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Generalized fractional integral operators on variable exponent Morrey type spaces over metric measure spaces

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Abstract. We prove the boundedness of generalized fractional integral operators $I_{\rho,\tau}$ on variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$ over non-doubling metric measure spaces X, where both $\rho(x,r)$ and $\omega(x,r)$ depend on $x \in X$. Our result extends the recent results by the authors.

1. Introduction

Sobolev inequalities for Riesz potentials were studied on Morrey spaces in [1], on generalized Morrey spaces in [24], on $L^{p(\cdot)}$ in [7,10,11], on variable exponent Morrey spaces in [2,12,19–21] and on local Morrey type spaces in [5], where Morrey spaces were introduced by Morrey [23] in 1938. See also [8,18,31,36].

Let G be a bounded open set in $\mathbf{R}^N$. For $f \in L^1(G)$ and a function $\rho \in (\rho)$ below, we define the generalized Riesz potential $I_\rho f$ by

$$I_\rho f(x) = \int_G \frac{\rho(x,|x-y|)}{|x-y|^N} f(y)\,dy.$$

If $\rho(x,r) = r^\alpha$ with $0 < \alpha < N$, then $I_\rho f$ is the usual Riesz potential $I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\,dy$. When $\rho(x,r) = \rho(r)$, $I_\rho f$ was studied in [13,25]. In the previous paper [28], we established Sobolev-type inequalities for $I_\rho f$ of functions in variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot),\omega}(G)$, as an extension of [22,30].

In the present paper, we work in a metric measure space $X = (X,d,\mu)$, where X is a bounded set, d is a metric on X and μ is a nonnegative Borel outer measure on X which is finite in every bounded set. For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the open ball in X centered at x with radius r and $d_X = \sup\{d(x,y) : x,y \in X\}$. We assume that $d_X < \infty$, $\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ has a so-called doubling condition.

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Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x,2r)) \le c_0\mu(B(x,r))$ for all $x \in \mathrm{supp}(\mu)(= X)$ and $r > 0$ (see [3]). Otherwise μ is said to be non-doubling. In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. In [29], we established Sobolev-type inequalities for $I_{\alpha,\tau} f$ of functions in Morrey type spaces $\mathcal{L}^{p,\omega,\theta_1}(X)$, where

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y)$$

in the constant exponent case.

Let us consider the family (ρ) of all functions ρ satisfying the following conditions: $\rho : X \times (0,\infty) \to (0,\infty)$ is a measurable function such that there exist constants $0 < k_1 < k_2$ and $C_\rho > 0$ such that

$$\sup_{r/2 \le s \le r} \rho(x,s) \le C_\rho \int_{k_1 r}^{k_2 r} \rho(x,s)\, \frac{ds}{s} \tag{1.1}$$

for all $r > 0$ and there exists a constant $C > 0$ such that

$$\int_0^{\max\{1,k_2\} d_X} \rho(x,s)\, \frac{ds}{s} \le C \tag{1.2}$$

for all $x \in X$. We do not assume the doubling condition on ρ. For $\tau \ge 1$ and a function $\rho \in (\rho)$, we define the generalized Riesz potential $I_{\rho,\tau} f$ for a locally integrable function f on X by

$$I_{\rho,\tau} f(x) = \int_X \frac{\rho(x,d(x,y)) f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y).$$

The operator $I_{\rho,\tau}$ is also called the generalized fractional integral operator. In the doubling metric measure setting,

$$I_\eta f(x) = \int_X \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\eta}}\, d\mu(y)$$

was studied in [9]. A variety of examples of ρ satisfy (1.1) ([37]).

- Let $\rho(x,r) = \mu(B(x,\tau r))^\eta$ for some $\eta \in (0,1)$, $\tau \ge 1$ and $0 < r < d_X$. Then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$. If μ satisfies the upper Ahlfors condition $\mu(B(x,r)) \le C r^Q$ $(x \in X, r > 0)$, then ρ satisfies (1.2). When μ is doubling, $I_{\rho,\tau} f(x) = I_\eta f(x)$.
- If ρ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that $C^{-1} \le \frac{\rho(x,r)}{\rho(x,s)} \le C$ for $x \in X$ and $1/2 \le r/s \le 2$, then ρ satisfies (1.1) whenever $2k_1 = k_2$.

- If ρ is increasing in the second variable, then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$. If $\alpha(\cdot)$ is a measurable function on X such that $\alpha^- := \inf_{x \in X} \alpha(x) > 0$ and $\rho(x, r) = r^{\alpha(x)}$, then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$.
- Let $x_0 \in X$. If $\rho(x, r) = (1 + d(x_0, x)/r)r^{\alpha}$ ($\alpha > 0$), then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$.
- If A is a positive measurable function on X, $\rho(x, r) = A(x)r^{\alpha}$ ($\alpha > 0$) for $0 < r < 1$ and $\rho(x, r)$ is $A(x)e^{-(r-1)}$ for $1 \le r < \infty$, then ρ satisfies (1.1) with $k_1 = 1/4$ and $k_2 = 1/2$.

In the present paper, we extend [28, Theorem 3.1] in the Euclidean setting to the non-doubling metric measure setting. In fact, we show that $I_{\rho,\tau}$ is bounded from variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)$ to Musielak–Orlicz–Morrey type spaces $\mathcal{L}^{\Psi,\omega,\theta_2}(X)$ over non-doubling metric measure spaces X (Theorem 5.1), as an extension of [28, 30] in the Euclidean case and [29] in the constant exponent case. See Sections 2 and 5 for the definitions. Theorem 5.1 is obtained applying Hedberg's trick [15] by the use of the (modified) Hardy–Littlewood maximal operator M_λ (Theorem 3.4). Theorem 3.4 extends [29, Theorem 2.4].

Throughout the paper, we let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on $a, b, \dots$ only. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

2. Variable exponent Morrey type spaces

Let us consider a real valued measurable function $p(\cdot)$ on X such that

(P1) $1 < p^- := \inf_{x \in X} p(x) \le \sup_{x \in X} p(x) =: p^+ < \infty$,

(P2) $p(\cdot)$ is log-Hölder continuous on X, namely

$$|p(x) - p(y)| \le \frac{C_p}{\log(e + 1/d(x, y))} \quad (x, y \in X)$$

with a constant $C_p \ge 0$.

We also consider a weight function $\omega(x, r) : X \times (0, \infty) \to (0, \infty)$ satisfying the following conditions:

(ω0) $\omega(\,\cdot\,, r)$ is measurable on X for each $r > 0$ and $\omega(x, \,\cdot\,)$ is continuous on $(0, \infty)$ for each $x \in X$;

(ω1) $r \mapsto \omega(x,r)$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $\tilde{c}_1 \ge 1$ such that

$$\omega(x,r_1) \le \tilde{c}_1 \omega(x,r_2)$$

for all $x \in X$ whenever $0 < r_1 < r_2 < \infty$;

(ω2) there exists a constant $\tilde{c}_2 > 1$ such that

$$\tilde{c}_2^{-1}\omega(x,r) \le \omega(x,2r) \le \tilde{c}_2 \omega(x,r)$$

for all $x \in X$ whenever $r > 0$;

(ω3) there exist constants $\omega_0 > 0$ and $\tilde{c}_3 \ge 1$ such that

$$\tilde{c}_3^{-1} r^{\omega_0} \le \omega(x,r) \le \tilde{c}_3$$

for all $x \in X$ and $0 < r \le 2d_X$.

Example 2.1. Let $\sigma(\cdot)$ and $\beta(\cdot)$ be measurable functions on X such that

$$0 < \sigma^- := \inf_{x\in X} \sigma(x) \le \sup_{x\in X} \sigma(x) =: \sigma^+ \le \omega_0$$

and $-c(\omega_0 - \sigma(x)) \le \beta(x) \le c$ for all $x \in X$ and some constant $c > 0$. Then

$$\omega(x,r) = r^{\sigma(x)}(\log(e+1/r))^{\beta(x)}$$

satisfies (ω0), (ω1), (ω2) and (ω3).

Recall that f is a locally integrable function on X if f is an integrable function on all balls B in X. Let $\theta \ge 1$. Given $p(x)$ and $\omega(x,r)$ as above, we define the $\mathcal{L}^{p(\cdot),\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta}(X)} = \inf\left\{\lambda > 0; \sup_{x\in X}\left(\int_0^{2d_X} \frac{\omega(x,r)}{\mu(B(x,\theta r))} \cdot \left(\int_{B(x,r)} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} d\mu(y)\right)\frac{dr}{r}\right) \le 1\right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$. The space $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$ is referred to as a variable exponent Morrey type space. Here, note that $2d_X$ can be replaced by κd_X with $\kappa > 1$. In case $p(x) \equiv p$, $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$ is denoted by $\mathcal{L}^{p,\omega,\theta}(X)$. See [29, Example 4.1] for an example of $\mathcal{L}^{p(\cdot),\omega,\theta_1}(X) \ne \{0\}$. In [32], Rafeiro and Samko introduced variable exponet Morrey type spaces $M^{p(\cdot),q(\cdot)}_{\omega(\cdot,\cdot)}(\mathbf{R}^N)$ in terms of norms instead of modular in the Euclidean setting. For Morrey type spaces related to our spaces, we refer to the survey [31] and e.g. [4–6, 33] in the Euclidean case.

3. Maximal operator

In this section, we show the boundedness of M_λ on $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$.

For a locally integrable function f on X and $\lambda \ge 1$, the (modified) Hardy–Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

For $\lambda \ge 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C > 0$ such that

$$\mu\big(\{x \in X : M_\lambda f(x) > k\}\big) \le \frac{C}{k} \int_X |f(y)|\, d\mu(y) \tag{3.1}$$

for all measurable functions $f \in L^1(X)$ and $k > 0$. In (3.1), we cannot reduce the number λ any more ([38]). Here, note that X satisfies $(M\lambda)$ for any $\lambda > 0$ if μ satisfies the doubling condition ([16]). For condition $(M\lambda)$, see [26,27,34,39], [14, Appendix] and [35, Section 5].

We know the following result. We refer to [33, Corollary 3.11].

Lemma 3.1 ([29, Theorem 2.4]). *Let $1 \le \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$. Assume that X satisfies $(M\lambda)$. Further suppose*

$(\omega 1')$ *$r \mapsto r^{-\varepsilon_1}\omega(x,r)$ is uniformly almost increasing in $(0, d_X]$ for some $\varepsilon_1 > 0$.*

If $p > 1$, then there is a constant $C > 0$ such that

$$\|M_\lambda f\|_{\mathcal{L}^{p,\omega,\theta_2}(X)} \le C\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Remark 3.2. Note that $(\omega 1')$ implies $(\omega 1)$. Letting $\omega(x,r) = r^{\sigma(x)}(\log(e+1/r))^{\beta(x)}$ be as in Example 2.1, then note that $(\omega 1')$ holds for $0 < \varepsilon_1 < \sigma^-$.

Lemma 3.3. *Let $1 \le \theta < \lambda$. Set*

$$I(f;x,r,\lambda) = \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,r)} f(y)\, d\mu(y)$$

and

$$J(f;x,r,\lambda) = \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,r)} f(y)^{p(y)/p^-}\, d\mu(y)$$

for $x \in X$ and $0 < r \le d_X$. Then, given $L \ge 1$, there exist constants $C = C(L) \ge 1$ such that

$$I(f;x,r,\lambda)^{p(x)/p^-} \le CJ(f;x,r,\lambda)$$

for all $x \in X$, $0 < r \le d_X$ and for all nonnegative measurable functions f on X such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in X$ and

$$\sup_{z\in X}\left(\int_0^{2d_X} \frac{\omega(z,t)}{\mu(B(z,\theta t))}\left(\int_{B(z,t)} f(y)^{p(y)}\,d\mu(y)\right)\frac{dt}{t}\right) \le L. \tag{3.2}$$

Proof. Given f as in the statement of the lemma, $x \in X$ and $0 < r < d_X$, set $I = I(f; x, r, \lambda)$ and $J = J(f; x, r, \lambda)$. Taking f, we see from (3.2) that

$$\begin{aligned}\frac{\omega(x,r)}{\mu(B(x,\lambda r))}&\int_{B(x,r)} f(y)^{p(y)}\,d\mu(y)\\ &\le C\int_r^{\lambda r/\theta} \frac{\omega(x,t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)} f(y)^{p(y)}\,d\mu(y)\right)\frac{dt}{t} \le C_1 L.\end{aligned}$$

Hence

$$J \le C_1^{1/p^-}\omega(x,r)^{-1/p^-}L^{1/p^-}. \tag{3.3}$$

Since $f(y)^{p(y)/p^-} \ge f(y)$ for all $y \in X$, it follows that $I \le J$. Thus, if $J \le 1$, then

$$I^{p(x)/p^-} = I \cdot I^{p(x)/p^- -1} \le J.$$

Next, suppose $J > 1$. Set $K = J^{p^-/p(x)}$. We obtain

$$\begin{aligned}\int_{B(x,r)} f(y)\,d\mu(y) &\le \int_{B(x,r)\cap\{y\in X: f(y)\le K\}} f(y)\,d\mu(y)\\ &\quad + \int_{B(x,r)\cap\{y\in X: f(y)>K\}} f(y)\,d\mu(y)\\ &\le K\mu(B(x,r)) + K\int_{B(x,r)} \frac{f(y)^{p(y)/p^-}}{K^{p(y)/p^-}}\,d\mu(y).\end{aligned}$$

By (3.3) and (ω3),

$$K^{p^-} \le J^{p^-} \le C_1\omega(x,r)^{-1}L \le C_1\tilde{c}_3 L r^{-\omega_0}$$

or $r \le \gamma K^{-p^-/\omega_0}$ with $\gamma = (C_1\tilde{c}_3 L)^{1/\omega_0}$. Thus, if $d(x,y) \le r$, then $d(x,y) \le \gamma K^{-p^-/\omega_0}$. Hence, by (P2) there is $\beta \ge 1$, independent of f, x, r, such that

$$K^{p(x)/p^-} \le \beta K^{p(y)/p^-} \quad \text{for all } y \in B(x,r).$$

Thus, we have

$$\begin{aligned}\int_{B(x,r)} f(y)\,d\mu(y) &\le K\mu(B(x,r)) + \frac{\beta K}{K^{p(x)/p^-}}\int_{B(x,r)} f(y)^{p(y)/p^-}\,d\mu(y)\\ &= K\mu(B(x,r)) + \beta K\mu(B(x,\lambda r))\frac{J}{K^{p(x)/p^-}}\\ &\le K\mu(B(x,\lambda r))(1+\beta),\end{aligned}$$

so that

$$I^{p(x)/p^-} \le CJ,$$

as required. ■

Now, we are ready to prove the boundedness of M_λ on $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$, as an extension of [29, Theorem 2.4] in the constant exponent case.

Theorem 3.4. *Let* $1 \le \theta_1 < \theta_2$ *and* $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$. *Assume that* X *satisfies* $(M\lambda)$ *and* $(\omega 1')$ *holds. Then there exists a constant* $C > 0$ *such that*

$$\|M_\lambda f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_2}(X)} \le C\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot),\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)} \le 1$. Let $f_1 = f\chi_{\{x\in X: f(x)\ge 1\}}$ and $f_2 = f - f_1$, where χ_E is the characteristic function of E. Set $g_1(y) = f_1(y)^{p(y)/p^-}$. Applying Lemma 3.3 to f_1 and $L = 1$, there exists a constant $C \ge 1$ such that

$$\{M_\lambda f_1(x)\}^{p(x)/p^-} \le CM_\lambda g_1(x) \tag{3.4}$$

for all $x \in X$.

Since $M_\lambda f_2 \le 1$, we have

$$\{M_\lambda f_2(x)\}^{p(x)} \le C \tag{3.5}$$

for all $x \in X$ with a constant $C > 0$ independent of f. Here, note from $(\omega 1')$ and $(\omega 3)$ that there exists a constant $C > 0$ such that

$$\int_0^{2d_X} \omega(z,r)\,\frac{dr}{r} = \int_0^{2d_X} r^{-\varepsilon_1}\omega(z,r)\cdot r^{\varepsilon_1}\,\frac{dr}{r} \le C\int_0^{2d_X} r^{\varepsilon_1}\,\frac{dr}{r} \le C \tag{3.6}$$

for all $z \in X$. Hence, we obtain by (3.4), (3.5), (3.6) and Lemma 3.1

$$\begin{aligned}
&\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\{M_\lambda f(x)\}^{p(x)}\,d\mu(x)\right)\frac{dr}{r}\\
&\quad\le C\left\{\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\{M_\lambda f_1(x)\}^{p(x)}\,d\mu(x)\right)\frac{dr}{r}\right.\\
&\qquad\left.+\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\{M_\lambda f_2(x)\}^{p(x)}\,d\mu(x)\right)\frac{dr}{r}\right\}\\
&\quad\le C\left\{\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)}\{M_\lambda g_1(x)\}^{p^-}\,d\mu(x)\right)\frac{dr}{r}\right.\\
&\qquad\left.+\int_0^{2d_X} \omega(z,r)\,\frac{dr}{r}\right\} \le C
\end{aligned}$$

for all $z \in X$, as required. ■

4. Lemmas

Set

$$\widetilde{\rho}(x,r) = \int_0^r \rho(x,s)\,\frac{ds}{s}$$

for $r > 0$.

We give an estimate inside a ball.

Lemma 4.1. *Let $1 \le \lambda < \tau$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \le C\widetilde{\rho}(x,k_2\delta)M_\lambda f(x)$$

for all $x \in X$, $0 < \delta < d_X/2$ and nonnegative $f \in L^1_{\mathrm{loc}}(X)$.

Proof. Let $f \in L^1_{\mathrm{loc}}(X)$ be a nonnegative measurable function on X. Let $x \in X$ and $0 < \delta < d_X/2$. Take $\gamma \in \mathbf{R}$ such that $1 < \gamma \le \min\{2, \tau/\lambda\}$. If $y \in B(x,\gamma^{-j+1}\delta) \setminus B(x,\gamma^{-j}\delta)$ for a positive integer j, then a geometric observation and (1.1) show

$$\begin{aligned}
\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} &\le \frac{1}{\mu(B(x,\tau\gamma^{-j}\delta))} \sup_{\gamma^{-j}\delta \le s \le \gamma^{-j+1}\delta} \rho(x,s) \\
&\le \frac{1}{\mu(B(x,\lambda\gamma^{-j+1}\delta))} \sup_{\gamma^{-j+1}\delta/2 \le s \le \gamma^{-j+1}\delta} \rho(x,s) \\
&\le \frac{C}{\mu(B(x,\lambda\gamma^{-j+1}\delta))} \int_{\gamma^{-j+1}k_1\delta}^{\gamma^{-j+1}k_2\delta} \rho(x,s)\,\frac{ds}{s}. \qquad (4.1)
\end{aligned}$$

Let j_1 be the smallest integer such that $k_1/k_2 \ge \gamma^{-j_1}$. Then note from (4.1) that

$$\begin{aligned}
&\int_{B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&= \sum_{j=1}^{\infty} \int_{B(x,\gamma^{-j+1}\delta)\setminus B(x,\gamma^{-j}\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&\le C\sum_{j=1}^{\infty} \int_{\gamma^{-j+1}k_1\delta}^{\gamma^{-j+1}k_2\delta} \rho(x,s)\,\frac{ds}{s} \times \frac{1}{\mu(B(x,\lambda\gamma^{-j+1}\delta))} \int_{B(x,\gamma^{-j+1}\delta)} f(y)\,d\mu(y) \\
&\le C\sum_{j=1}^{\infty} \int_{\gamma^{-j-j_1+1}k_2\delta}^{\gamma^{-j+1}k_2\delta} \rho(x,s)\,\frac{ds}{s} \times M_\lambda f(x) \\
&\le C\int_0^{k_2\delta} \rho(x,s)\,\frac{ds}{s} \times M_\lambda f(x) = C\widetilde{\rho}(x,k_2\delta)M_\lambda f(x),
\end{aligned}$$

as required. ∎

We consider the following condition:

$(\rho\omega)$ there exist constants $\varepsilon_2 > 0$ and $A \geq 1$ such that

$$r_2^{\varepsilon_2} \rho(x, r_2)\omega(x, r_2)^{-1/p(x)} \leq A r_1^{\varepsilon_2} \rho(x, r_1)\omega(x, r_1)^{-1/p(x)}$$

for all $x \in X$ whenever $0 < r_1 < r_2 < \max\{1, 2k_2\}d_X$.

Remark 4.2. We have by $(\rho\omega)$

$$t^{\varepsilon_2} \rho(x, t)\omega(x, t)^{-1/p(x)} \leq A s^{\varepsilon_2} \rho(x, s)\omega(x, s)^{-1/p(x)}$$

for all $x \in X$ whenever $0 < t/2 \leq s \leq t \leq \max\{1, 2k_2\}d_X$. Then, by $(\omega 1)$ and $(\omega 2)$, we obtain

$$\rho(x, t) \leq C\rho(x, s)$$

for all $x \in X$ whenever $0 < t/2 \leq s \leq t \leq \max\{1, 2k_2\}d_X$.

Remark 4.3. Let $\alpha(\cdot)$ be a measurable function on X such that $\alpha^- > 0$ and set $\rho(x, r) = r^{\alpha(x)}$. Let

$$\omega(x, r) = r^{\sigma(x)}(\log(e + 1/r))^{\beta(x)}$$

be as in Example 2.1. Condition $(\rho\omega)$ is satisfied if

$$\inf_{x \in X} \left(\frac{\sigma(x)}{p(x)} - \alpha(x)\right) > 0.$$

The next lemma concerns an estimate outside a ball.

Lemma 4.4. *Let $1 \leq \theta < \tau$. Assume that $(\rho\omega)$ holds. Then there exists a constant $C > 0$ such that*

$$\int_{X \setminus B(x,\delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C\rho(x, \min\{1, k_1\}\delta)\omega(x, \min\{1, k_1\}\delta)^{-1/p(x)}$$

for all $x \in X$, $0 < \delta < d_X/2$ and nonnegative $f \in \mathcal{L}^{p(\cdot),\omega,\theta}(X)$ with

$$\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta}(X)} \leq 1.$$

Proof. Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta}(X)} \leq 1$. Let $x \in X$ and $0 < \delta < d_X/2$. In view of $(\omega 3)$, we have

$$d(x, y) \leq \tilde{c}_3^{1/\omega_0} \omega(x, d(x, y))^{1/\omega_0} \tag{4.2}$$

for all $x, y \in X$. It follows from (ω3), (P2) and (4.2) that

$$\begin{aligned} \omega(x,d(x,y))^{(p(y)-p(x))/p(x)} &\le C\omega(x,d(x,y))^{-\frac{1}{p(x)}\frac{C_p}{\log(e+1/d(x,y))}} \\ &\le C\omega(x,d(x,y))^{-\frac{1}{p(x)}\frac{C_p}{\log\left(e+\tilde{c}_3^{-1/\omega_0}\omega(x,d(x,y))^{-1/\omega_0}\right)}} \\ &\le C. \end{aligned} \tag{4.3}$$

Hence, by (P1), we find

$$\begin{aligned} &\int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &= \int_{\{y\in X: f(y)\le\omega(x,d(x,y))^{-1/p(x)}\}\setminus B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\quad + \int_{\{y\in X: f(y)>\omega(x,d(x,y))^{-1/p(x)}\}\setminus B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\le \int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))\omega(x,d(x,y))^{-1/p(x)}}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\quad + \int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,\frac{f(y)^{p(y)-1}}{\omega(x,d(x,y))^{-(p(y)-1)/p(x)}}\,d\mu(y) \\ &\le \int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))\omega(x,d(x,y))^{-1/p(x)}}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\quad + C\int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))\omega(x,d(x,y))^{1-1/p(x)}f(y)^{p(y)}}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &= I_1 + CI_2. \end{aligned}$$

Here we used (4.3) in the last inequality.

Take $\gamma_1 \in \mathbf{R}$ such that $1 < \gamma_1 \le \min\{\tau, 2\}$. Let j_1 be the smallest integer such that $\gamma_1^{j_1}\delta \ge d_X$ and let j_0 be the smallest integer such that $k_2/k_1 \le \gamma_1^{j_0}$. As in the proof of Lemma 4.1, we have by (4.1)

$$\begin{aligned} I_1 &\le \sum_{j=1}^{j_1} \int_{B(x,\gamma_1^j\delta)\setminus B(x,\gamma_1^{j-1}\delta)} \frac{\rho(x,d(x,y))\omega(x,d(x,y))^{-1/p(x)}}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\le C\sum_{j=1}^{j_1} \omega(x,\gamma_1^j\delta)^{-1/p(x)} \int_{\gamma_1^j k_1\delta}^{\gamma_1^j k_2\delta} \rho(x,s)\,\frac{ds}{s} \\ &\le C\sum_{j=1}^{j_1} \int_{\gamma_1^j k_1\delta}^{\gamma_1^j k_2\delta} \rho(x,s)\omega(x,s)^{-1/p(x)}\,\frac{ds}{s} \end{aligned}$$

since we find by (ω1) and (ω2) that

$$\omega(x,\gamma_1^j\delta)^{-1/p(x)} \le C\omega(x,s)^{-1/p(x)}$$

for all $1 \le j \le j_1$ and $\gamma_1^j k_1\delta \le s \le \gamma_1^j k_2\delta$. Further, we see from the definitions of j_0 and j_1 that

$$\begin{aligned} I_1 &\le C\sum_{j=1}^{j_1}\int_{\gamma_1^j k_1\delta}^{\gamma_1^j k_2\delta} \rho(x,s)\omega(x,s)^{-1/p(x)}\,\frac{ds}{s} \\ &\le Cj_0\int_{\gamma_1 k_1\delta}^{\gamma_1^{j_1} k_2\delta} \rho(x,s)\omega(x,s)^{-1/p(x)}\,\frac{ds}{s} \\ &\le C\int_{k_1\delta}^{2k_2 d_X} \rho(x,s)\omega(x,s)^{-1/p(x)}\,\frac{ds}{s}, \end{aligned}$$

so that we have by ($\rho\omega$)

$$\begin{aligned} I_1 &\le C(k_1\delta)^{\varepsilon_2}\rho(x,k_1\delta)\omega(x,k_1\delta)^{-1/p(x)}\int_{k_1\delta}^{2k_2 d_X} s^{-\varepsilon_2}\,\frac{ds}{s} \\ &\le C\rho(x,k_1\delta)\omega(x,k_1\delta)^{-1/p(x)}. \end{aligned}$$

Next, take $\gamma_2 \in \mathbf{R}$ such that $1 < \gamma_2 \le \min\{\tau/\theta, 2\}$. Let j_2 be the smallest integer such that $\gamma_2^{j_2/2}\delta \ge d_X$. For I_2, we have by ($\rho\omega$)

$$\begin{aligned} I_2 &\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)}\int_{X\setminus B(x,\delta)} \frac{\omega(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)^{p(y)}\,d\mu(y) \\ &\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)} \\ &\quad \cdot \sum_{j=1}^{j_2}\int_{B(x,\gamma_2^{j/2}\delta)\setminus B(x,\gamma_2^{(j-1)/2}\delta)} \frac{\omega(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)^{p(y)}\,d\mu(y). \end{aligned}$$

Hence, by (ω1),

$$\begin{aligned} I_2 &\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)}\sum_{j=1}^{j_2}\frac{\omega(x,\gamma_2^{j/2}\delta)}{\mu(B(x,\gamma_2^{(j+1)/2}\theta\delta))}\int_{B(x,\gamma_2^{j/2}\delta)} f(y)^{p(y)}\,d\mu(y) \\ &\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)} \\ &\quad \cdot \sum_{j=1}^{j_2}\int_{\gamma_2^{j/2}\delta}^{\gamma_2^{(j+1)/2}\delta} \frac{\omega(x,t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)} f(y)^{p(y)}\,d\mu(y)\right)\frac{dt}{t} \\ &\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)}\int_{\gamma_2^{1/2}\delta}^{\gamma_2 d_X} \frac{\omega(x,t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)} f(y)^{p(y)}\,d\mu(y)\right)\frac{dt}{t} \end{aligned}$$

$$\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)} \int_0^{2d_X} \frac{\omega(x,t)}{\mu(B(x,\theta t))}\left(\int_{B(x,t)} f(y)^{p(y)}\,d\mu(y)\right)\frac{dt}{t}$$
$$\le C\rho(x,\delta)\omega(x,\delta)^{-1/p(x)}.$$

Thus, we obtain the required result by ($\rho\omega$). ∎

5. Sobolev-type inequality

To state Theorem 5.1 below we give the assumptions for the function appearing in the theorem. We consider a function

$$\Psi(x,t) : X \times [0,\infty) \to [0,\infty)$$

which satisfies

(Ψ1) $\Psi(\cdot,t)$ is measurable on X for each $t \ge 0$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Ψ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \Psi(x,1) \le A_1 \quad \text{for all } x \in X;$$

(Ψ3) $t \mapsto \Psi(x,t)/t$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\Psi(x,t_1)/t_1 \le A_2\Psi(x,t_2)/t_2 \quad \text{for all } x \in X \text{ whenever } 0 < t_1 < t_2;$$

(Ψ4) there exists a constant $A_3 \ge 1$ such that

$$\Psi\big(x, t\tilde{\rho}(x,\omega^{-1}(x,t^{-p(x)}))\big) \le A_3 t^{p(x)}$$

for all $x \in X$ and $t \ge 1$.

We write

$$\bar{\psi}(x,t) = \sup_{0<s\le t} \frac{\Psi(x,s)}{s}$$

and

$$\overline{\Psi}(x,t) = \int_0^t \bar{\psi}(x,r)\,dr$$

for $x \in X$ and $t \ge 0$. Then $\overline{\Psi}(x,\cdot)$ is convex and

$$\Psi(x,t/2) \le \overline{\Psi}(x,t) \le A_2\Psi(x,t) \tag{5.1}$$

for all $x \in X$ and $t \ge 0$.

Let $\theta \geq 1$. Given $\Psi(x,t)$ and $\omega(x,r)$ as above, we define the $\mathcal{L}^{\Psi,\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{\Psi,\omega,\theta}(X)} = \inf\left\{\lambda > 0; \sup_{x\in X}\left(\int_0^{2d_X} \frac{\omega(x,r)}{\mu(B(x,\theta r))} \cdot \left(\int_{B(x,r)} \overline{\Psi}(x,|f(y)|/\lambda)\,d\mu(y)\right)\frac{dr}{r}\right) \leq 1\right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{\Psi,\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{\Psi,\omega,\theta}(X)$.

Note from Remark 4.2 that

$$\rho(x,t) \leq C\int_{t/2}^{t} \rho(x,s)\,\frac{ds}{s} \leq C\widetilde{\rho}(x,t) \tag{5.2}$$

for all $x \in X$ and all $0 < t \leq \max\{1, 2k_2\}d_X$.

As an extension of [28, Theorem 3.1] and [29,30], we give a Sobolev-type inequality for $I_{\rho,\tau}f$ of functions in $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$.

Theorem 5.1. *Let* $1 \leq \theta_1 < \theta_2$ *and* $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda < \tau$. *Assume that* X *satisfies* ($M\lambda$) *and* ($\omega 1'$) *and* ($\rho\omega$) *hold. Then there exists a constant* $C > 0$ *such that*

$$\|I_{\rho,\tau}f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \leq C\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot),\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X such that

$$\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)} \leq 1.$$

Let $x \in X$ and $0 < \delta < d_X/2$. By Lemmas 4.1 and 4.4 and (5.2), we find

$$\begin{aligned}
I_{\rho,\tau}f(x) &= \int_{B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&\quad + \int_{X\setminus B(x,\delta)} \frac{\rho(x,d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&\leq C\left\{\widetilde{\rho}(x,k_2\delta)M_\lambda f(x) + \rho(x,\min\{1,k_1\}\delta)\omega(x,\min\{1,k_1\}\delta)^{-1/p(x)}\right\} \\
&\leq C\left\{\widetilde{\rho}(x,k_2\delta)M_\lambda f(x) + \widetilde{\rho}(x,\min\{1,k_1\}\delta)\omega(x,\min\{1,k_1\}\delta)^{-1/p(x)}\right\}.
\end{aligned}$$

Set

$$\omega^{-1}(x,r) = \sup\{t > 0; \omega(x,t) < r\}.$$

Here, note from [17, Lemma 5.1 (3)] that

$$\omega(x,\omega^{-1}(x,r)) = r. \tag{5.3}$$

If $k_2^{-1}\omega^{-1}(x,\{M_\lambda f(x)\}^{-p(x)}) \ge d_X/2$, then, taking $\delta = d_X/2$, we have

$$I_{\rho,\tau} f(x) \le C\widetilde{\rho}(x, \max\{1,k_2\}d_X/2) \le C$$

by (5.3), (ω1) and (ω3). If $k_2^{-1}\omega^{-1}(x,\{M_\lambda f(x)\}^{-p(x)}) < d_X/2$, then take

$$\delta = k_2^{-1}\omega^{-1}\big(x,\{M_\lambda f(x)\}^{-p(x)}\big).$$

Then we have

$$I_{\rho,\tau} f(x) \le C M_\lambda f(x)\widetilde{\rho}\big(x,\omega^{-1}(x,\{M_\lambda f(x)\}^{-p(x)})\big)$$

by (5.3), (ω1) and (ω2). Thus, we have

$$I_{\rho,\tau} f(x) \le C_1 \max\Big\{M_\lambda f(x)\widetilde{\rho}\big(x,\omega^{-1}(x,\{M_\lambda f(x)\}^{-p(x)})\big), 1\Big\}.$$

In view of (Ψ4), we obtain

$$\begin{aligned}\Psi(x, I_{\rho,\tau} f(x)/C_1) &\le C\Big\{\Psi\big(x, M_\lambda f(x)\widetilde{\rho}\big(x,\omega^{-1}(x,\{M_\lambda f(x)\}^{-p(x)})\big)\big) + 1\Big\}\\ &\le C\big[\{M_\lambda f(x)\}^{p(x)} + 1\big].\end{aligned}$$

Hence, we see from Theorem 3.4 and (3.6)

$$\begin{aligned}&\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \Psi(x, I_{\rho,\tau} f(x)/C_1)\,d\mu(x)\right)\frac{dr}{r}\\ &\le C\left\{\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \{M_\lambda f(x)\}^{p(x)}\,d\mu(x)\right)\frac{dr}{r}\right.\\ &\qquad\left. + \int_0^{2d_X} \omega(z,r)\,\frac{dr}{r}\right\}\\ &\le C_2\end{aligned}$$

for all $z \in X$, so that by the convexity of $\overline{\Psi}$ and (5.1)

$$\int_0^{2d_X} \frac{\omega(z,r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \overline{\Psi}\big(x, I_{\rho,\tau} f(x)/(A_2 C_1 C_2)\big)\,d\mu(x)\right)\frac{dr}{r} \le 1$$

for all $z \in X$. This completes the proof. ■

Example 5.2. Let $\alpha(\cdot)$ and $\gamma(\cdot)$ be measurable functions on X such that $\alpha^- > 0$ and

$$-\infty < \gamma^- := \inf_{x\in X} \gamma(x) \le \sup_{x\in X} \gamma(x) =: \gamma^+ < \infty.$$

Let

$$\rho(x,r) = r^{\alpha(x)}(\log(e+1/r))^{\gamma(x)}$$

and let

$$\omega(x,r) = r^{\sigma(x)}(\log(e+1/r))^{\beta(x)}$$

be as in Example 2.1, see Remark 3.2. We know that condition $(\rho\omega)$ holds if

$$\inf_{x\in X}(\sigma(x)/p(x) - \alpha(x)) > 0$$

(Remark 4.3). Set

$$\Psi(x,t) = \left\{t(\log(e+t))^{\alpha(x)\beta(x)/\sigma(x)-\gamma(x)}\right\}^{p^*(x)},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$. Then note that $\Psi(x,t)$ satisfies conditions $(\Psi 1)$–$(\Psi 4)$, see [28, Example 4.1].

Applying Theorem 5.1 to special ρ and ω given in Example 5.2, we obtain the following corollaries.

Corollary 5.3. *Let $\rho(x,r)$ and $\omega(x,r)$ be as in Example 5.2. Let $1 \le \theta_1 < \theta_2$ and $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda < \tau$. Assume that X satisfies $(M\lambda)$. Further, suppose $\inf_{x\in X}(\sigma(x)/p(x) - \alpha(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\|I_{\rho,\tau} f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \le C\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot),\omega,\theta_1}(X)$, where $\Psi(x,t) = \{t(\log(e+t))^{\alpha(x)\beta(x)/\sigma(x)-\gamma(x)}\}^{p^*(x)}$ with $p^*(x)$ given by $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$.

Corollary 5.4. *Let X be a doubling metric measure space. Let $\rho(x,r)$ and $\omega(x,r)$ be as in Example 5.2. Suppose $\inf_{x\in X}(\sigma(x)/p(x) - \alpha(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\|I_{\rho,1} f\|_{\mathcal{L}^{\Psi,\omega,1}(X)} \le C\|f\|_{\mathcal{L}^{p(\cdot),\omega,1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot),\omega,1}(X)$, where $\Psi(x,t) = \{t(\log(e+t))^{\alpha(x)\beta(x)/\sigma(x)-\gamma(x)}\}^{p^*(x)}$ with $p^*(x)$ given by $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$.

Remark 5.5. We write that $L_c(t) = \log(c+t)$ for $c > 1$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. We may consider

$$\rho(x,r) = r^{\alpha(x)}\prod_{j=1}^{k}(L_e^{(j)}(1/r))^{\gamma_j(x)}$$

and

$$\omega(x,r) = r^{\sigma(x)}\prod_{j=1}^{k}(L_e^{(j)}(1/r))^{\beta_j(x)}.$$

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