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Uniform regularity for the flow of a chemically reacting gaseous mixture

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Abstract. Uniform regularity plays an important role in the global existence of strong solutions and large time behavior of global solutions. In this work, we prove the uniform regularity of smooth solutions to the compressible flow of a chemically reacting gaseous mixture in $\mathbb{T}^3$.

1. Introduction

The flow of chemically reacting gaseous mixture is associated with a variety of phenomena and processes: pollutant formation, biotechnology, fuel droplets in combustion, sprays, and astrophysical plasma. Due to its numerous applications, the chemically reacting gaseous mixture have been the subject of many theoretical research and engineering. However, it is formidable and complex to describe and analyze this model mathematically due to the complicate radiating and thermonuclear processes, as well as a number of physical hypothesis. Therefore, simplification should be introduced. It is well known that there are three classical different ways for simplification: simplifying the reactive process, simplifying the fluid dynamics and simplifying the coupling relationship. More precisely, a wide variety of nuclear reactions take place inside the star and produce the burning of the constitutive elements, giving rise to a self-consistent production of energy. As pointed out by Bebernes et al. [2], Ducomet [10], Feireisl et al. [15], people introduce a simple reacting process with first-order kinetics and it is coupled with a compressible Navier–Stokes–Possion equations system. Inspired by [10, 15], we consider the following system of simple reacting compressible flows in astrophysics:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = -\rho \nabla \phi, \tag{1.2}$$

$$\partial_t(\rho Y) + \operatorname{div}(\rho u Y) - \varepsilon \Delta Y + k\rho Y = 0, \tag{1.3}$$

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$$-\Delta\phi + e^{\phi} = \rho \quad \text{in } \mathbb{T}^3 \times (0,\infty), \tag{1.4}$$

$$(\rho, u, Y)(\cdot, 0) = (\rho_0, u_0, Y_0)(\cdot) \quad \text{in } \mathbb{T}^3. \tag{1.5}$$

Here, ρ denotes the density, u the velocity field, Y the reactant fraction, and ϕ is a potential function, λ and μ are two viscosity constants satisfying

$$\mu > 0 \quad \text{and} \quad \lambda + \frac{2}{3}\mu \geq 0,$$

ε and k are positive constants; and $p := a\rho^{\gamma}$ is the pressure with the constants $a > 0$ and $\gamma \geq 1$. Compared with [10,15], we neglect the temperature of the mixture and consider a simple diffusion flux with a more complex and general Poisson equation (1.4). This simplification leads to a possibility of considering the above system as isentropic compressible models from the mathematical viewpoint.

Generally speaking, the highly complex nonlinearity and coupling shows well-posedness is not an easy issue. In recent years, the corresponding study have paid a lot of attentions. Let us recall some known results. Feireisl and his coauthors [16] studied a multi-dimensional model for the dynamic combustion of a viscous, compressible, radiative and reactive gas with higher order kinetics. They obtained the global existence of a weak solution, which relies on the concept of a variational solution. Based on the seminal work of Feireisl, Donatelli and Trivisa [8] established the global existence of weak solutions with large initial data for a multi-dimensional combustion model. And they [7] extended the result to a more general situation, where the heat conductivity and viscosity depend on the temperature, pressure depends on the density, temperature and reactant. We refer readers to [3,9,10,17,19,20,22–25] for more details and other results.

For the one-dimensional case, Ducomet [11] established the global existence and exponential decay in H^1 of solutions to the one-dimensional model for $q \geq 4$. Later on, Ducomet and Zlotnik [12–14] established the existence of global solutions to the one-dimensional model under rather general assumptions on q. Moreover, they obtained the exponential stabilization for solutions by constructing new Lyapunov functionals. Chen, Hoff and Trivisa [4–6] studied the discontinuous solutions with large discontinuous initial data for the one-dimensional model.

Before stating our main results, we recall the local existence of smooth solutions to the problem (1.1)–(1.5). Since the system (1.1)–(1.5) is a parabolic-hyperbolic one, the results in [27] imply the following.

Proposition 1.1 ([27]). *Let $s > \frac{5}{2}$ be an integer and assume that the initial data satisfy*

$$\rho_0, u_0, Y_0 \in H^s \quad \textit{and} \quad \frac{1}{C_0} \leq \rho_0 \tag{1.6}$$

for a positive constant C_0. Then the problem (1.1)–(1.5) *has a unique smooth solution* (ρ, u, Y) *satisfying*

$$\rho \in C^{\ell}([0,T); H^{s-\ell}), \quad u, Y \in C^{\ell}([0,T); H^{s-2\ell}), \quad \ell = 0, 1; \quad \frac{1}{C} \le \rho \tag{1.7}$$

for some $0 < T \le \infty$.

The aim of this paper is to prove uniform regularity estimates in $(\lambda, \mu, \varepsilon)$. We will prove the following theorem.

Theorem 1.1. *Let* $0 < \mu < 1$, $0 < \lambda + \mu < 1$, $0 < \varepsilon < 1$, $0 < \frac{1}{C_0} \le \rho_0$, $0 \le Y_0$, $\rho_0, u_0, Y_0 \in H^3(\mathbb{T}^3)$. *Let* (ρ, u, Y, ϕ) *be the unique local smooth solutions to the problem* (1.1)–(1.5). *Then*

$$\|(\rho, u, Y)(\cdot, t)\|_{H^3} \le C \quad \text{and} \quad \|\phi(\cdot, t)\|_{H^5} \le C \quad \text{in } [0, T] \tag{1.8}$$

hold true for some positive constants C *and* T_0 $(\le T)$ *independent of* λ, μ *and* ε.

We define

$$M(t) := 1 + \sup_{0 \le \tau \le t} \Big\{ \|(\rho, u, Y, p)(\cdot, \tau)\|_{H^3} + \|\partial_t u(\cdot, \tau)\|_{L^2} + \|\partial_t Y(\cdot, \tau)\|_{L^2} + \Big\| \frac{1}{\rho}(\cdot, \tau) \Big\|_{L^\infty} \Big\}. \tag{1.9}$$

We can prove:

Theorem 1.2. *For any* $t \in [0, T_0)$, *we have that*

$$M(t) \le C_0(M_0) \exp(t C(M)) \tag{1.10}$$

for some nondecreasing continuous functions $C_0(\cdot)$ *and* $C(\cdot)$.

It follows from (1.10), see [1,21], that

$$M(t) \le C. \tag{1.11}$$

If we can prove Theorem 1.2, then Theorem 1.1 follows immediately. Therefore, we only need to show Theorem 1.2.

In the following proofs, we will use the bilinear commutator and product estimates due to Kato–Ponce [18],

$$\|D^s(fg) - f D^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|D^{s-1} g\|_{L^{q_1}} + \|g\|_{L^{p_2}} \|D^s f\|_{L^{q_2}}), \tag{1.12}$$

$$\|D^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}} \|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \tag{1.13}$$

with $D = (-\triangle)^{\frac{1}{2}}$, $s > 0$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

2. Proof of Theorem 1.2

First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^q \mathrm{d}x = \Big(1-\frac{1}{q}\Big)\int \rho^q \,\mathrm{div}\, u \mathrm{d}x \le \|\mathrm{div}\, u\|_{L^\infty}\int \rho^q \mathrm{d}x,$$

and it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q} \le \|\mathrm{div}\, u\|_{L^\infty}\|\rho\|_{L^q}.$$

A routine computation gives rise to

$$\|\rho\|_{L^q} \le \|\rho_0\|_{L^q} \exp\Big(\int_0^t \|\mathrm{div}\, u\|_{L^\infty}\mathrm{d}\tau\Big). \tag{2.1}$$

Taking $q \to +\infty$, we get

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty} \exp(tC(M)). \tag{2.2}$$

Due to the mass equation (1.1), it follows that

$$\partial_t \frac{1}{\rho} + u\cdot\nabla\frac{1}{\rho} - \frac{1}{\rho}\,\mathrm{div}\, u = 0. \tag{2.3}$$

Proceeding as (2.1), we multiply (2.3) by $(\frac{1}{\rho})^{q-1}$ and get the following:

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \Big(\frac{1}{\rho}\Big)^q \mathrm{d}x = \Big(1+\frac{1}{q}\Big)\int \Big(\frac{1}{\rho}\Big)^q \mathrm{div}\, u \mathrm{d}x \le \Big(1+\frac{1}{q}\Big)\Big\|\frac{1}{\rho}\Big\|_{L^q}^q \|\mathrm{div}\, u\|_{L^\infty}.$$

It is obvious to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{1}{\rho}\Big\|_{L^q} \le \Big(1+\frac{1}{q}\Big)\Big\|\frac{1}{\rho}\Big\|_{L^q}\|\mathrm{div}\, u\|_{L^\infty},$$

which gives

$$\Big\|\frac{1}{\rho}\Big\|_{L^q} \le \Big\|\frac{1}{\rho_0}\Big\|_{L^q} \exp\Big(\Big(1+\frac{1}{q}\Big)\int_0^t \|\mathrm{div}\, u\|_{L^\infty}\mathrm{d}\tau\Big).$$

Sending $q \to +\infty$ leads that

$$\Big\|\frac{1}{\rho}\Big\|_{L^\infty} \le \Big\|\frac{1}{\rho_0}\Big\|_{L^\infty} \exp(tC(M)). \tag{2.4}$$

Thus, combining (2.2) and (2.4), we have

$$\|p\|_{L^\infty} + \Big\|\frac{1}{p}\Big\|_{L^\infty} \le C_0(M_0)\exp(tC(M)). \tag{2.5}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x = 2\int u\partial_t u\mathrm{d}x \le 2\|u\|_{L^2}\|\partial_t u\|_{L^2} \le C(M),$$

which implies

$$\|u\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.6}$$

Testing (1.5) by $-\Delta\phi$ yields the following estimate:

$$\begin{aligned}\|\Delta\phi\|_{L^2}^2 + \int e^{\phi}|\nabla\phi|^2\mathrm{d}x &= -\int \rho\Delta\phi\mathrm{d}x\\ &\le \frac{1}{2}\int \rho^2\mathrm{d}x + \frac{1}{2}\int |\Delta\phi|^2\mathrm{d}x,\end{aligned}$$

which gives

$$\|\Delta\phi\|_{L^2} \le C\|\rho\|_{L^2}. \tag{2.7}$$

We obtain the following equation by integrating (1.5) over $\mathbb{T}^3$:

$$\int e^{\phi}\mathrm{d}x = \int \rho\mathrm{d}x = \int \rho_0\mathrm{d}x. \tag{2.8}$$

Recalling the following well-known Poincaré inequality

$$\left\|\phi - \ln\int e^{\phi}\mathrm{d}x\right\|_{L^2} \le C\|\nabla\phi\|_{L^2} \quad (|\mathbb{T}^3| = |[0,1]^3| = 1) \tag{2.9}$$

and

$$\|\nabla\phi\|_{L^2}^2 \le \|\phi\|_{L^2}\|\Delta\phi\|_{L^2}, \tag{2.10}$$

we can prove that

$$\begin{aligned}\|\phi\|_{L^\infty} &\le C\|\phi\|_{H^2} = C\|\phi\|_{L^2} + C\|\nabla^2\phi\|_{L^2}\\ &\le C\|\phi\|_{L^2} + C\|\Delta\phi\|_{L^2}\\ &\le C + C\|\rho\|_{L^2}.\end{aligned} \tag{2.11}$$

It is evident that

$$Y \ge 0 \quad \text{in } \mathbb{T}^3 \times (0,\infty). \tag{2.12}$$

Multiplying (1.3) by Y and using (1.1), we observe that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho Y^2\mathrm{d}x + \varepsilon\int |\nabla Y|^2\mathrm{d}x + k\int \rho Y^2 = 0,$$

whence

$$\int Y^2\mathrm{d}x + \varepsilon\int_0^T\int |\nabla Y|^2\mathrm{d}x\mathrm{d}t \le C_0(M_0)\exp(tC(M)). \tag{2.13}$$

We now turn to obtain the higher regularity. First, we establish the higher estimate for density. It is straightforward to show that

$$\frac{1}{\gamma p}\partial_t p + \frac{1}{\gamma p} u \cdot \nabla p + \operatorname{div} u = 0. \tag{2.14}$$

With the help of the proceeding inequalities (1.12) and (1.13), applying D^3 to (2.14) and testing by $D^3 p$ give rise to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{\gamma p}(D^3 p)^2 \mathrm{d}x + \int D^3 p D^3 \operatorname{div} u \mathrm{d}x \\
&= \frac{1}{2}\int (D^3 p)^2 \Big[\operatorname{div}\Big(\frac{u}{\gamma p}\Big) - \frac{1}{\gamma p^2}\partial_t p\Big]\mathrm{d}x \\
&\quad - \int \Big(D^3\Big(\frac{1}{\gamma p}\partial_t p\Big) - \frac{1}{\gamma p} D^3 \partial_t p\Big) D^3 p \mathrm{d}x \\
&\quad - \int D^3\Big(\frac{u}{\gamma p}\cdot \nabla p - \frac{u}{\gamma p}\cdot \nabla D^3 p\Big) D^3 p \mathrm{d}x \\
&\le C\|D^3 p\|_{L^2}^2 \Big\| \operatorname{div}\Big(\frac{u}{\gamma p}\Big) - \frac{1}{\gamma p^2}\partial_t p\Big\|_{L^\infty} \\
&\quad + C\|\partial_t p\|_{L^\infty}\Big\|D^3\Big(\frac{1}{\gamma p}\Big)\Big\|_{L^2}\|D^3 p\|_{L^2} \\
&\quad + C\Big\|\nabla \frac{1}{\gamma p}\Big\|_{L^\infty}\|D^2 \partial_t p\|_{L^2}\|D^3 p\|_{L^2} \\
&\quad + C\|\nabla p\|_{L^\infty}\Big\|D^3\Big(\frac{u}{\gamma p}\Big)\Big\|_{L^2}\|D^3 p\|_{L^2} + \Big\|\nabla \frac{u}{\gamma p}\Big\|_{L^\infty}\|D^3 p\|_{L^2}^2 \\
&\le C(M) + C(M)\|\partial_t p\|_{L^\infty} + C(M)\|D^2 \partial_t p\|_{L^2} \\
&\le C(M) + C(M)\|u\cdot \nabla p + \gamma p \operatorname{div} u\|_{L^\infty} \\
&\quad + C(M)\|D^2(u\cdot\nabla p + \gamma p \operatorname{div} u)\|_{L^2} \\
&\le C(M).
\end{aligned} \tag{2.15}$$

Here we have used the following estimate [26]:

$$\Big\|D^3 \frac{1}{p}\Big\|_{L^2} \le C(M)\|D^3 p\|_{L^2} \le C(M). \tag{2.16}$$

Next, it is clear that

$$\int_0^t \int |\partial_t u|^2 \mathrm{d}x\mathrm{d}\tau \le t \sup \int |\partial_t u|^2 \mathrm{d}x \le t C(M). \tag{2.17}$$

Operating D^2 to (1.2) and testing by $D^2\partial_t u$, one gets by some direct calculations that

$$\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|D^3u|^2\mathrm{d}x+\frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(D^2\operatorname{div}u)^2\mathrm{d}x+\int\rho|D^2\partial_t u|^2\mathrm{d}x\\
&\quad=-\int D^2\nabla p\cdot D^2\partial_t u\mathrm{d}x-\int D^2(\rho u\cdot\nabla u)\cdot D^2\partial_t u\mathrm{d}x\\
&\qquad-\int[D^2(\rho\partial_t u)-\rho D^2\partial_t u]D^2\partial_t u\mathrm{d}x-\int D^2(\rho\nabla\phi)D^2\partial_t u\mathrm{d}x\\
&\quad\le C\|D^3p\|_{L^2}\|D^2\partial_t u\|_{L^2}+C\|\rho\|_{H^2}\|u\|_{H^3}^2\|D^2\partial_t u\|_{L^2}\\
&\qquad+C\big(\|\nabla\rho\|_{L^\infty}\|D\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|D^2\rho\|_{L^2}\big)\|D^2\partial_t u\|_{L^2}\\
&\qquad+\|D^2(\rho\nabla\phi)\|_{L^2}\|D^2\partial_t u\|_{L^2}\\
&\quad\le C(M)\|D^2\partial_t u\|_{L^2}+C(M)\big(\|D\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\big)\|D^2\partial_t u\|_{L^2}\\
&\quad\le C(M)\|D^2\partial_t u\|_{L^2}+C(M)\\
&\qquad\cdot\big(\|\partial_t u\|_{L^2}^{\frac12}\|D^2\partial_t u\|_{L^2}^{\frac12}+\|\partial_t u\|_{L^2}+\|\partial_t u\|_{L^2}^{\frac14}\|D^2\partial_t u\|_{L^2}^{\frac34}\big)\|D^2\partial_t u\|_{L^2}\\
&\quad\le C(M)\|D^2\partial_t u\|_{L^2}+C(M)\big(\|D^2\partial_t u\|_{L^2}^{\frac12}+\|D^2\partial_t u\|_{L^2}^{\frac34}\big)\|D^2\partial_t u\|_{L^2}\\
&\quad\le\frac12\int\rho|D^2\partial_t u|^2\mathrm{d}x+C(M),
\end{aligned}$$

where we have used (1.12) and (1.13). Integrating the above inequality gives that

$$\int_0^t\int|D^2\partial_t u|^2\mathrm{d}x\mathrm{d}\tau\le C_0(M_0)\exp(tC(M)).\tag{2.18}$$

Then, performing D^3 to (1.2), multiplying by D^3u, it follows from (1.1), (1.12) and (1.13) that

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|D^3u|^2\mathrm{d}x+\mu\int|D^4u|^2\mathrm{d}x+(\lambda+\mu)\int(D^3\operatorname{div}u)^2\mathrm{d}x\\
&\qquad+\int D^3\nabla p\cdot D^3u\mathrm{d}x\\
&\quad=-\int(D^3(\rho\partial_t u)-\rho D^3\partial_t u)D^3u\mathrm{d}x\\
&\qquad-\int(D^3(\rho u\cdot\nabla u)-\rho u\cdot\nabla D^3u)D^3u\mathrm{d}x-\int D^3(\rho\nabla\phi)D^3u\mathrm{d}x\\
&\quad\le C\big(\|\nabla\rho\|_{L^\infty}\|D^2\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|D^3\rho\|_{L^2}\big)\|D^3u\|_{L^2}\\
&\qquad+C\big(\|\nabla u\|_{L^\infty}\|D^3(\rho u)\|_{L^2}+\|\nabla(\rho u)\|_{L^\infty}\|D^3u\|_{L^2}\big)\|D^3u\|_{L^2}\\
&\qquad+C\big(\|\rho\|_{L^\infty}\|D^4\phi\|_{L^2}+\|\nabla\phi\|_{L^\infty}\|D^3\rho\|_{L^2}\big)\|D^3u\|_{L^2}\\
&\quad\le C(M)+C(M)(\|D^2\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\\
&\quad\le C(M)+\|D^2\partial_t u\|_{L^2}^2.
\end{aligned}\tag{2.19}$$

Now, it turns to the reaction fraction Y. The argument is analogous to that in (2.18), we apply D^2 to (1.3) and test by $D^2\partial_t Y$, then we derive

$$
\begin{aligned}
&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (D^3Y)^2\mathrm{d}x + \int \rho(D^2\partial_t Y)^2\mathrm{d}x \\
&\quad= -\int D^2(k\rho Y)\cdot D^2\partial_t Y\,\mathrm{d}x - \int D^2(\rho u\cdot\nabla Y)\cdot D^2\partial_t Y\,\mathrm{d}x \\
&\qquad - \int \big(D^2(\rho\partial_t Y) - \rho D^2\partial_t Y\big)D^2\partial_t Y\,\mathrm{d}x \\
&\quad\le C\|D^2(\rho Y)\|_{L^2}\|D^2\partial_t Y\|_{L^2} \\
&\qquad + C\big(\|\rho u\|_{L^\infty}\|D^3Y\|_{L^2} + \|\nabla Y\|_{L^\infty}\|D^2(\rho u)\|_{L^2}\big)\|D^2\partial_t Y\|_{L^2} \\
&\qquad + C\big(\|\nabla\rho\|_{L^\infty}\|D\partial_t Y\|_{L^2} + \|\partial_t Y\|_{L^\infty}\|D^2\rho\|_{L^2}\big)\|D^2\partial_t Y\|_{L^2} \\
&\quad\le C(M)\|D^2\partial_t Y\|_{L^2} + C(M)(\|D\partial_t Y\|_{L^2} + \|\partial_t Y\|_{L^\infty})\|D^2\partial_t Y\|_{L^2} \\
&\quad\le C(M)\|D^2\partial_t Y\|_{L^2} + C(M)\big(\|\partial_t Y\|_{L^2}^{\frac12}\|D^2\partial_t Y\|_{L^2}^{\frac12} + \|\partial_t Y\|_{L^2} \\
&\qquad + \|\partial_t Y\|_{L^2}^{\frac14}\|D^2\partial_t Y\|_{L^2}^{\frac34}\big)\|D^2\partial_t Y\|_{L^2} \\
&\quad\le C(M)\|D^2\partial_t Y\|_{L^2} + C(M)\big(\|D^2\partial_t Y\|_{L^2}^{\frac12} + \|D^2\partial_t Y\|_{L^2}^{\frac34}\big)\|D^2\partial_t Y\|_{L^2} \\
&\quad\le \frac12\int \rho|D^2\partial_t Y|^2\mathrm{d}x + C(M),
\end{aligned}
$$

which gives

$$
\int_0^t\int |D^2\partial_t Y|^2\mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.20}
$$

Similar to (2.19), performing D^3 to (1.3) and multiplying by D^3u yield that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|D^3Y|^2\mathrm{d}x + \varepsilon\int |D^4Y|^2\mathrm{d}x \\
&\quad= -\int \big(D^3(\rho\partial_t Y) - \rho D^3\partial_t Y\big)D^3Y\,\mathrm{d}x \\
&\qquad - \int \big(D^3(\rho u\cdot\nabla Y) - \rho u\cdot\nabla D^3Y\big)D^3Y\,\mathrm{d}x \\
&\qquad - k\int D^3(\rho Y)\cdot D^3Y\,\mathrm{d}x \\
&\quad\le C\big(\|\nabla\rho\|_{L^\infty}\|D^2\partial_t Y\|_{L^2} + \|\partial_t Y\|_{L^\infty}\|D^3\rho\|_{L^2}\big)\|D^3Y\|_{L^2} \\
&\qquad + C\big(\|\nabla(\rho u)\|_{L^\infty}\|D^3Y\|_{L^2} + \|\nabla Y\|_{L^\infty}\|D^3(\rho u)\|_{L^2}\big)\|D^3Y\|_{L^2} \\
&\qquad + C\|D^3(\rho Y)\|_{L^2}\|D^3Y\|_{L^2} \\
&\quad\le C(M) + C(M)\big(\|D^2\partial_t Y\|_{L^2} + \|\partial_t Y\|_{L^\infty}\big) \\
&\quad\le C(M) + \|D^2\partial_t Y\|_{L^2}^2.
\end{aligned} \tag{2.21}
$$

Summing up (2.15), (2.19) and (2.21), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{1}{\gamma p}(D^3p)^2+\rho|D^3u|^2+\rho|D^3Y|^2\Big)\mathrm{d}x+\mu\int|D^4u|^2\mathrm{d}x\\
&\quad+(\lambda+\mu)\int(D^3\operatorname{div}u)^2\mathrm{d}x+\varepsilon\int(D^4Y)^2\mathrm{d}x\\
&\quad+\int\big(D^3pD^3\operatorname{div}u+D^3\nabla pD^3u\big)\mathrm{d}x\\
&\le C(M)+\|D^2\partial_tu\|_{L^2}^2+\|D^2\partial_tY\|_{L^2}^2.
\end{aligned}\tag{2.22}$$

Noting that the last term of the left-hand side of (2.22) is zero, using (2.18) and (2.20), we have

$$\|D^3(p,u,Y)(\cdot,t)\|_{L^2}\le C_0(M_0)\exp(tC(M)).\tag{2.23}$$

On the other hand, from (1.2), it can be easily be shown that

$$\begin{aligned}
\|\partial_tu\|_{L^2}&=\Big\|\frac{1}{\rho}\big(-\rho\nabla\phi+\mu\Delta u+(\lambda+\mu)\nabla\operatorname{div}u-\nabla p-\rho u\cdot\nabla u\big)\Big\|_{L^2}\\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned}\tag{2.24}$$

According to the estimate in [26],

$$\|D^3\rho\|_{L^2}\le C(1+\|p\|_{L^\infty})^3\|f\|_{W^{3,\infty}(I)}\|D^3p\|_{L^2}\tag{2.25}$$

with $\rho=f(p):=(\frac{p}{a})^{\frac{1}{\gamma}}$, and

$$I\subset\Big(\frac{1}{C_0(M_0)}\exp(-tC(M)),C_0(M_0)\exp(tC(M))\Big),$$

it follows that

$$\|D^3\rho\|_{L^2}\le C_0(M_0)\exp(tC(M)).\tag{2.26}$$

Using a similar argument, we have

$$\|\partial_tY\|_{L^2}\le C_0(M_0)\exp(tC(M)).\tag{2.27}$$

Combining (2.4), (2.5), (2.6), (2.23), (2.24), (2.26) and (2.27), we conclude that (1.10) holds true.

This completes the proof. ∎

3. Data availability

Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

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