

Generalized fractional integral operators on variable exponent Morrey type spaces over metric measure spaces

Takao Ohno and Tetsu Shimomura

Abstract. We prove the boundedness of generalized fractional integral operators $I_{\rho,\tau}$ on variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$ over non-doubling metric measure spaces X , where both $\rho(x,r)$ and $\omega(x,r)$ depend on $x \in X$. Our result extends the recent results by the authors.

1. Introduction

Sobolev inequalities for Riesz potentials were studied on Morrey spaces in [1], on generalized Morrey spaces in [24], on $L^{p(\cdot)}$ in [7, 10, 11], on variable exponent Morrey spaces in [2, 12, 19–21] and on local Morrey type spaces in [5], where Morrey spaces were introduced by Morrey [23] in 1938. See also [8, 18, 31, 36].

Let G be a bounded open set in \mathbf{R}^N . For $f \in L^1(G)$ and a function $\rho \in (\rho)$ below, we define the generalized Riesz potential $I_\rho f$ by

$$I_\rho f(x) = \int_G \frac{\rho(x, |x-y|)}{|x-y|^N} f(y) dy.$$

If $\rho(x,r) = r^\alpha$ with $0 < \alpha < N$, then $I_\rho f$ is the usual Riesz potential $I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y) dy$. When $\rho(x,r) = \rho(r)$, $I_\rho f$ was studied in [13, 25]. In the previous paper [28], we established Sobolev-type inequalities for $I_\rho f$ of functions in variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot),\omega}(G)$, as an extension of [22, 30].

In the present paper, we work in a metric measure space $X = (X, d, \mu)$, where X is a bounded set, d is a metric on X and μ is a nonnegative Borel outer measure on X which is finite in every bounded set. For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the open ball in X centered at x with radius r and $d_X = \sup\{d(x,y) : x,y \in X\}$. We assume that $d_X < \infty$, $\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ has a so-called doubling condition.

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Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \leq c_0\mu(B(x, r))$ for all $x \in \text{supp}(\mu)(= X)$ and $r > 0$ (see [3]). Otherwise μ is said to be non-doubling. In connection with the $5r$ -covering lemma, the doubling condition had been a key condition in harmonic analysis. In [29], we established Sobolev-type inequalities for $I_{\alpha,\tau}f$ of functions in Morrey type spaces $\mathcal{L}^{p,\omega,\theta_1}(X)$, where

$$I_{\alpha,\tau}f(x) = \int_X \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y)$$

in the constant exponent case.

Let us consider the family (ρ) of all functions ρ satisfying the following conditions: $\rho : X \times (0, \infty) \rightarrow (0, \infty)$ is a measurable function such that there exist constants $0 < k_1 < k_2$ and $C_\rho > 0$ such that

$$\sup_{r/2 \leq s \leq r} \rho(x, s) \leq C_\rho \int_{k_1 r}^{k_2 r} \rho(x, s) \frac{ds}{s} \tag{1.1}$$

for all $r > 0$ and there exists a constant $C > 0$ such that

$$\int_0^{\max\{1, k_2\}d_X} \rho(x, s) \frac{ds}{s} \leq C \tag{1.2}$$

for all $x \in X$. We do not assume the doubling condition on ρ . For $\tau \geq 1$ and a function $\rho \in (\rho)$, we define the generalized Riesz potential $I_{\rho,\tau}f$ for a locally integrable function f on X by

$$I_{\rho,\tau}f(x) = \int_X \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y).$$

The operator $I_{\rho,\tau}$ is also called the generalized fractional integral operator. In the doubling metric measure setting,

$$I_\eta f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\eta}} d\mu(y)$$

was studied in [9]. A variety of examples of ρ satisfy (1.1) ([37]).

- Let $\rho(x, r) = \mu(B(x, \tau r))^\eta$ for some $\eta \in (0, 1)$, $\tau \geq 1$ and $0 < r < d_X$. Then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$. If μ satisfies the upper Ahlfors condition $\mu(B(x, r)) \leq Cr^Q$ ($x \in X, r > 0$), then ρ satisfies (1.2). When μ is doubling, $I_{\rho,\tau}f(x) = I_\eta f(x)$.
- If ρ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that $C^{-1} \leq \frac{\rho(x,r)}{\rho(x,s)} \leq C$ for $x \in X$ and $1/2 \leq r/s \leq 2$, then ρ satisfies (1.1) whenever $2k_1 = k_2$.

- If ρ is increasing in the second variable, then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$. If $\alpha(\cdot)$ is a measurable function on X such that $\alpha^- := \inf_{x \in X} \alpha(x) > 0$ and $\rho(x, r) = r^{\alpha(x)}$, then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$.
- Let $x_0 \in X$. If $\rho(x, r) = (1 + d(x_0, x)/r)r^\alpha$ ($\alpha > 0$), then ρ satisfies (1.1) with $k_1 = 1$ and $k_2 = 2$.
- If A is a positive measurable function on X , $\rho(x, r) = A(x)r^\alpha$ ($\alpha > 0$) for $0 < r < 1$ and $\rho(x, r)$ is $A(x)e^{-(r-1)}$ for $1 \leq r < \infty$, then ρ satisfies (1.1) with $k_1 = 1/4$ and $k_2 = 1/2$.

In the present paper, we extend [28, Theorem 3.1] in the Euclidean setting to the non-doubling metric measure setting. In fact, we show that $I_{\rho, \tau}$ is bounded from variable exponent Morrey type spaces $\mathcal{L}^{p(\cdot), \omega, \theta_1}(X)$ to Musielak–Orlicz–Morrey type spaces $\mathcal{L}^{\Psi, \omega, \theta_2}(X)$ over non-doubling metric measure spaces X (Theorem 5.1), as an extension of [28, 30] in the Euclidean case and [29] in the constant exponent case. See Sections 2 and 5 for the definitions. Theorem 5.1 is obtained applying Hedberg’s trick [15] by the use of the (modified) Hardy–Littlewood maximal operator M_λ (Theorem 3.4). Theorem 3.4 extends [29, Theorem 2.4].

Throughout the paper, we let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on a, b, \dots only. The symbol $g \sim h$ means that $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$.

2. Variable exponent Morrey type spaces

Let us consider a real valued measurable function $p(\cdot)$ on X such that

(P1) $1 < p^- := \inf_{x \in X} p(x) \leq \sup_{x \in X} p(x) =: p^+ < \infty$,

(P2) $p(\cdot)$ is log-Hölder continuous on X , namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(e + 1/d(x, y))} \quad (x, y \in X)$$

with a constant $C_p \geq 0$.

We also consider a weight function $\omega(x, r) : X \times (0, \infty) \rightarrow (0, \infty)$ satisfying the following conditions:

- ($\omega 0$) $\omega(\cdot, r)$ is measurable on X for each $r > 0$ and $\omega(x, \cdot)$ is continuous on $(0, \infty)$ for each $x \in X$;

($\omega 1$) $r \mapsto \omega(x, r)$ is uniformly almost increasing on $(0, \infty)$, namely there exists a constant $\tilde{c}_1 \geq 1$ such that

$$\omega(x, r_1) \leq \tilde{c}_1 \omega(x, r_2)$$

for all $x \in X$ whenever $0 < r_1 < r_2 < \infty$;

($\omega 2$) there exists a constant $\tilde{c}_2 > 1$ such that

$$\tilde{c}_2^{-1} \omega(x, r) \leq \omega(x, 2r) \leq \tilde{c}_2 \omega(x, r)$$

for all $x \in X$ whenever $r > 0$;

($\omega 3$) there exist constants $\omega_0 > 0$ and $\tilde{c}_3 \geq 1$ such that

$$\tilde{c}_3^{-1} r^{\omega_0} \leq \omega(x, r) \leq \tilde{c}_3$$

for all $x \in X$ and $0 < r \leq 2d_X$.

Example 2.1. Let $\sigma(\cdot)$ and $\beta(\cdot)$ be measurable functions on X such that

$$0 < \sigma^- := \inf_{x \in X} \sigma(x) \leq \sup_{x \in X} \sigma(x) =: \sigma^+ \leq \omega_0$$

and $-c(\omega_0 - \sigma(x)) \leq \beta(x) \leq c$ for all $x \in X$ and some constant $c > 0$. Then

$$\omega(x, r) = r^{\sigma(x)} (\log(e + 1/r))^{\beta(x)}$$

satisfies ($\omega 0$), ($\omega 1$), ($\omega 2$) and ($\omega 3$).

Recall that f is a locally integrable function on X if f is an integrable function on all balls B in X . Let $\theta \geq 1$. Given $p(x)$ and $\omega(x, r)$ as above, we define the $\mathcal{L}^{p(\cdot), \omega, \theta}$ norm by

$$\|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta}(X)} = \inf \left\{ \lambda > 0; \sup_{x \in X} \left(\int_0^{2d_X} \frac{\omega(x, r)}{\mu(B(x, \theta r))} \cdot \left(\int_{B(x, r)} \left(\frac{|f(y)|}{\lambda} \right)^{p(y)} d\mu(y) \right) \frac{dr}{r} \right) \leq 1 \right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta}(X)} < \infty$ is denoted by $\mathcal{L}^{p(\cdot), \omega, \theta}(X)$. The space $\mathcal{L}^{p(\cdot), \omega, \theta}(X)$ is referred to as a variable exponent Morrey type space. Here, note that $2d_X$ can be replaced by κd_X with $\kappa > 1$. In case $p(x) \equiv p$, $\mathcal{L}^{p(\cdot), \omega, \theta}(X)$ is denoted by $\mathcal{L}^{p, \omega, \theta}(X)$. See [29, Example 4.1] for an example of $\mathcal{L}^{p(\cdot), \omega, \theta_1}(X) \neq \{0\}$. In [32], Rafeiro and Samko introduced variable exponent Morrey type spaces $M_{\omega(\cdot, \cdot)}^{p(\cdot), q(\cdot)}(\mathbf{R}^N)$ in terms of norms instead of modular in the Euclidean setting. For Morrey type spaces related to our spaces, we refer to the survey [31] and e.g. [4–6, 33] in the Euclidean case.

3. Maximal operator

In this section, we show the boundedness of M_λ on $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$.

For a locally integrable function f on X and $\lambda \geq 1$, the (modified) Hardy–Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} |f(y)| d\mu(y).$$

For $\lambda \geq 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C > 0$ such that

$$\mu(\{x \in X : M_\lambda f(x) > k\}) \leq \frac{C}{k} \int_X |f(y)| d\mu(y) \tag{3.1}$$

for all measurable functions $f \in L^1(X)$ and $k > 0$. In (3.1), we cannot reduce the number λ any more ([38]). Here, note that X satisfies $(M\lambda)$ for any $\lambda > 0$ if μ satisfies the doubling condition ([16]). For condition $(M\lambda)$, see [26,27,34,39], [14, Appendix] and [35, Section 5].

We know the following result. We refer to [33, Corollary 3.11].

Lemma 3.1 ([29, Theorem 2.4]). *Let $1 \leq \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2 + 1)/(\theta_2 - \theta_1)$. Assume that X satisfies $(M\lambda)$. Further suppose*

$(\omega 1')$ $r \mapsto r^{-\varepsilon_1} \omega(x, r)$ is uniformly almost increasing in $(0, d_X]$ for some $\varepsilon_1 > 0$.

If $p > 1$, then there is a constant $C > 0$ such that

$$\|M_\lambda f\|_{\mathcal{L}^{p,\omega,\theta_2}(X)} \leq C \|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Remark 3.2. Note that $(\omega 1')$ implies $(\omega 1)$. Letting $\omega(x, r) = r^{\sigma(x)} (\log(e + 1/r))^{\beta(x)}$ be as in Example 2.1, then note that $(\omega 1')$ holds for $0 < \varepsilon_1 < \sigma^-$.

Lemma 3.3. *Let $1 \leq \theta < \lambda$. Set*

$$I(f; x, r, \lambda) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} f(y) d\mu(y)$$

and

$$J(f; x, r, \lambda) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)} f(y)^{p(y)/p^-} d\mu(y)$$

for $x \in X$ and $0 < r \leq d_X$. Then, given $L \geq 1$, there exist constants $C = C(L) \geq 1$ such that

$$I(f; x, r, \lambda)^{p(x)/p^-} \leq C J(f; x, r, \lambda)$$

for all $x \in X$, $0 < r \leq d_X$ and for all nonnegative measurable functions f on X such that $f(y) \geq 1$ or $f(y) = 0$ for each $y \in X$ and

$$\sup_{z \in X} \left(\int_0^{2d_X} \frac{\omega(z, t)}{\mu(B(z, \theta t))} \left(\int_{B(z, t)} f(y)^{p(y)} d\mu(y) \right) \frac{dt}{t} \right) \leq L. \tag{3.2}$$

Proof. Given f as in the statement of the lemma, $x \in X$ and $0 < r < d_X$, set $I = I(f; x, r, \lambda)$ and $J = J(f; x, r, \lambda)$. Taking f , we see from (3.2) that

$$\begin{aligned} & \frac{\omega(x, r)}{\mu(B(x, \lambda r))} \int_{B(x, r)} f(y)^{p(y)} d\mu(y) \\ & \leq C \int_r^{\lambda r/\theta} \frac{\omega(x, t)}{\mu(B(x, \theta t))} \left(\int_{B(x, t)} f(y)^{p(y)} d\mu(y) \right) \frac{dt}{t} \leq C_1 L. \end{aligned}$$

Hence

$$J \leq C_1^{1/p^-} \omega(x, r)^{-1/p^-} L^{1/p^-}. \tag{3.3}$$

Since $f(y)^{p(y)/p^-} \geq f(y)$ for all $y \in X$, it follows that $I \leq J$. Thus, if $J \leq 1$, then

$$I^{p(x)/p^-} = I \cdot I^{p(x)/p^- - 1} \leq J.$$

Next, suppose $J > 1$. Set $K = J^{p^-/p(x)}$. We obtain

$$\begin{aligned} \int_{B(x, r)} f(y) d\mu(y) & \leq \int_{B(x, r) \cap \{y \in X: f(y) \leq K\}} f(y) d\mu(y) \\ & \quad + \int_{B(x, r) \cap \{y \in X: f(y) > K\}} f(y) d\mu(y) \\ & \leq K\mu(B(x, r)) + K \int_{B(x, r)} \frac{f(y)^{p(y)/p^-}}{K^{p(y)/p^-}} d\mu(y). \end{aligned}$$

By (3.3) and (ω3),

$$K^{p^-} \leq J^{p^-} \leq C_1 \omega(x, r)^{-1} L \leq C_1 \tilde{c}_3 L r^{-\omega_0}$$

or $r \leq \gamma K^{-p^-/\omega_0}$ with $\gamma = (C_1 \tilde{c}_3 L)^{1/\omega_0}$. Thus, if $d(x, y) \leq r$, then $d(x, y) \leq \gamma K^{-p^-/\omega_0}$. Hence, by (P2) there is $\beta \geq 1$, independent of f, x, r , such that

$$K^{p(x)/p^-} \leq \beta K^{p(y)/p^-} \quad \text{for all } y \in B(x, r).$$

Thus, we have

$$\begin{aligned} \int_{B(x, r)} f(y) d\mu(y) & \leq K\mu(B(x, r)) + \frac{\beta K}{K^{p(x)/p^-}} \int_{B(x, r)} f(y)^{p(y)/p^-} d\mu(y) \\ & = K\mu(B(x, r)) + \beta K\mu(B(x, \lambda r)) \frac{J}{K^{p(x)/p^-}} \\ & \leq K\mu(B(x, \lambda r))(1 + \beta), \end{aligned}$$

so that

$$I^{p(x)/p^-} \leq CJ,$$

as required. ■

Now, we are ready to prove the boundedness of M_λ on $\mathcal{L}^{p(\cdot),\omega,\theta}(X)$, as an extension of [29, Theorem 2.4] in the constant exponent case.

Theorem 3.4. *Let $1 \leq \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2 + 1)/(\theta_2 - \theta_1)$. Assume that X satisfies (M_λ) and $(\omega 1')$ holds. Then there exists a constant $C > 0$ such that*

$$\|M_\lambda f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_2}(X)} \leq C \|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot),\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{p(\cdot),\omega,\theta_1}(X)} \leq 1$. Let $f_1 = f\chi_{\{x \in X: f(x) \geq 1\}}$ and $f_2 = f - f_1$, where χ_E is the characteristic function of E . Set $g_1(y) = f_1(y)^{p(y)/p^-}$. Applying Lemma 3.3 to f_1 and $L = 1$, there exists a constant $C \geq 1$ such that

$$\{M_\lambda f_1(x)\}^{p(x)/p^-} \leq CM_\lambda g_1(x) \tag{3.4}$$

for all $x \in X$.

Since $M_\lambda f_2 \leq 1$, we have

$$\{M_\lambda f_2(x)\}^{p(x)} \leq C \tag{3.5}$$

for all $x \in X$ with a constant $C > 0$ independent of f . Here, note from $(\omega 1')$ and $(\omega 3)$ that there exists a constant $C > 0$ such that

$$\int_0^{2d_X} \omega(z, r) \frac{dr}{r} = \int_0^{2d_X} r^{-\varepsilon_1} \omega(z, r) \cdot r^{\varepsilon_1} \frac{dr}{r} \leq C \int_0^{2d_X} r^{\varepsilon_1} \frac{dr}{r} \leq C \tag{3.6}$$

for all $z \in X$. Hence, we obtain by (3.4), (3.5), (3.6) and Lemma 3.1

$$\begin{aligned} & \int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z, r)} \{M_\lambda f(x)\}^{p(x)} d\mu(x) \right) \frac{dr}{r} \\ & \leq C \left\{ \int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z, r)} \{M_\lambda f_1(x)\}^{p(x)} d\mu(x) \right) \frac{dr}{r} \right. \\ & \quad \left. + \int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z, r)} \{M_\lambda f_2(x)\}^{p(x)} d\mu(x) \right) \frac{dr}{r} \right\} \\ & \leq C \left\{ \int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z, r)} \{M_\lambda g_1(x)\}^{p^-} d\mu(x) \right) \frac{dr}{r} \right. \\ & \quad \left. + \int_0^{2d_X} \omega(z, r) \frac{dr}{r} \right\} \leq C \end{aligned}$$

for all $z \in X$, as required. ■

4. Lemmas

Set

$$\tilde{\rho}(x, r) = \int_0^r \rho(x, s) \frac{ds}{s}$$

for $r > 0$.

We give an estimate inside a ball.

Lemma 4.1. *Let $1 \leq \lambda < \tau$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x, \delta)} \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \leq C \tilde{\rho}(x, k_2 \delta) M_\lambda f(x)$$

for all $x \in X$, $0 < \delta < d_X/2$ and nonnegative $f \in L^1_{\text{loc}}(X)$.

Proof. Let $f \in L^1_{\text{loc}}(X)$ be a nonnegative measurable function on X . Let $x \in X$ and $0 < \delta < d_X/2$. Take $\gamma \in \mathbf{R}$ such that $1 < \gamma \leq \min\{2, \tau/\lambda\}$. If $y \in B(x, \gamma^{-j+1}\delta) \setminus B(x, \gamma^{-j}\delta)$ for a positive integer j , then a geometric observation and (1.1) show

$$\begin{aligned} \frac{\rho(x, d(x, y))}{\mu(B(x, \tau d(x, y)))} &\leq \frac{1}{\mu(B(x, \tau \gamma^{-j}\delta))} \sup_{\gamma^{-j}\delta \leq s \leq \gamma^{-j+1}\delta} \rho(x, s) \\ &\leq \frac{1}{\mu(B(x, \lambda \gamma^{-j+1}\delta))} \sup_{\gamma^{-j+1}\delta/2 \leq s \leq \gamma^{-j+1}\delta} \rho(x, s) \\ &\leq \frac{C}{\mu(B(x, \lambda \gamma^{-j+1}\delta))} \int_{\gamma^{-j+1}k_1\delta}^{\gamma^{-j+1}k_2\delta} \rho(x, s) \frac{ds}{s}. \end{aligned} \tag{4.1}$$

Let j_1 be the smallest integer such that $k_1/k_2 \geq \gamma^{-j_1}$. Then note from (4.1) that

$$\begin{aligned} &\int_{B(x, \delta)} \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &= \sum_{j=1}^{\infty} \int_{B(x, \gamma^{-j+1}\delta) \setminus B(x, \gamma^{-j}\delta)} \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\leq C \sum_{j=1}^{\infty} \int_{\gamma^{-j+1}k_1\delta}^{\gamma^{-j+1}k_2\delta} \rho(x, s) \frac{ds}{s} \times \frac{1}{\mu(B(x, \lambda \gamma^{-j+1}\delta))} \int_{B(x, \gamma^{-j+1}\delta)} f(y) d\mu(y) \\ &\leq C \sum_{j=1}^{\infty} \int_{\gamma^{-j-j_1+1}k_2\delta}^{\gamma^{-j+1}k_2\delta} \rho(x, s) \frac{ds}{s} \times M_\lambda f(x) \\ &\leq C \int_0^{k_2\delta} \rho(x, s) \frac{ds}{s} \times M_\lambda f(x) = C \tilde{\rho}(x, k_2 \delta) M_\lambda f(x), \end{aligned}$$

as required. ■

We consider the following condition:

$(\rho\omega)$ there exist constants $\varepsilon_2 > 0$ and $A \geq 1$ such that

$$r_2^{\varepsilon_2} \rho(x, r_2) \omega(x, r_2)^{-1/p(x)} \leq A r_1^{\varepsilon_2} \rho(x, r_1) \omega(x, r_1)^{-1/p(x)}$$

for all $x \in X$ whenever $0 < r_1 < r_2 < \max\{1, 2k_2\}d_X$.

Remark 4.2. We have by $(\rho\omega)$

$$t^{\varepsilon_2} \rho(x, t) \omega(x, t)^{-1/p(x)} \leq A s^{\varepsilon_2} \rho(x, s) \omega(x, s)^{-1/p(x)}$$

for all $x \in X$ whenever $0 < t/2 \leq s \leq t \leq \max\{1, 2k_2\}d_X$. Then, by $(\omega 1)$ and $(\omega 2)$, we obtain

$$\rho(x, t) \leq C \rho(x, s)$$

for all $x \in X$ whenever $0 < t/2 \leq s \leq t \leq \max\{1, 2k_2\}d_X$.

Remark 4.3. Let $\alpha(\cdot)$ be a measurable function on X such that $\alpha^- > 0$ and set $\rho(x, r) = r^{\alpha(x)}$. Let

$$\omega(x, r) = r^{\sigma(x)} (\log(e + 1/r))^{\beta(x)}$$

be as in Example 2.1. Condition $(\rho\omega)$ is satisfied if

$$\inf_{x \in X} \left(\frac{\sigma(x)}{p(x)} - \alpha(x) \right) > 0.$$

The next lemma concerns an estimate outside a ball.

Lemma 4.4. Let $1 \leq \theta < \tau$. Assume that $(\rho\omega)$ holds. Then there exists a constant $C > 0$ such that

$$\begin{aligned} & \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ & \leq C \rho(x, \min\{1, k_1\}\delta) \omega(x, \min\{1, k_1\}\delta)^{-1/p(x)} \end{aligned}$$

for all $x \in X$, $0 < \delta < d_X/2$ and nonnegative $f \in \mathcal{L}^{p(\cdot), \omega, \theta}(X)$ with

$$\|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta}(X)} \leq 1.$$

Proof. Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta}(X)} \leq 1$. Let $x \in X$ and $0 < \delta < d_X/2$. In view of $(\omega 3)$, we have

$$d(x, y) \leq \tilde{c}_3^{1/\omega_0} \omega(x, d(x, y))^{1/\omega_0} \tag{4.2}$$

for all $x, y \in X$. It follows from $(\omega 3)$, $(P2)$ and (4.2) that

$$\begin{aligned} \omega(x, d(x, y))^{(p(y)-p(x))/p(x)} &\leq C \omega(x, d(x, y))^{-\frac{1}{p(x)} \frac{Cp}{\log(e+1/d(x, y))}} \\ &\leq C \omega(x, d(x, y))^{-\frac{1}{p(x)} \frac{Cp}{\log(e+\tilde{\epsilon}_3^{-1/\omega_0} \omega(x, d(x, y))^{-1/\omega_0})}} \\ &\leq C. \end{aligned} \tag{4.3}$$

Hence, by $(P1)$, we find

$$\begin{aligned} &\int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &= \int_{\{y \in X: f(y) \leq \omega(x, d(x, y))^{-1/p(x)}\} \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\quad + \int_{\{y \in X: f(y) > \omega(x, d(x, y))^{-1/p(x)}\} \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\leq \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) \omega(x, d(x, y))^{-1/p(x)}}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\quad + \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) f(y)}{\mu(B(x, \tau d(x, y)))} \frac{f(y)^{p(y)-1}}{\omega(x, d(x, y))^{-(p(y)-1)/p(x)}} d\mu(y) \\ &\leq \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) \omega(x, d(x, y))^{-1/p(x)}}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\quad + C \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y)) \omega(x, d(x, y))^{1-1/p(x)} f(y)^{p(y)}}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &= I_1 + CI_2. \end{aligned}$$

Here we used (4.3) in the last inequality.

Take $\gamma_1 \in \mathbf{R}$ such that $1 < \gamma_1 \leq \min\{\tau, 2\}$. Let j_1 be the smallest integer such that $\gamma_1^{j_1} \delta \geq d_X$ and let j_0 be the smallest integer such that $k_2/k_1 \leq \gamma_1^{j_0}$. As in the proof of Lemma 4.1, we have by (4.1)

$$\begin{aligned} I_1 &\leq \sum_{j=1}^{j_1} \int_{B(x, \gamma_1^j \delta) \setminus B(x, \gamma_1^{j-1} \delta)} \frac{\rho(x, d(x, y)) \omega(x, d(x, y))^{-1/p(x)}}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\leq C \sum_{j=1}^{j_1} \omega(x, \gamma_1^j \delta)^{-1/p(x)} \int_{\gamma_1^j k_1 \delta}^{\gamma_1^j k_2 \delta} \rho(x, s) \frac{ds}{s} \\ &\leq C \sum_{j=1}^{j_1} \int_{\gamma_1^j k_1 \delta}^{\gamma_1^j k_2 \delta} \rho(x, s) \omega(x, s)^{-1/p(x)} \frac{ds}{s} \end{aligned}$$

since we find by (ω1) and (ω2) that

$$\omega(x, \gamma_1^j \delta)^{-1/p(x)} \leq C \omega(x, s)^{-1/p(x)}$$

for all $1 \leq j \leq j_1$ and $\gamma_1^j k_1 \delta \leq s \leq \gamma_1^j k_2 \delta$. Further, we see from the definitions of j_0 and j_1 that

$$\begin{aligned} I_1 &\leq C \sum_{j=1}^{j_1} \int_{\gamma_1^j k_1 \delta}^{\gamma_1^j k_2 \delta} \rho(x, s) \omega(x, s)^{-1/p(x)} \frac{ds}{s} \\ &\leq C j_0 \int_{\gamma_1 k_1 \delta}^{\gamma_1^{j_1} k_2 \delta} \rho(x, s) \omega(x, s)^{-1/p(x)} \frac{ds}{s} \\ &\leq C \int_{k_1 \delta}^{2^{k_2} d_X} \rho(x, s) \omega(x, s)^{-1/p(x)} \frac{ds}{s}, \end{aligned}$$

so that we have by (ρω)

$$\begin{aligned} I_1 &\leq C (k_1 \delta)^{\varepsilon_2} \rho(x, k_1 \delta) \omega(x, k_1 \delta)^{-1/p(x)} \int_{k_1 \delta}^{2^{k_2} d_X} s^{-\varepsilon_2} \frac{ds}{s} \\ &\leq C \rho(x, k_1 \delta) \omega(x, k_1 \delta)^{-1/p(x)}. \end{aligned}$$

Next, take $\gamma_2 \in \mathbf{R}$ such that $1 < \gamma_2 \leq \min\{\tau/\theta, 2\}$. Let j_2 be the smallest integer such that $\gamma_2^{j_2/2} \delta \geq d_X$. For I_2 , we have by (ρω)

$$\begin{aligned} I_2 &\leq C \rho(x, \delta) \omega(x, \delta)^{-1/p(x)} \int_{X \setminus B(x, \delta)} \frac{\omega(x, d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y)^{p(y)} d\mu(y) \\ &\leq C \rho(x, \delta) \omega(x, \delta)^{-1/p(x)} \\ &\quad \cdot \sum_{j=1}^{j_2} \int_{B(x, \gamma_2^{j/2} \delta) \setminus B(x, \gamma_2^{(j-1)/2} \delta)} \frac{\omega(x, d(x, y))}{\mu(B(x, \tau d(x, y)))} f(y)^{p(y)} d\mu(y). \end{aligned}$$

Hence, by (ω1),

$$\begin{aligned} I_2 &\leq C \rho(x, \delta) \omega(x, \delta)^{-1/p(x)} \sum_{j=1}^{j_2} \frac{\omega(x, \gamma_2^{j/2} \delta)}{\mu(B(x, \gamma_2^{(j+1)/2} \theta \delta))} \int_{B(x, \gamma_2^{j/2} \delta)} f(y)^{p(y)} d\mu(y) \\ &\leq C \rho(x, \delta) \omega(x, \delta)^{-1/p(x)} \\ &\quad \cdot \sum_{j=1}^{j_2} \int_{\gamma_2^{j/2} \delta}^{\gamma_2^{(j+1)/2} \delta} \frac{\omega(x, t)}{\mu(B(x, \theta t))} \left(\int_{B(x, t)} f(y)^{p(y)} d\mu(y) \right) \frac{dt}{t} \\ &\leq C \rho(x, \delta) \omega(x, \delta)^{-1/p(x)} \int_{\gamma_2^{1/2} \delta}^{\gamma_2^{j_2/2} d_X} \frac{\omega(x, t)}{\mu(B(x, \theta t))} \left(\int_{B(x, t)} f(y)^{p(y)} d\mu(y) \right) \frac{dt}{t} \end{aligned}$$

$$\begin{aligned} &\leq C\rho(x, \delta)\omega(x, \delta)^{-1/p(x)} \int_0^{2d_X} \frac{\omega(x, t)}{\mu(B(x, \theta t))} \left(\int_{B(x, t)} f(y)^{p(y)} d\mu(y) \right) \frac{dt}{t} \\ &\leq C\rho(x, \delta)\omega(x, \delta)^{-1/p(x)}. \end{aligned}$$

Thus, we obtain the required result by $(\rho\omega)$. ■

5. Sobolev-type inequality

To state Theorem 5.1 below we give the assumptions for the function appearing in the theorem. We consider a function

$$\Psi(x, t) : X \times [0, \infty) \rightarrow [0, \infty)$$

which satisfies

(Ψ1) $\Psi(\cdot, t)$ is measurable on X for each $t \geq 0$ and $\Psi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in X$;

(Ψ2) there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \Psi(x, 1) \leq A_1 \quad \text{for all } x \in X;$$

(Ψ3) $t \mapsto \Psi(x, t)/t$ is uniformly almost increasing on $(0, \infty)$, namely there exists a constant $A_2 \geq 1$ such that

$$\Psi(x, t_1)/t_1 \leq A_2\Psi(x, t_2)/t_2 \quad \text{for all } x \in X \text{ whenever } 0 < t_1 < t_2;$$

(Ψ4) there exists a constant $A_3 \geq 1$ such that

$$\Psi(x, t\tilde{\rho}(x, \omega^{-1}(x, t^{-p(x)}))) \leq A_3t^{p(x)}$$

for all $x \in X$ and $t \geq 1$.

We write

$$\bar{\psi}(x, t) = \sup_{0 < s \leq t} \frac{\Psi(x, s)}{s}$$

and

$$\bar{\Psi}(x, t) = \int_0^t \bar{\psi}(x, r) dr$$

for $x \in X$ and $t \geq 0$. Then $\bar{\Psi}(x, \cdot)$ is convex and

$$\Psi(x, t/2) \leq \bar{\Psi}(x, t) \leq A_2\Psi(x, t) \tag{5.1}$$

for all $x \in X$ and $t \geq 0$.

Let $\theta \geq 1$. Given $\Psi(x, t)$ and $\omega(x, r)$ as above, we define the $\mathcal{L}^{\Psi, \omega, \theta}$ norm by

$$\|f\|_{\mathcal{L}^{\Psi, \omega, \theta}(X)} = \inf \left\{ \lambda > 0; \sup_{x \in X} \left(\int_0^{2d_X} \frac{\omega(x, r)}{\mu(B(x, \theta r))} \cdot \left(\int_{B(x, r)} \bar{\Psi}(x, |f(y)|/\lambda) d\mu(y) \right) \frac{dr}{r} \right) \leq 1 \right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{\Psi, \omega, \theta}(X)} < \infty$ is denoted by $\mathcal{L}^{\Psi, \omega, \theta}(X)$.

Note from Remark 4.2 that

$$\rho(x, t) \leq C \int_{t/2}^t \rho(x, s) \frac{ds}{s} \leq C \tilde{\rho}(x, t) \tag{5.2}$$

for all $x \in X$ and all $0 < t \leq \max\{1, 2k_2\}d_X$.

As an extension of [28, Theorem 3.1] and [29,30], we give a Sobolev-type inequality for $I_{\rho, \tau} f$ of functions in $\mathcal{L}^{p(\cdot), \omega, \theta}(X)$.

Theorem 5.1. *Let $1 \leq \theta_1 < \theta_2$ and $\theta_1(\theta_2 + 1)/(\theta_2 - \theta_1) < \lambda < \tau$. Assume that X satisfies $(M\lambda)$ and $(\omega 1')$ and $(\rho\omega)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_{\rho, \tau} f\|_{\mathcal{L}^{\Psi, \omega, \theta_2}(X)} \leq C \|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta_1}(X)}$$

for all $f \in \mathcal{L}^{p(\cdot), \omega, \theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X such that

$$\|f\|_{\mathcal{L}^{p(\cdot), \omega, \theta_1}(X)} \leq 1.$$

Let $x \in X$ and $0 < \delta < d_X/2$. By Lemmas 4.1 and 4.4 and (5.2), we find

$$\begin{aligned} I_{\rho, \tau} f(x) &= \int_{B(x, \delta)} \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\quad + \int_{X \setminus B(x, \delta)} \frac{\rho(x, d(x, y))f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \\ &\leq C \{ \tilde{\rho}(x, k_2\delta) M_\lambda f(x) + \rho(x, \min\{1, k_1\}\delta) \omega(x, \min\{1, k_1\}\delta)^{-1/p(x)} \} \\ &\leq C \{ \tilde{\rho}(x, k_2\delta) M_\lambda f(x) + \tilde{\rho}(x, \min\{1, k_1\}\delta) \omega(x, \min\{1, k_1\}\delta)^{-1/p(x)} \}. \end{aligned}$$

Set

$$\omega^{-1}(x, r) = \sup\{t > 0; \omega(x, t) < r\}.$$

Here, note from [17, Lemma 5.1 (3)] that

$$\omega(x, \omega^{-1}(x, r)) = r. \tag{5.3}$$

If $k_2^{-1}\omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)}) \geq d_X/2$, then, taking $\delta = d_X/2$, we have

$$I_{\rho,\tau}f(x) \leq C\tilde{\rho}(x, \max\{1, k_2\}d_X/2) \leq C$$

by (5.3), (ω1) and (ω3). If $k_2^{-1}\omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)}) < d_X/2$, then take

$$\delta = k_2^{-1}\omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)}).$$

Then we have

$$I_{\rho,\tau}f(x) \leq CM_\lambda f(x)\tilde{\rho}(x, \omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)}))$$

by (5.3), (ω1) and (ω2). Thus, we have

$$I_{\rho,\tau}f(x) \leq C_1 \max \left\{ M_\lambda f(x)\tilde{\rho}(x, \omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)})), 1 \right\}.$$

In view of (Ψ4), we obtain

$$\begin{aligned} \Psi(x, I_{\rho,\tau}f(x)/C_1) &\leq C \left\{ \Psi(x, M_\lambda f(x)\tilde{\rho}(x, \omega^{-1}(x, \{M_\lambda f(x)\}^{-p(x)}))) + 1 \right\} \\ &\leq C [\{M_\lambda f(x)\}^{p(x)} + 1]. \end{aligned}$$

Hence, we see from Theorem 3.4 and (3.6)

$$\begin{aligned} &\int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z,r)} \Psi(x, I_{\rho,\tau}f(x)/C_1) d\mu(x) \right) \frac{dr}{r} \\ &\leq C \left\{ \int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z,r)} \{M_\lambda f(x)\}^{p(x)} d\mu(x) \right) \frac{dr}{r} \right. \\ &\quad \left. + \int_0^{2d_X} \omega(z, r) \frac{dr}{r} \right\} \\ &\leq C_2 \end{aligned}$$

for all $z \in X$, so that by the convexity of $\bar{\Psi}$ and (5.1)

$$\int_0^{2d_X} \frac{\omega(z, r)}{\mu(B(z, \theta_2 r))} \left(\int_{B(z,r)} \bar{\Psi}(x, I_{\rho,\tau}f(x)/(A_2 C_1 C_2)) d\mu(x) \right) \frac{dr}{r} \leq 1$$

for all $z \in X$. This completes the proof. ■

Example 5.2. Let $\alpha(\cdot)$ and $\gamma(\cdot)$ be measurable functions on X such that $\alpha^- > 0$ and

$$-\infty < \gamma^- := \inf_{x \in X} \gamma(x) \leq \sup_{x \in X} \gamma(x) =: \gamma^+ < \infty.$$

Let

$$\rho(x, r) = r^{\alpha(x)} (\log(e + 1/r))^{\gamma(x)}$$

and let

$$\omega(x, r) = r^{\sigma(x)} (\log(e + 1/r))^{\beta(x)}$$

be as in Example 2.1, see Remark 3.2. We know that condition $(\rho\omega)$ holds if

$$\inf_{x \in X} (\sigma(x)/p(x) - \alpha(x)) > 0$$

(Remark 4.3). Set

$$\Psi(x, t) = \{t(\log(e + t))^{\alpha(x)\beta(x)/\sigma(x) - \gamma(x)}\}^{p^*(x)},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$. Then note that $\Psi(x, t)$ satisfies conditions $(\Psi1)$ – $(\Psi4)$, see [28, Example 4.1].

Applying Theorem 5.1 to special ρ and ω given in Example 5.2, we obtain the following corollaries.

Corollary 5.3. *Let $\rho(x, r)$ and $\omega(x, r)$ be as in Example 5.2. Let $1 \leq \theta_1 < \theta_2$ and $\theta_1(\theta_2 + 1)/(\theta_2 - \theta_1) < \lambda < \tau$. Assume that X satisfies $(M\lambda)$. Further, suppose $\inf_{x \in X} (\sigma(x)/p(x) - \alpha(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\|I_{\rho, \tau} f\|_{\mathcal{L}^{\Psi, \omega, \theta_2}(X)} \leq C \|f\|_{\mathcal{L}^{p^{(\cdot)}, \omega, \theta_1}(X)}$$

for all $f \in \mathcal{L}^{p^{(\cdot)}, \omega, \theta_1}(X)$, where $\Psi(x, t) = \{t(\log(e + t))^{\alpha(x)\beta(x)/\sigma(x) - \gamma(x)}\}^{p^*(x)}$ with $p^*(x)$ given by $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$.

Corollary 5.4. *Let X be a doubling metric measure space. Let $\rho(x, r)$ and $\omega(x, r)$ be as in Example 5.2. Suppose $\inf_{x \in X} (\sigma(x)/p(x) - \alpha(x)) > 0$. Then there exists a constant $C > 0$ such that*

$$\|I_{\rho, 1} f\|_{\mathcal{L}^{\Psi, \omega, 1}(X)} \leq C \|f\|_{\mathcal{L}^{p^{(\cdot)}, \omega, 1}(X)}$$

for all $f \in \mathcal{L}^{p^{(\cdot)}, \omega, 1}(X)$, where $\Psi(x, t) = \{t(\log(e + t))^{\alpha(x)\beta(x)/\sigma(x) - \gamma(x)}\}^{p^*(x)}$ with $p^*(x)$ given by $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$.

Remark 5.5. We write that $L_c(t) = \log(c + t)$ for $c > 1$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. We may consider

$$\rho(x, r) = r^{\alpha(x)} \prod_{j=1}^k (L_e^{(j)}(1/r))^{\gamma_j(x)}$$

and

$$\omega(x, r) = r^{\sigma(x)} \prod_{j=1}^k (L_e^{(j)}(1/r))^{\beta_j(x)}.$$

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Takao Ohno

Faculty of Education, Oita University, Dannoharu Oita-city 870-1192, Japan;
t-ohno@oita-u.ac.jp

Tetsu Shimomura

Department of Mathematics, Graduate School of Humanities and Social Sciences, Hiroshima University, Higashi-Hiroshima 739-8524, Japan; tshimo@hiroshima-u.ac.jp