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A remark on Kashin's discrepancy argument and partial coloring in the Komlós conjecture

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Abstract. In this expository note, we discuss an early partial coloring result of B. Kashin (1985). Although this result only implies Spencer's six standard deviations (1985) up to a $\log\log n$ factor, Kashin's argument gives a simple proof of the existence of a constant discrepancy partial coloring in the setup of the Komlós conjecture.

A seminal result of Spencer [10] states that if $x_1, \ldots, x_n$ are vectors in $\{0,1\}^n$, then there is a vector $\varepsilon \in \{1,-1\}^n$ such that

$$\max_{i\in\{1,\ldots,n\}} |\langle \varepsilon, x_i\rangle| \le 6\sqrt{n}. \tag{1}$$

A few years later this result was shown independently by E. Gluskin [3] who used convex geometry arguments. In fact, the arguments of Gluskin improve upon earlier papers by B. Kashin [5, 6], which imply Spencer's bound up to a multiplicative $\log\log n$ factor.

To the best of our knowledge, the work [5] has not been translated. In this expository note, we present this original argument and discuss some of its immediate applications. We provide a slightly more general statement (namely, we replace the bound on the individual norms of the vectors by a bound on the sum of these norms squared) whose proof follows from the arguments in [5].

Theorem 1 (Kashin's partial coloring theorem [5]). *There is an absolute constant $c > 0$ such that the following holds. Assume that $x_1, \ldots, x_m \in \mathbb{R}^n$ satisfy*

$$\sum_{i=1}^m \|x_i\|_2^2 \le m.$$

There is a choice $\varepsilon \in \{-1,0,1\}^n$ such that $\|\varepsilon\|_1 \ge n/6$ and

$$\max_{i\in\{1,\ldots,m\}} |\langle \varepsilon, x_i\rangle| \le c\sqrt{\frac{m}{n}}.$$

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In comparison, the classic partial coloring result of Gluskin [3, Theorem 3], which implies Spencer's result (1), requires as stated $\max_{i\in\{1,\dots,m\}} \|x_i\|_2 \le 1$, instead of $\sum_{i=1}^m \|x_i\|_2^2 \le m$ required in Theorem 1. Under this slightly stronger assumption, Gluskin's result claims the existence of $\varepsilon \in \{-1,0,1\}^n$ such that $\|\varepsilon\|_1 \ge c_1 n$ for some constant $c_1 \in (0,1)$ and

$$\max_{i\in\{1,\dots,m\}} |\langle \varepsilon, x_i\rangle| \le c_2\sqrt{1+\log(m/n)}$$

for some constant $c_2 > 0$. Despite the worse dependence on m/n in Theorem 1, the partial coloring result of Theorem 1 implies (1) up to a multiplicative $\log\log n$ factor.

Corollary 1 (A full coloring argument of Kashin, [6, Proposition 2]). *There is an absolute constant $c > 0$ such that the following holds. Assume that $x_1,\dots,x_n \in \mathbb{R}^n$ each satisfy $\|x_i\|_\infty \le 1$. There is $\varepsilon \in \{-1,1\}^n$ such that*

$$\max_{i\in\{1,\dots,n\}} |\langle \varepsilon, x_i\rangle| \le c\sqrt{n}\log\log n.$$

Proof. We repeat the proof of [6, Proposition 2], whose translated version is available, for the sake of completeness. Applying Theorem 1 to $\{x_i/\sqrt{n}\}_{i=1}^n$, we get a partial coloring $\varepsilon \in \{-1,0,1\}^n$ with at most $(5/6)n$ zeros satisfying $\max_{i\in[n]} |\langle \varepsilon, x_i\rangle| \le c\sqrt{n}$. Let $S = \{j \in [n] : \varepsilon_j = 0\}$. Denoting $n_1 = |S|$, we have $n_1 \le (5/6)n$. When we restrict the set of coordinates of $\{x_i\}_{i=1}^n$ to S, we get vectors $x_i^{(1)} \in \mathbb{R}^{n_1}$, still satisfying $\|x_i^{(1)}\|_\infty \le 1$. We can thus apply Theorem 1 recursively to $\{x_i^{(1)}/\sqrt{n_1}\}_{i=1}^n$: at step k, we work with a set S_k such that $n_k = |S_k| \le (5/6)^k n$, and with corresponding vectors $x_i^{(k)} \in \mathbb{R}^{n_k}$ for $i = 1,\dots,n$. After s steps, we have at most $(5/6)^s n$ zero remaining coordinates in the coloring, and a total discrepancy (by the triangle inequality) of at most $cs\sqrt{n}$. We set $s = \lceil \log\log n/\log(6/5)\rceil$, so that $n_s = n(5/6)^s \le n/\log n$. For the remaining n_s coordinates we use a random value of $\varepsilon^{(s)} \in \{\pm 1\}^{n_s}$. By the standard Hoeffding's inequality and union bound argument, the discrepancy of this random coloring is

$$\max_{i\in[n]} |\langle \varepsilon^{(s)}, x_i^{(s)}\rangle| \le C\sqrt{\log n}\max_{i\in[n]} \|x_i^{(s)}\|_2 \le C\sqrt{n_s\log n} \le C\sqrt{n}, \tag{2}$$

for a universal constant $C > 0$. Therefore, by the triangle inequality, the total discrepancy of the resulting coloring $\varepsilon \in \{\pm 1\}^n$ is

$$\max_{i\in[n]} |\langle \varepsilon, x_i\rangle| \le c\sqrt{n}\lceil \log\log n/\log(6/5)\rceil + C\sqrt{n} \le C_2\sqrt{n}\log\log n, \tag{3}$$

for some universal constant $C_2 > 0$. The claim follows. ∎

A previously unnoticed[1] corollary of Theorem 1 is the following partial coloring result. This result appears in the work of Spencer [10, Theorem 16], though its proof there is quite involved.

Corollary 2 (Komlós partial coloring). *There is a constant $K > 0$ such that the following holds. Assume that $u_1, \dots, u_n \in \mathbb{R}^m$ satisfy $\max_{i\in\{1,\dots,n\}} \|u_i\|_2 \le 1$. There is $\varepsilon \in \{-1, 0, 1\}^n$ such that $\|\varepsilon\|_1 \ge n/6$ and*

$$\Big\| \sum_{i=1}^{n} \varepsilon_i u_i \Big\|_\infty \le K.$$

Proof. Consider the vectors $a_1, \dots, a_m \in \mathbb{R}^n$ defined by $a_j = (u_1^{(j)}, \dots, u_n^{(j)})$ for $j = 1, \dots, m$, where $u_k^{(j)}$ denotes the j-th coordinate of u_k. We have $\sum_{j=1}^{m} \|a_j\|_2^2 = \sum_{i=1}^{n} \|u_i\|_2^2 \le n$, or equivalently $\sum_{j=1}^{m} \|a_j \sqrt{\frac{m}{n}}\|_2^2 \le m$. Applying Theorem 1 to $\{a_j \sqrt{\frac{m}{n}}\}_{j=1}^{m}$, we obtain for some $\varepsilon \in \{-1, 0, 1\}^n$ with $\|\varepsilon\|_1 \ge n/6$

$$\Big\| \sum_{i=1}^{n} \varepsilon_i u_i \Big\|_\infty = \max_{j\in\{1,\dots,m\}} |\langle \varepsilon, a_j\rangle| = \sqrt{\frac{n}{m}} \max_{j\in\{1,\dots,m\}} \Big|\Big\langle \varepsilon, a_j \sqrt{\frac{m}{n}}\Big\rangle\Big| \le K. \qquad \blacksquare$$

The Komlós conjecture asks if the result of Corollary 2 holds for some full coloring $\varepsilon \in \{-1, 1\}^n$ instead. A different proof of Corollary 2, based on the ideas of Gluskin, appears in the work of Giannopoulos [2]. Recent results on algorithmic discrepancy imply a similar result. However, instead of the *true partial coloring* taking its values in $\{-1, 0, 1\}$, one usually considers *fractional colorings* where the colors take values in $[-1, 1]$, though many of the colors are close to $\{-1, 1\}$ (see e.g., [8]). Regarding the fractional coloring results, we note that an earlier version of Theorem 1, where $\varepsilon \in [-1, 1]^n$ and $\|\varepsilon\|_1 \ge cn$ for some constant $c \in (0, 1)$ appears in [4] (see also [3]).

Remark 1 (Partial coloring iterations for Komlós). We note that, for any s, iterating Corollary 2 implies a partial coloring with at most $n(5/6)^s$ zeros and discrepancy Ks. For example, with s scaling as $\log\log n$ this yields a partial coloring with only a $O(1/\log(n))$ fraction of zeros and discrepancy $O(\log\log n)$ and with s scaling as $\log n$ one can get a full coloring with discrepancy $O(\log(n))$.

[1] After this note was made public, the authors were made aware that a similar argument, based on Vaaler's theorem, was used by Y. Lonke to prove the result of Corollary 2. This proof was presented in the Ph.D. dissertation of Y. Lonke (1998, Hebrew University, in Hebrew) [7]. Remarkably, the argument of Y. Lonke is independent of B. Kashin's earlier application of Vaaler's Theorem.

We note that in the setup of Komlós, partial colorings are only known to imply the discrepancy bound $O(\log n)$, whereas the best known discrepancy bound due to Banaszczyk [1] scales as $O(\sqrt{\log n})$. The remainder of the note is devoted to the proof of Theorem 1.

Theorem 2 (Vaaler's theorem [11]). *Let* $Q_d = [-1/2, 1/2]^d$, *and let* $A \in \mathbb{R}^{d\times n}$ *satisfy* $\mathrm{rank}(A) = n$. *Then, the following inequality holds*

$$\frac{1}{\sqrt{\det(A^\top A)}} \le \int_{\mathbb{R}^n} \mathbf{1}\{Ax \in Q_d\} dx.$$

Let us provide some intuition behind this result. Let P_U be a uniform distribution on $[-1/2, 1/2]$, and P_G be a Gaussian measure with density $\exp(-\pi x^2)$. It is clear that for any closed, convex, symmetric set $A \subseteq [-1/2, 1/2]$, it holds that $P_G(A) \le P_U(A)$. In this case, we say that the measure P_U is *more peaked* than P_G. The proof of Theorem 2 first shows that the uniform measure in $[-1/2, 1/2]^d$ is more peaked than the Gaussian measure with density $\exp(-\pi \|x\|_2^2)$. The proof is then generalized to an arbitrary dimension d, so that the final statement is essentially a comparison between the Gaussian measure and the volume of the convex polytope induced by A. Lower bounding the Gaussian measure has continued to be a fundamental approach in modern discrepancy.

For $\lambda > 0$ consider the set

$$E_\lambda = \left\{ z \in \mathbb{R}^n : \|z\|_\infty \le 1 \quad \text{and} \quad \max_{i\in\{1,\dots,m\}} |\langle z, x_i\rangle| \le \frac{1}{\lambda}\sqrt{\frac{m}{n}} \right\}.$$

We want to lower bound the volume of this set using the bound of Theorem 2. Let A in this theorem be defined as follows. Set $d = m + n$. The first m rows of A are $\frac{\lambda x_i}{2}\sqrt{\frac{n}{m}}$, where $i = 1, \dots, m$. The remaining n rows are the normalized standard basis vectors $e_i/2$, where $i = 1, \dots, n$. Clearly, the rank of A is equal to n. It is straightforward to see that $\mathbf{1}\{Az \in Q_d\} = \mathbf{1}\{z \in E_\lambda\}$ for all $z \in \mathbb{R}^n$. Therefore,

$$\int_{\mathbb{R}^n} \mathbf{1}\{Ax \in Q_d\} dx = \mathrm{vol}(E_\lambda).$$

Observe that $A^\top A \in \mathbb{R}^{n\times n}$ is positive semi-definite, and by the standard upper bound we have

$$\begin{aligned} \det(A^\top A) &\le \Big(\frac{\mathrm{Tr}(A^\top A)}{n}\Big)^n = \Big(\frac{\mathrm{Tr}(AA^\top)}{n}\Big)^n \\ &= \Big(\sum_{i=1}^m \frac{\lambda^2\|x_i\|_2^2}{4}\frac{1}{m} + \frac{1}{4}\Big)^n \le \Big(\frac{\lambda^2}{4} + \frac{1}{4}\Big)^n. \end{aligned}$$

Fix $\delta = 0.01$ and choose $\lambda > 0$ small enough so that $(1+\lambda^2)^{-n/2} > (1-\delta)^n$. By Theorem 2 we have

$$2^n(1-\delta)^n < \frac{1}{\sqrt{\det(A^\top A)}} \leq \mathrm{vol}(E_\lambda).$$

In particular, for the set $(2-\delta)E_\lambda$, we have $\mathrm{vol}((2-\delta)E_\lambda) > 2^n(2-\delta)^n(1-\delta)^n$. This set is also convex, and symmetric around the origin. By Minkowski's convex body theorem [9, Chapter 2, Ex. 1], we have that the set $(2-\delta)E_\lambda$ contains at least $2(2-\delta)^n(1-\delta)^n$ non-zero points with integer coordinates. By construction, all these integer points belong to $\{-1,0,1\}^n$. Finally, the size of the set $\{x \in \{-1,0,1\}^n : \|x\|_1 \leq n/6\}$ is

$$\sum_{r=0}^{\lfloor n/6 \rfloor} \binom{n}{r} 2^r \leq 2^{\lfloor n/6 \rfloor} \sum_{r=0}^{\lfloor n/6 \rfloor} \binom{n}{r} \leq (12e)^{n/6}.$$

We check that for our choice of δ, for all n it holds that $(12e)^{n/6} \leq 2(2-\delta)^n(1-\delta)^n$. Thus, by the counting argument, there is at least one $\varepsilon \in \{-1,0,1\}^n \cap (2-\delta)E_\lambda$ such that $\|\varepsilon\|_1 \geq n/6$. The claim of Theorem 1 follows. ■

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