mathcal{Q}}$-non martingale.*

Proof. (1) $\Rightarrow$ (3) Let $X \in \mathrm{spm}_2(\mathcal{Q})$, then by Theorem 5.2 we get $X = X_0 + \pi_0^{\mathcal{Q}}(X) + \pi_1^{\mathcal{Q}}(X)$ with $\pi_0^{\mathcal{Q}}(X) \in m_2([\mathcal{Q}])$, so $\pi_0^{\mathcal{Q}}(X) = 0$ and $X = X_0 + \pi_1^{\mathcal{Q}}(X)$.

(3) $\Rightarrow$ (2) We have $\mathcal{A}^{\perp} \subseteq \mathcal{A}$, and for the reverse implication let $X \in \mathrm{spm}_2(\mathcal{Q})$, then $X = X_0 + \pi_1^{\mathcal{Q}}(X)$ and therefore $X \in \mathrm{spm}_2(\mathcal{Q}^{\perp})$.

(2) $\Rightarrow$ (1) By assertion (2) in Proposition 5.5, we get $[\mathcal{A}] = [\mathcal{A}^{\perp}] = \mathcal{L}_{-}^{\infty}$. ∎

Finally, we generalise Theorem 5.2. We denote $\mathcal{M}\mathrm{art}$, to be the set of all martingale sets and define $\mathcal{M}\mathrm{art}(\mathcal{A}) := \{\mathcal{B} \in \mathcal{M}\mathrm{art} : \mathcal{B} \subseteq \mathcal{A}\}$ for $\mathcal{A} \in \mathbb{A}$.

Theorem 5.9. *Let $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$ and $\mathcal{B} \in \mathcal{M}\mathrm{art}(\mathcal{A})$. Then there exists a unique element $\mathcal{C} \in \mathbb{A}^{\mathrm{st}}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$.*

Proof. We define the family $\mathcal{V} := \{\pi_1^{\widetilde{\mathcal{Q}}}(X) : X \in \mathrm{spm}_2(\mathcal{Q})\}$, where $\mathcal{Q} = \mathcal{Q}^{\mathcal{A}}$ and $\widetilde{\mathcal{Q}} = \mathcal{Q}^{\mathcal{B}}$, we define the set $\mathcal{H} = \mathcal{M}\mathrm{sp}(\mathcal{V})$ and $\mathcal{C} = \mathcal{A}^{\mathcal{H}}$. The rest of the proof is identical to that of Theorem 5.2. ∎

Definition 5.10. Let $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$ and $\mathcal{B} \in \mathcal{M}\mathrm{art}(\mathcal{A})$. The set $\mathcal{C} \in \mathbb{A}^{\mathrm{st}}$ in Theorem 5.9 is called the orthogonal complementary set of $\mathcal{B}$ in $\mathcal{A}$ and denoted by $c(\mathcal{A}, \mathcal{B})$. So the set $\mathcal{A}^{\perp}$ for $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$, introduced in Theorem 5.2, is the orthogonal complementary set of $[\mathcal{A}]$ in $\mathcal{A}$.

Corollary 5.11. *Let $\mathcal{A} \in \mathbb{A}^{\mathrm{st}}$ and $\mathcal{B} \in \mathcal{M}\mathrm{art}(\mathcal{A})$. Then $c(\mathcal{A}, \mathcal{B}) = \mathcal{A}^{\perp} \oplus c([\mathcal{A}], \mathcal{B})$.*

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