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The solution of a type of absolute value equations using two new matrix splitting iterative techniques

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Abstract. Finding the solution of an absolute value vector equation (AVE) of the form

$$Ax - |x| = b$$

is an important subject in scientific computing, operations research, engineering, management science, and economic applications. This paper proposes two new iterative techniques to solve such AVEs. Both techniques are based on a decomposition of the coefficient matrix and a fixed-point principle. The convergence of the proposed techniques under appropriate assumptions is examined. We present results of numerical simulations to verify our theoretical findings and demonstrate the efficiency of our techniques.

1. Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$, consider the AVE

$$Ax - |x| = b, \tag{1}$$

where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix or strictly diagonally dominant matrix, $|x| \in \mathbb{R}^n$ is the vector with the components $|x_1|, \dots, |x_n|$. One can also consider the AVE of general form

$$Ax + B|x| = b, \tag{2}$$

where $B \in \mathbb{R}^{n\times n}$. Equation (2) reduces to equation (1) when $B = -I$, where I is the $n \times n$ unit matrix.

The AVE (1) occurs in various scientific computing problems as well as engineering fields, including convex quadratic programming, network prices, linear complementarity problems (LCPs), linear programming, and modeling of journal bearing lubrication [5, 15, 21, 22, 26].

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Numerical techniques for AVEs focus on the structure of solutions, algebraic constructions, mathematical hypotheses, and the precise implementations of high-quality preconditioners and high-performance numerical methods. In addition, in recent years saw considerable interest in numerical techniques for AVEs, and many papers have suggested a variety of methods. For example, Li [18] proposed a preconditioner AOR (accelerated over-relaxation) approach for solving equation (1) and provided conditions ensuring the convergence of his technique. Wu and Li [28] analyzed a novel approach based on the shift splitting procedure for solving equation (1). Fakhazadeh and Shams [10] proposed a mixed-type splitting procedure for computing solutions of the AVE (1) and discussed its convergence properties. Zainali and Lotfi [29] investigated the Newton technique and developed a stable as well as a quadratically convergent solution for equation (1). Feng with Liu [11, 12] presented a two-step technique as well as an improved generalized Newton technique. Saheya et al. [27] explored smoothing-type schemes for equation (1) and analyzed the convergence properties for the proposed schemes. According to Abdalah et al. [1], the AVE problem can be reformulated as an LCP and its solutions can be computed by applying a smoothing strategy. The unique features of AVEs and their association with LCPs have been studied by Prokopyev [25]. Ke and Ma [17] considered an SOR-like (successive over-relaxation) technique for equation (1). The work of Chen et al. [6] has extensively investigated the approach of [17] and proposed a series of optimal parameters for an SOR-like system. Zamani and Hladík [30] developed a novel concave minimization strategy for equation (1), which eliminates some of the deficiencies of the earlier methods proposed in [20] and other works; see [4, 8, 9, 13, 14, 16] and the references therein.

To solve LCPs, Miao and Zhang [24], Li et al. [19], Dehghan and Hajarian [7] and Mao et al. [23] have recently suggested different techniques based on a system of fixed-point type. It is the goal of the analysis performed in the present paper to extend the fixed-point principle to AVEs and develop effective iterative algorithms for computing solutions of equation (1). To this end, we decompose the coefficient matrix A of the equation into three different parts and use this decomposition to derive two fixed-point formulas for the solution of the AVE. The new techniques are obtained from these formulas. Moreover, we examine the convergence properties of our procedures under new circumstances.

The paper is organized as follows. Section 2 presents the proposed techniques for computing solutions of the AVE (1) as well as their convergence properties. Section 3 presents results of numerical simulation, and Section 4 draws final conclusions.

2. Iterative schemes

In this section, we outline the proposed techniques for computing solutions of equation (1). Initially, we go over a few preparatory results.

Throughout the paper, $\mathrm{Td}(A)$, $\|A\|_\infty = \max_{1\le i\le m}\sum_{j=1}^{m}|a_{ij}|$, $\rho(A)$ and $|A| = (|a_{ij}|)$ stand for the tridiagonal part, the infinity norm, the spectral radius and the absolute value of the matrix A, respectively. The square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are all nonpositive. The matrix A is called an M-matrix if it is a Z-matrix and nonsingular with $A^{-1} \ge 0$. Furthermore, the matrix A is said to be strictly row diagonally dominant if $\sum_{j=1,j\ne i}^{n}|a_{ij}| < |a_{ii}|$, $i = 1, 2, \ldots, n$.

To introduce and analyze the new techniques, we decompose the matrix A as

$$A = W - \Theta, \tag{3}$$

with

$$W = D - U + U^{\mathsf{T}} \qquad \text{and} \qquad \Theta = L + U^{\mathsf{T}}.$$

Here $A = D - U - L$ with D, U and L are diagonal, strictly upper triangular and strictly lower triangular, respectively, and $^{\mathsf{T}}$ stands for transposition. The AVE (1) is equivalent to the fixed-point problem of solving (see [3] for more details)

$$x = Q(x),$$

where

$$Q(x) = x - \beta E[Ax - |x| - b], \tag{4}$$

$0 < \beta \le 1$, and $E = D^{-1}$ is a diagonal matrix consisting of positive diagonal elements (see [2, 3]). Equations (3) and (4) imply that

$$x = x - \beta E[Wx - \Theta x - |x| - b],$$

or, equivalently,

$$(I - \beta E\Theta)x = x - \beta E[Wx - |x| - b].$$

Now we can formulate our iterative Technique I for equation (1) as

$$x^{k+1} = (I - \beta E\Theta)^{-1}\{x^k - \beta E[Wx^k - |x^k| - b]\}, \quad k = 0, 1, 2, \ldots \tag{5}$$

The next result deals with the convergence of Technique I.

Theorem 2.1. *Let $\{x^k\}$ be the sequence generated by the recipe (5) of Technique* I *and let $\ddot{x}$ denote the solution of the AVE (1). Then*

$$|x^{k+1} - \ddot{x}| \le |G^{-1}|T|x^k - \ddot{x}|,$$

where $G = I - \beta E\Theta$ and $T = \beta E + |I - \beta EW|$. Moreover, if $\rho(|G^{-1}|T) < 1$, then the sequence $\{x^k\}$ converges to the unique solution $\ddot{x}$ of the AVE (1).

Proof. Let $\ddot{x}$ be a solution of (1). Then

$$\ddot{x} = (I - \beta E\Theta)^{-1}\big\{\ddot{x} - \beta E[W\ddot{x} - |\ddot{x}| - b]\big\}. \tag{6}$$

Subtracting (6) from (5), we get

$$x^{k+1} - \ddot{x} = (I - \beta E\Theta)^{-1}\big\{(x^k - \ddot{x}) - \beta EW(x^k - \ddot{x}) + \beta E(|x^k| - |\ddot{x}|)\big\},$$

or

$$x^{k+1} - \ddot{x} = (I - \beta E\Theta)^{-1}\big\{(I - \beta EW)(x^k - \ddot{x}) + \beta E(|x^k| - |\ddot{x}|)\big\}.$$

Using absolute values on both sides, we obtain successively the inequalities

$$|x^{k+1} - \ddot{x}| \leq |(I - \beta E\Theta)^{-1}|\big\{|I - \beta EW||x^k - \ddot{x}| + \beta E\big||x^k| - |\ddot{x}|\big|\big\},$$
$$|x^{k+1} - \ddot{x}| \leq |(I - \beta E\Theta)^{-1}|\big\{|I - \beta EW||x^k - \ddot{x}| + \beta E|x^k - \ddot{x}|\big\},$$
$$|x^{k+1} - \ddot{x}| \leq |(I - \beta E\Theta)^{-1}|\big\{(|I - \beta EW| + \beta E)|x^k - \ddot{x}|\big\},$$

or, equivalently,

$$|x^{k+1} - \ddot{x}| \leq |(I - \beta E\Theta)^{-1}|\big(\beta E + |I - \beta EW|\big)|x^k - \ddot{x}|.$$

It follows that

$$|x^{k+1} - \ddot{x}| \leq |G^{-1}|T|x^k - \ddot{x}|.$$

In the present case, the matrix $|G^{-1}|T$ is nonnegative. According to [2, 3], if $\rho(|G^{-1}|T) < 1$, the iteration sequence $\{x^k\}$ of Technique I converges to the solution $\ddot{x}$ of the AVE (1).

To establish the uniqueness, suppose that $\ddot{y}$ is another solution of equation (1). Thus,

$$A\ddot{x} - |\ddot{x}| = b,$$

and

$$A\ddot{y} - |\ddot{y}| = b,$$

which we rewrite as

$$\ddot{x} = (I - \beta E\Theta)^{-1}\big\{\ddot{x} - \beta E[W\ddot{x} - (|\ddot{x}| + b)]\big\},$$

and

$$\ddot{y} = (I - \beta E\Theta)^{-1}\big\{\ddot{y} - \beta E[W\ddot{y} - (|\ddot{y}| + b)]\big\},$$

respectively. It follows that

$$|\ddot{x} - \ddot{y}| \leq |G^{-1}|T|\ddot{x} - \ddot{y}|,$$

where $G = I - \beta E\Theta$ and $T = \beta E + |I - \beta EW|$. Since $\rho(|G^{-1}|T) < 1$, this implies that $\ddot{x} = \ddot{y}$. This completes the proof. ∎

We next address Technique II. Recall that the equation (1) can be rewritten as equation (4),

$$x = x - \beta E[Ax - |x| - b],$$

or, equivalently, as

$$x = \beta\{x - E[Ax - |x| - b]\} + (1 - \beta)x. \tag{7}$$

Let $\Delta = \Upsilon I$, where I is the unit matrix and $0 < \Upsilon \leq 1$. Equations (3) and (7), imply successively that

$$x = \beta\{x - E[(W + \Delta)x - (\Theta + \Delta)x - |x| - b]\} + (1 - \beta)x,$$
$$x - \beta E(\Theta + \Delta)x = \beta\{x - E[(W + \Delta)x - |x| - b]\} + (1 - \beta)x,$$

and finally

$$(I - \beta E(\Theta + \Delta))x = \beta\{x - E[(W + \Delta)x - |x| - b]\} + (1 - \beta)x.$$

The iteration sequence generated by our Technique II for solving equation (1) is defined by

$$x^{k+1} = (I - \beta E(\Theta + \Delta))^{-1}\{\beta\{x^k - E[(W + \Delta)x^k - |x^k| - b]\} + (1 - \beta)x^k\}, \tag{8}$$

for $k = 0, 1, 2, \ldots$

Next, we focus on the convergence of Technique II by using the subsequent theorem.

Theorem 2.2. *Let $\{x^k\}$ be the iterative sequences generated by Technique* II *and $\ddot{x}$ denote the solution of the AVE* (1). *Then*

$$|x^{k+1} - \ddot{x}| \leq |\bar{B}^{-1}|C|x^k - \ddot{x}|,$$

where

$$\bar{B} = I - \beta E(\Theta + \Delta) \qquad \text{and} \qquad C = \beta E + |I - \beta E(W + \Delta)|.$$

Moreover, if $\rho(|\bar{B}^{-1}|C) < 1$, then the sequence $\{x^k\}$ converges to the unique solution $\ddot{x}$ of the AVE (1).

Proof. Let $\ddot{x}$ be a solution of (1). Then

$$\ddot{x} = (I - \beta E(\Theta + \Delta))^{-1}\{\beta\{\ddot{x} - E[(W + \Delta)\ddot{x} - |\ddot{x}| - b]\} + (1 - \beta)\ddot{x}\}. \tag{9}$$

Upon subtracting equation (9) from equation (8), we get

$$x^{k+1} - \ddot{x} = (I - \beta E(\Theta + \Delta))^{-1}\{\beta\{(x^k - \ddot{x}) - E(W + \Delta)(x^k - \ddot{x}) + E(|x^k| - |\ddot{x}|\} + (1 - \beta)(x^k - \ddot{x})\}.$$

Simplifying the right-hand side, we get

$$x^{k+1} - \ddot{x} = (I - \beta E(\Theta + \Delta))^{-1}\big\{(I - \beta E(W + \Delta))(x^k - \ddot{x}) + \beta E(|x^k| - |\ddot{x}|)\big\}.$$

Taking absolute values on both sides, we successively obtain

$$|x^{k+1} - \ddot{x}| \leq |I - \beta E(\Theta + \Delta))^{-1}|\big\{|I - \beta E(W + \Delta)||x^k - \ddot{x}| + \beta E||x^k| - |\ddot{x}||\big\},$$

hence

$$|x^{k+1} - \ddot{x}| \leq |I - \beta E(\Theta + \Delta))^{-1}|\big\{|I - \beta E(W + \Delta)||x^k - \ddot{x}| + \beta E|x^k - \ddot{x}|\big\},$$

or

$$|x^{k+1} - \ddot{x}| \leq |I - \beta E(\Theta + \Delta))^{-1}|\big\{(\beta E + |I - \beta E(W + \Delta)|)|x^k - \ddot{x}|\big\},$$

ending with

$$|x^{k+1} - \ddot{x}| \leq |\bar{B}^{-1}|C|x^k - \ddot{x}|,$$

where

$$\bar{B} = I - \beta E(\Theta + \Delta) \qquad \text{and} \qquad C = \beta E + |I - \beta E(W + \Delta)|.$$

Clearly, if $\rho(|\bar{B}^{-1}|C) < 1$, then the iterative sequence $\{x^k\}$ generated by Technique II is convergent. The proof of the uniqueness is omitted, since it is similar to the proof in Theorem 2.1. ■

3. Numerical experiments

In this section, we report results of several numerical investigations which demonstrate the efficiency of the newly developed techniques with respect to the number of iteration steps (Iters), CPU time (CPU), as well as the relative residual error (RRE), Where 'RRE' is defined as

$$\text{RRE} := \frac{\|Ax^k - |x^k| - b\|_2}{\|b\|_2}$$

and is subject to the bound RRE $\leq 10^{-6}$.

Ahn [2] used the fixed-point principle to solve the linear complementarity problems. Some authors select the value of β differently. For instance, Dehghan and Hajarian [7] took $\beta = 1$ in their work problems, while Mao et al. [23] divided the parameter range into intervals. In this work, we used the same idea for AVEs and took the parameters in the interval $0 < \Upsilon, \beta \leq 1$.

Examples	n	Technique I $\rho(\lvert G^{-1}\rvert T)$	Technique II $\rho(\lvert \bar{B}^{-1}\rvert C)$
Example 3.1	100	0.4356	0.4286
	1600	0.4828	0.4286
Example 3.2	64	0.5587	0.5546
	1024	0.6199	0.6163
Example 3.3	1000	0.4926	0.5047
	3000	0.4973	0.5098
Example 3.4	16	0.1375	0.2199
	49	0.1444	0.2274

Table 1. Conditions for convergence of Theorems 2.1 and 2.2.

Initially, we performed numerical experiments in order to satisfy the convergence conditions $\rho(|G^{-1}|T) < 1$ and $\rho(|\bar{B}^{-1}|C) < 1$. Table 1 displays the results.

As shown in Table 1, we performed numerical experiments to test the convergence conditions for both theorems. The results indicate that both approaches work well. To evaluate the implementation of our newly developed techniques, the following tests were conducted.

Example 3.1. Let $A \in \mathbb{R}^{n\times n}$ be of the form $A = M + I$ and $b \in \mathbb{R}^n$ be given by $A\ddot{x} - |\ddot{x}| = b$ with

$$M = \mathrm{Td}(-1.5I, J, -0.5I) \in \mathbb{R}^{n\times n} \qquad \text{and} \qquad \ddot{x} = (1.2, \ldots, 1.2)^{\mathsf{T}} \in \mathbb{R}^n$$

where $J = \mathrm{Td}(-1.5, 4, -0.5) \in \mathbb{R}^{f\times f}$, $I \in \mathbb{R}^{f\times f}$ is the unit matrix, and $f^2 = n$. In Example 3.1 and Example 3.2 below, we compare the proposed techniques I and II with the AOR approach [18] and the mixed-type (MT) splitting approach [10]. Table 2 displays the numerical data.

Example 3.2. Let $A \in \mathbb{R}^{n\times n}$ be of the form $A = M + 4I$ and let $b \in \mathbb{R}^n$ be given by $A\ddot{x} - |\ddot{x}| = b$ with

$$M = \mathrm{Td}(-I, J, -I) \in \mathbb{R}^{n\times n} \qquad \text{and} \qquad \ddot{x} = ((-1)^1, \ldots, (-1)^n))^{\mathsf{T}} \in \mathbb{R}^n,$$

where $J = \mathrm{Td}(-1, 4, -1) \in \mathbb{R}^{f\times f}$, $I \in \mathbb{R}^{f\times f}$ is the unit matrix and $f^2 = n$. Table 3 displays the results.

Tables 2 and 3 compare numerical features of the AOR, MT, and the newly developed techniques. The results demonstrate that the proposed techniques I and II are superior to both the AOR and MT procedures considered.

Techniques	n	100	400	900	1600	4900
AOR	Iters	97	190	336	706	384
	CPU	0.4721	2.8203	3.2174	6.3887	9.2344
	RRE	9.80e–07	9.61e–07	9.73e–07	9.84e–07	9.37e–07
MT	Iters	88	157	250	386	342
	CPU	0.4042	1.7954	3.0218	5.7627	8.8966
	RRE	8.92e–07	9.66e–07	9.19e–07	9.57e–07	9.88e–07
Technique I	Iters	42	62	79	94	103
	CPU	0.1821	0.3227	0.9642	1.3403	1.9528
	RRE	9.66e–07	9.79e–07	8.66e–07	8.83e–07	8.82e–07
Technique II	Iters	19	26	31	35	52
	CPU	0.1027	0.1632	0.7241	1.0971	1.4852
	RRE	9.06e–07	8.81e–07	8.18e–07	8.67e–07	8.86e–07

Table 2. The effects of Example 3.1 with $\Upsilon = 0.8$ and $\beta = 1$.

Techniques	n	64	256	1024	4096
AOR	Iters	14	14	15	35
	CPU	0.3483	1.9788	2.3871	5.8097
	RRE	5.21e-07	6.29e-07	6.54e-07	8.75e-07
MT	Iters	14	14	15	25
	CPU	0.3169	1.0953	1.9648	2.2195
	RRE	4.32e-07	5.47e-07	5.07e-07	9.39e-07
Technique I	Iters	11	11	10	9
	CPU	0.1328	0.5326	1.6768	2.0363
	RRE	9.44e-07	4.87e-07	6.32e-07	8.95e-07
Technique II	Iters	11	11	10	9
	CPU	0.1202	0.3573	0.8627	1.3492
	RRE	9.83e-07	5.07e–07	6.54e–07	9.38e–07

Table 3. The outcomes of Example 3.2 with $\Upsilon = 0.05$ and $\beta = 1$.

Example 3.3. Suppose

$$A = \mathrm{Td}(-1, 4, -1) \in \mathbb{R}^{n\times n}, \qquad \ddot{x} = ((-1)^{-1}, \ldots, (-1)^{-n}))^{\top} \in \mathbb{R}^{n}$$

and $A\ddot{x} - |\ddot{x}| = b$. For Example 3.3 and Example 3.4 below, we present a comparison between the proposed techniques and the optimal parameter SOR-like approach [6] (reported as SLA) and the shift splitting approach [28] (reported as SSA). Table 4 displays the results.

Example 3.4. Let $\bar{h} = 1/n$ and $\bar{\Omega} = n^2$. Consider the matrix

$$A = I \otimes Q + P \otimes I \in \mathbb{R}^{\bar{\Omega}\times\bar{\Omega}},$$

where $I \in \mathbb{R}^{n\times n}$ denotes again the unit matrix and $\otimes$ stands for the Kronecker product. Moreover, Q and P are $n \times n$ tridiagonal matrices defined as follows:

$$\begin{cases} Q = \mathrm{Td}\Big(\dfrac{2+\bar{h}}{8}, 8, \dfrac{2-\bar{h}}{8}\Big), \\ P = \mathrm{Td}\Big(\dfrac{1+\bar{h}}{4}, 4, \dfrac{1-\bar{h}}{4}\Big). \end{cases}$$

The vector is given by $b = A\ddot{x} - |\ddot{x}|$, where $\ddot{x} = (1, 1, 1, \ldots, 1, 1) \in \mathbb{R}^{\bar{\Omega}}$. The results of computations are listed in Table 5.

Techniques	n	1000	2000	3000	4000
SLA	Iters	18	18	18	18
	CPU	3.0156	13.1249	33.9104	65.1345
	RRE	6.12e–07	6.13e–07	6.13e–07	6.15e–07
SSA	Iters	14	14	14	14
	CPU	2.8129	9.0955	17.3029	29.1645
	RRE	8.92e–07	8.93e–07	8.94e–07	8.94e–07
Technique I	Iters	11	11	11	11
	CPU	2.1304	5.1618	9.2085	16.6924
	RRE	7.48e–07	7.43e–07	7.41e–07	7.40e–07
Technique II	Iters	12	12	12	12
	CPU	1.0815	2.1425	3.2091	5.3009
	RRE	6.30e–07	6.31e–07	6.32e–07	6.33e–07

Table 4. The outcomes of Example 3.3 with $\Upsilon = 0.05$ and $\beta = 1$.

Techniques	$\bar{\Omega}$	256	1296	2401	4096
SLA	Iters	12	12	12	12
	CPU	1.6492	3.1458	14.5518	82.6151
	RRE	3.77e–07	3.74e–07	3.73e–07	3.72e–07
SSA	Iters	8	8	8	8
	CPU	0.3707	4.5734	22.7075	117.6810
	RRE	1.54e–07	1.55e–07	1.56e–07	1.56e–07
Technique I	Iters	6	6	6	6
	CPU	0.1322	1.1309	7.0435	15.3732
	RRE	2.26e–09	2.16e–09	2.14e–07	2.16e–09
Technique II	Iters	4	4	4	4
	CPU	0.0569	1.2082	1.9334	3.3582
	RRE	2.18e–07	1.50e–07	1.29e–07	1.13e–07

Table 5. The outcomes for Example 3.4 with $\Upsilon = 0.9$ and $\beta = 1$.

Tables 4 and 5 demonstrate that all of the tested techniques achieve an accurate calculation of solutions of the AVE (1). In comparison with the existing techniques, the ‘Iters’ and ‘CPU’ values in the proposed techniques I and II are superior. This allows us to conclude that the proposed techniques are both highly effective and implementable.

4. Conclusions

This paper introduced two new iterative techniques for computing solutions of equation (1). We confirmed that the proposed approaches lead to the solution of the AVE (1) under suitable selections of the involved parameters. We reported results of some numerical investigations which indicate that the presented techniques are implementable and effective. Theoretical comparisons as well as research of these iterative procedures are attractive topics for future research.

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