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Deformations of Legendrian curves

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Abstract. We construct versal and equimultiple versal deformations of the parametrization of a Legendrian curve.

1. Introduction

Legendrian varieties are analytic subsets of the projective cotangent bundle of a smooth manifold or, more generally, of a contact manifold. They are projectivizations of conic Lagrangian varieties. These are specifically important in $\mathcal{D}$-modules theory and microlocal analysis (see [6–8]). Its deformation theory is still an almost virgin territory (see [2, 10]).

In Sections 2, 3 and 4, we introduce the languages of contact geometry and deformation theory. In Sections 5 and 6, we construct the semiuniversal and equimultiple semiuniversal deformations of the parametrization of a germ of a Legendrian curve, extending to Legendrian curves previous results on deformations of germs of plane curves (see [5]). We show in the proof of Theorem 5.4 that the category of Legendrian curves verifies Schlessinger's condition for formal versality. We will follow the definitions and notations of [5].

The results will be useful to the study of equisingular deformations of Legendrian curves and its moduli spaces in forthcoming articles.

2. Contact geometry

Let $(X, \mathcal{O}_X)$ be a complex manifold of dimension 3. A differential form ω of degree 1 is said to be a *contact form* if $\omega \wedge d\omega$ never vanishes. Let ω be a contact form. By Darboux's theorem for contact forms there is locally a system of coordinates (x, y, p) such that $\omega = dy - p dx$. If ω is a contact form and f is a holomorphic function that

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never vanishes, $f\omega$ is also a contact form. We say that a locally free subsheaf $\mathcal{L}$ of Ω^1_X is a *contact structure* on X if $\mathcal{L}$ is locally generated by a contact form. If $\mathcal{L}$ is a contact structure on X the pair $(X, \mathcal{L})$ is said to be a *contact manifold*. Let $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$ be contact manifolds. Let $\chi : X_1 \to X_2$ be a holomorphic map. We say that χ is a *contact transformation* if $\chi^*\omega$ is a local generator of $\mathcal{L}_1$ for a local generator ω of $\mathcal{L}_2$.

Let $\theta = \xi dx + \eta dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Let

$$\pi : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$$

be the *projective cotangent bundle* of $\mathbb{C}^2$, given by $\pi(x, y; \xi : \eta) = (x, y)$. Let U $[V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0$ $[\xi \neq 0]$. Then θ/η $[\theta/\xi]$ defines a contact form $dy - pdx$ $[dx - qdy]$ on U $[V]$, where $p = -\xi/\eta$ $[q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define a structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

Further, if $\Phi(x, y) = (a(x, y), b(x, y))$ with $a, b \in \mathbb{C}\{x, y\}$ is an automorphism of $(\mathbb{C}^2, (0, 0))$, we associate to Φ the germ of contact transformation

$$\chi : (\mathbb{P}^*\mathbb{C}^2, (0, 0; 0 : 1)) \to \big(\mathbb{P}^*\mathbb{C}^2, (0, 0; -\partial_x b(0, 0) : \partial_x a(0, 0))\big)$$

defined by

$$\chi(x, y; \xi : \eta) = \big(a(x, y), b(x, y); \partial_y b\xi - \partial_x b\eta : -\partial_y a\xi + \partial_x a\eta\big). \tag{2.1}$$

If $D\Phi_{(0,0)}$ leaves invariant $\{y = 0\}$, then $\partial_x b(0, 0) = 0$, $\partial_x a(0, 0) \neq 0$ as well as $\chi(0, 0; 0 : 1) = (0, 0; 0 : 1)$. Moreover,

$$\chi(x, y, p) = \big(a(x, y), b(x, y), (\partial_y bp + \partial_x b)/(\partial_y ap + \partial_x a)\big).$$

Let $(X, \mathcal{L})$ be a contact manifold. A curve L in X is said to be *Legendrian* if $\iota^*\omega = 0$ for each section ω of $\mathcal{L}$, where $\iota : L \hookrightarrow X$.

Let Z be the germ at $(0, 0)$ of an irreducible plane curve parametrized by

$$\varphi(t) = (x(t), y(t)). \tag{2.2}$$

We define the *conormal* of Z as the curve parametrized by

$$\psi(t) = (x(t), y(t); -y'(t) : x'(t)). \tag{2.3}$$

The conormal of Z is the germ of a Legendrian curve of $\mathbb{P}^*\mathbb{C}^2$. We will denote the conormal of Z by $\mathbb{P}^*_Z\mathbb{C}^2$ and the parametrization (2.3) by $\mathcal{C}\mathrm{on}\,\varphi$.

Assume that the tangent cone $C(Z)$ is defined by the equation $ax + by = 0$, with $(a, b) \neq (0, 0)$. Then $\mathbb{P}^*_Z\mathbb{C}^2$ is a germ of a Legendrian curve at $(0, 0; a : b)$. Let $f \in \mathbb{C}\{t\}$. We say the f has order k and write $\operatorname{ord} f = k$ or $\operatorname{ord}_t f = k$ if f/t^k is a unit of $\mathbb{C}\{t\}$.

Remark 2.1. Let Z be the plane curve parametrized by (2.2). Let $L = \mathbb{P}^*_Z\mathbb{C}^2$. Then

(i) $C(Z) = \{y = 0\}$ if and only if $\operatorname{ord} y > \operatorname{ord} x$. If $C(Z) = \{y = 0\}$, L admits the parametrization

$$\psi(t) = (x(t), y(t), y'(t)/x'(t))$$

on the chart (x, y, p).

(ii) $C(Z) = \{y = 0\}$ and $C(L) = \{x = y = 0\}$ if and only if $\operatorname{ord} x < \operatorname{ord} y < 2 \operatorname{ord} x$;

(iii) $C(Z) = \{y = 0\}$ and $\{x = y = 0\} \not\subset C(L) \subset \{y = 0\}$ if and only if $\operatorname{ord} y \geq 2 \operatorname{ord} x$;

(iv) $C(L) = \{y = p = 0\}$ if and only if $\operatorname{ord} y > 2 \operatorname{ord} x$;

(v) $\operatorname{mult} L \leq \operatorname{mult} Z$. Moreover, $\operatorname{mult} L = \operatorname{mult} Z$ if and only if $\operatorname{ord} y \geq 2 \operatorname{ord} x$.

If L is the germ of a Legendrian curve at $(0, 0; a : b)$, $\pi(L)$ is a germ of a plane curve of $(\mathbb{C}^2, (0, 0))$. Notice that all branches of $\pi(L)$ have the same tangent cone.

If Z is the germ of a plane curve with irreducible tangent cone, the union L of the conormal of the branches of Z is a germ of a Legendrian curve. We say that L is the *conormal* of Z.

If $C(Z)$ has several components, the union of the conormals of the branches of Z is a union of several germs of Legendrian curves.

If L is a germ of Legendrian curve, L is the conormal of $\pi(L)$.

We consider the symplectic form $dp \wedge dx$ in the vector space $\mathbb{C}^2$, with coordinates x, p. We associate to each symplectic linear automorphism

$$(p, x) \mapsto (\alpha p + \beta x, \gamma p + \delta x)$$

of $\mathbb{C}^2$ the contact transformation

$$(x, y, p) \mapsto \big(\gamma p + \delta x, y + \frac{1}{2}\alpha\gamma p^2 + \beta\gamma x p + \frac{1}{2}\beta\delta x^2, \alpha p + \beta x\big). \tag{2.4}$$

We say that (2.4) is a *paraboloidal contact transformation*.

In the case $\alpha = \delta = 0$ and $\gamma = -\beta = 1$, we get the so called *Legendre* transformation

$$\Psi(x, y, p) = (p, y - px, -x).$$

We say that a germ of a Legendrian curve L of $(\mathbb{P}^*\mathbb{C}^2, (0, 0; a : b))$ is in *generic position* if $C(L) \not\supset \pi^{-1}(0, 0)$.

Remark 2.2. Let L be the germ of Legendrian curve on a contact manifold $(X, \mathcal{L})$ at a point o. By Darboux's theorem for contact forms there is a germ of a contact transformation $\chi : (X, o) \to (U, (0, 0, 0))$, where $U = \{\eta \neq 0\}$ is the open subset of $\mathbb{P}^*\mathbb{C}^2$ considered above. Hence $C(\pi(\chi(L))) = \{y = 0\}$. Applying a convenient paraboloidal transformation to $\chi(L)$ we can assume that $C(\chi(L)) = D\chi(o)(C(L)) \not\supset \{x = y = 0\}$. Hence $\chi(L)$ is in generic position. If $C(L)$ is irreducible, we can assume $C(\chi(L)) = \{y = p = 0\}$.

Following the above remark, from now on we will always assume that every Legendrian curve germ is embedded in $(\mathbb{C}^3_{(x,y,p)}, \omega)$, where $\omega = dy - pdx$.

Example 2.3. The plane curve $Z = \{y^2 - x^3 = 0\}$ admits a parametrization $\varphi(t) = (t^2, t^3)$. The conormal L of Z admits the parametrization $\psi(t) = (t^2, t^3, \frac{3}{2}t)$. Hence $C(L) = \pi^{-1}(0,0)$ and L is not in generic position. If χ is the Legendre transformation, $C(\chi(L)) = \{y = p = 0\}$ and $\chi(L)$ is in generic position. Moreover, $\pi(\chi(L))$ is a smooth curve.

Example 2.4. The plane curve $Z = \{(y^2 - x^3)(y^2 - x^5) = 0\}$ admits a parametrization given by

$$\varphi_1(t_1) = ({t_1}^2, {t_1}^3), \qquad \varphi_2(t_2) = ({t_2}^2, {t_2}^5).$$

The conormal L of Z admits the parametrization given by

$$\psi_1(t_1) = \left({t_1}^2, {t_1}^3, \frac{3}{2}t_1\right), \qquad \psi_2(t_2) = \left({t_2}^2, {t_2}^5, \frac{5}{2}{t_2}^3\right).$$

Hence $C(L_1) = \pi^{-1}(0,0)$ and L is not in generic position. If χ is the paraboloidal contact transformation

$$\chi : (x, y, p) \mapsto (x + p, y + \frac{1}{2}p^2, p),$$

then $\chi(L)$ has branches with parametrization given by

$$\chi(\psi_1)(t_1) = \left({t_1}^2 + \frac{3}{2}t_1, {t_1}^3 + \frac{9}{8}{t_1}^2, \frac{3}{2}t_1\right),$$
$$\chi(\psi_2)(t_2) = \left({t_2}^2 + \frac{5}{2}{t_2}^3, {t_2}^5 + \frac{25}{8}{t_2}^6, \frac{5}{2}{t_2}^3\right).$$

Then

$$C(\chi(L_1)) = \{y = p - x = 0\}, \qquad C(\chi(L_2)) = \{y = p = 0\}$$

and L is in generic position.

3. Relative contact geometry

Set $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{z} = (z_1, \ldots, z_m)$. Let I be an ideal of the ring $\mathbb{C}\{\mathbf{z}\}$. Let $\widetilde{I}$ be the ideal of $\mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ generated by I.

Lemma 3.1. (a) *Let $f \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$, $f = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ with $a_\alpha \in \mathbb{C}\{\mathbf{z}\}$. Then $f \in \widetilde{I}$ if and only if $a_\alpha \in I$ for each α.*

(b) *If $f \in \widetilde{I}$, then $\partial_{x_i} f \in \widetilde{I}$ for $1 \leq i \leq n$.*

(c) *Let $a_1, \dots, a_{n-1} \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$. Let $b, \beta_0 \in \widetilde{I}$. Assume that $\partial_{x_n} \beta_0 = 0$. If β is the solution of the Cauchy problem*

$$\partial_{x_n} \beta - \sum_{i=1}^{n-1} a_i \partial_{x_i} \beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\} x_n, \tag{3.1}$$

then $\beta \in \widetilde{I}$.

Proof. There are $g_1, \dots, g_\ell \in \mathbb{C}\{\mathbf{z}\}$ such that $I = (g_1, \dots, g_\ell)$. If $a_\alpha \in I$ for each α, there are $h_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $a_\alpha = \sum_{i=1}^{\ell} h_{i,\alpha} g_i$. Hence

$$f = \sum_{i=1}^{\ell} \Big(\sum_\alpha h_{i,\alpha} \mathbf{x}^\alpha \Big) g_i \in \widetilde{I}.$$

If $f \in \widetilde{I}$, there are $H_i \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ such that $f = \sum_{i=1}^{\ell} H_i g_i$. There are $b_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $H_i = \sum_\alpha b_{i,\alpha} \mathbf{x}^\alpha$. Therefore, $a_\alpha = \sum_{i=1}^{\ell} b_{i,\alpha} g_i \in I$.

We can perform a change of variables that rectifies the vector field

$$\partial_{x_n} - \sum_{i=1}^{n-1} a_i \partial_{x_i},$$

leaving invariant the hypersurface $\{x_n = 0\}$ and reducing the Cauchy problem (3.1) to the Cauchy problem

$$\partial_{x_n} \beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\} x_n.$$

Hence, statements (b) and (c) follow from (a). ∎

Let J be an ideal of $\mathbb{C}\{\mathbf{z}\}$ contained in I. Let X, S and T be analytic spaces with local rings $\mathbb{C}\{\mathbf{x}\}$, $\mathbb{C}\{\mathbf{z}\}/I$ and $\mathbb{C}\{\mathbf{z}\}/J$. Hence, $X \times S$ and $X \times T$ have local rings $\mathcal{O} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{J}$. Let $\mathbf{a_1}, \dots, \mathbf{a_{n-1}}, \mathbf{b} \in \mathcal{O}$ and $\mathbf{g} \in \mathcal{O}/x_n\mathcal{O}$. Let $a_i, b \in \widetilde{\mathcal{O}}$ and $g \in \widetilde{\mathcal{O}}/x_n\widetilde{\mathcal{O}}$ be representatives of $\mathbf{a_i}$, $\mathbf{b}$ and $\mathbf{g}$. Consider the Cauchy problems

$$\partial_{x_n} f + \sum_{i=1}^{n-1} a_i \partial_{x_i} f = b, \qquad f + x_n \widetilde{\mathcal{O}} = g \tag{3.2}$$

and

$$\partial_{x_n} \mathbf{f} + \sum_{i=1}^{n-1} \mathbf{a_i} \partial_{x_i} \mathbf{f} = \mathbf{b}, \qquad \mathbf{f} + x_n \mathcal{O} = \mathbf{g}. \tag{3.3}$$

Theorem 3.2. *The following statements hold.*

(a) *There is one and only one solution of the Cauchy problem* (3.3).

(b) *If f is a solution of* (3.2), $\mathbf{f} = f + \widetilde{I}$ *is a solution of* (3.3).

(c) *If* $\mathbf{f}$ *is a solution of* (3.3) *there is a representative f of* $\mathbf{f}$ *that is a solution of* (3.2).

Proof. By Lemma 3.1, $\partial_{x_i}\widetilde{I} = \widetilde{I}$. Hence (b) holds.

Assume $J = (0)$. The existence and uniqueness of the solution of (3.2) is a special case of the classical Cauchy–Kowalevski Theorem. There is one and only one formal solution of (3.2). Its convergence follows from the majorant method.

The existence of a solution of (3.3) follows from (b).

Let $\mathbf{f_1}, \mathbf{f_2}$ be two solutions of (3.3). Let f_j be a representative of $\mathbf{f_j}$ for $j = 1, 2$. Then

$$\partial_{x_n}(f_2 - f_1) + \sum_{i=1}^{n-1} a_i \partial_{x_i}(f_2 - f_1) \in \widetilde{I}$$

and

$$f_2 - f_1 + x_n\widetilde{\mathcal{O}} \in \widetilde{I} + x_n\widetilde{\mathcal{O}}.$$

By Lemma 3.1, $f_2 - f_1 \in \widetilde{I}$. Therefore $\mathbf{f_1} = \mathbf{f_2}$. This ends the proof of statement (a). Statement (c) follows from statements (a) and (b). ∎

Set $\Omega^1_{X|S} = \bigoplus_{i=1}^n \mathcal{O}dx_i$. We say that the elements of $\Omega^1_{X|S}$ are *germs of relative differential forms* on $X \times S$. The map $d : \mathcal{O} \to \Omega^1_{X|S}$ given by $df = \sum_{i=1}^n \partial_{x_i} f dx_i$ is said to be the *relative differential* of f.

Assume that $\dim X = 3$ and let $\mathcal{L}$ be a contact structure on X. Let $\rho : X \times S \to X$ be the first projection. Let ω be a generator of $\mathcal{L}$. We will denote by $\mathcal{L}_S$ the sub $\mathcal{O}$-module of $\Omega^1_{X|S}$ generated by $\rho^*\omega$. We say that $\mathcal{L}_S$ is a *relative contact structure* of $X \times S$. The pair $(X \times S, \mathcal{L}_S)$ is called a relative contact manifold. We say that an isomorphism of analytic spaces

$$\chi : X \times S \to X \times S \tag{3.4}$$

is a *relative contact transformation* if $\chi(\mathbf{0}, s) = (\mathbf{0}, s)$, $\chi^*\omega \in \mathcal{L}_S$ for each $\omega \in \mathcal{L}_S$ and the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] & X \\ X \times S \arrow[u, hook] \arrow[r, "\chi"] \arrow[d] & X \times S \arrow[u, hook] \arrow[d] \\ S \arrow[r, "\mathrm{id}_S"] & S. \end{tikzcd} \tag{3.5}$$

The demand of the commutativity of diagram (3.5) is a very restrictive condition but these are the only relative contact transformations we will need. We can and will assume that the local ring of X equals $\mathbb{C}\{x, y, p\}$ and that $\mathcal{L}$ is generated by $dy - p\,dx$.

Set $\mathcal{O} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{J}$. Let $\mathfrak{m}_X$ be the maximal ideal of $\mathbb{C}\{x, y, p\}$. Let $\mathfrak{m}$ [$\widetilde{\mathfrak{m}}$] be the maximal ideal of $\mathbb{C}\{\mathbf{z}\}/I$ [$\mathbb{C}\{\mathbf{z}\}/J$]. Let $\mathfrak{n}$ [$\widetilde{\mathfrak{n}}$] be the ideal of $\mathcal{O}$ [$\widetilde{\mathcal{O}}$] generated by $\mathfrak{m}_X\mathfrak{m}$ [$\mathfrak{m}_X\widetilde{\mathfrak{m}}$].

Remark 3.3. If (3.4) is a relative contact transformation, there are $\alpha, \beta, \gamma \in \mathfrak{n}$ such that $\partial_x\beta \in \mathfrak{n}$ and

$$\chi(x, y, p, \mathbf{z}) = (x + \alpha, y + \beta, p + \gamma, \mathbf{z}). \tag{3.6}$$

Theorem 3.4. (a) *Let $\chi : X \times S \to X \times S$ be the relative contact transformation (3.6). There is $\beta_0 \in \mathfrak{n}$ such that $\partial_p\beta_0 = 0$, $\partial_x\beta_0 \in \mathfrak{n}$ and β is the solution of the Cauchy problem*

$$\left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)\frac{\partial\beta}{\partial p} - p\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x} = p\frac{\partial\alpha}{\partial p}, \qquad \beta - \beta_0 \in p\mathcal{O} \tag{3.7}$$

and

$$\gamma = \left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)^{-1}\left(\frac{\partial\beta}{\partial x} + p\left(\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x} - p\frac{\partial\alpha}{\partial y}\right)\right). \tag{3.8}$$

(b) *Given $\alpha, \beta_0 \in \mathfrak{n}$ such that $\partial_p\beta_0 = 0$ and $\partial_x\beta_0 \in \mathfrak{n}$, there is a unique contact transformation χ verifying the conditions of statement (a). We will denote χ by χ_{α,β_0}.*

(c) *Assume S and T are the analytic spaces defined right after Lemma 3.1. Given a relative contact transformation $\widetilde{\chi} : X \times T \to X \times T$ there is one and only one contact transformation $\chi : X \times S \to X \times S$ such that the diagram*

$$\begin{tikzcd} X \times S \arrow[r, "\chi"] \arrow[d, hook] & X \times S \arrow[d, hook] \\ X \times T \arrow[r, "\widetilde{\chi}"] & X \times T \end{tikzcd} \tag{3.9}$$

commutes.

(d) *Given $\alpha, \beta_0 \in \mathfrak{n}$ and $\widetilde{\alpha}, \widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ such that $\partial_p\beta_0 = 0$, $\partial_p\widetilde{\beta}_0 = 0$, $\partial_x\beta_0 \in \mathfrak{n}$, $\partial_x\widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ and $\widetilde{\alpha}, \widetilde{\beta}_0$ are representatives of α, β_0, set $\chi = \chi_{\alpha,\beta_0}$ and $\widetilde{\chi} = \chi_{\widetilde{\alpha},\widetilde{\beta}_0}$. Then diagram (3.9) commutes.*

Proof. Statements (a) and (b) are a relative version of [1, Theorem 3.2] (see also [9]). In [1] we assume $S = \{0\}$. The proof works as long S is smooth. The proof in the singular case depends on the singular variant of the Cauchy–Kowalevski Theorem introduced in Theorem 3.2. Statement (c) follows from statement (b) of Theorem 3.2. Statement (d) follows from statement (c) of Theorem 3.2. ∎

Remark 3.5. (1) The inclusion $S \hookrightarrow T$ is said to be a *small extension* if the surjective map $\mathcal{O}_T \twoheadrightarrow \mathcal{O}_S$ has one-dimensional kernel. If the kernel is generated by ε, we have that, as complex vector spaces, $\mathcal{O}_T = \mathcal{O}_S \oplus \varepsilon\mathbb{C}$. Every extension of Artinian local rings factors through small extensions.

Theorem 3.6. *Let $S \hookrightarrow T$ be a small extension such that $\mathcal{O}_S \cong \mathbb{C}\{\mathbf{z}\}$ and*

$$\mathcal{O}_T \cong \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m) = \mathbb{C}\{\mathbf{z}\} \oplus \mathbb{C}\varepsilon.$$

Assume $\chi : X \times S \to X \times S$ is a relative contact transformation given at the ring level by

$$(x, y, p) \mapsto (H_1, H_2, H_3),$$

$\alpha, \beta_0 \in \mathfrak{m}_X$, such that $\partial_p \beta_0 = 0$ and $\beta_0 \in (x^2, y)$. Then, there are uniquely determined $\beta, \gamma \in \mathfrak{m}_X$ such that $\beta - \beta_0 \in p\mathcal{O}_X$ and $\widetilde{\chi} :\ X \times T \to X \times T$, given by

$$\widetilde{\chi}(x, y, p, \mathbf{z}, \varepsilon) = (H_1 + \varepsilon\alpha, H_2 + \varepsilon\beta, H_3 + \varepsilon\gamma, \mathbf{z}, \varepsilon),$$

is a relative contact transformation extending χ (see diagram (3.9)). Moreover, the Cauchy problem (3.7) for $\widetilde{\chi}$ takes the simplified form

$$\frac{\partial\beta}{\partial p} = p\frac{\partial\alpha}{\partial p}, \qquad \beta - \beta_0 \in \mathbb{C}\{x, y, p\}p \tag{3.10}$$

and

$$\gamma = \frac{\partial\beta}{\partial x} + p\Big(\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x}\Big) - p^2\frac{\partial\alpha}{\partial y}. \tag{3.11}$$

Proof. We have that $\widetilde{\chi}$ is a relative contact transformation if and only if there is $f := f' + \varepsilon f'' \in \mathcal{O}_T\{x, y, p\}$ with $f \notin (x, y, p)\mathcal{O}_T\{x, y, p\}$, $f' \in \mathcal{O}_S\{x, y, p\}$, $f'' \in \mathbb{C}\{x, y, p\} = \mathcal{O}_X$ such that

$$d(H_2 + \varepsilon\beta) - (H_3 + \varepsilon\gamma)d(H_1 + \varepsilon\alpha) = f(dy - p\,dx). \tag{3.12}$$

Since χ is a relative contact transformation we can suppose that

$$dH_2 - H_3\,dH_1 = f'(dy - p\,dx).$$

Using the fact that $\varepsilon\mathfrak{m}_{\mathcal{O}_T} = 0$ we see that (3.12) is equivalent to

$$\frac{\partial\beta}{\partial p} = p\frac{\partial\alpha}{\partial p}, \qquad \gamma = \frac{\partial\beta}{\partial x} + p\Big(\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x}\Big) - p^2\frac{\partial\alpha}{\partial y}, \qquad f'' = \frac{\partial\beta}{\partial y} - p\frac{\partial\alpha}{\partial y}.$$

As $\beta - \beta_0 \in (p)\mathbb{C}\{x, y, p\}$ we have that β, and consequently γ, are completely determined by α and β_0. ∎

Remark 3.7. Set $\alpha = \sum_k \alpha_k p^k$, $\beta = \sum_k \beta_k p^k$, $\gamma = \sum_k \gamma_k p^k$, where $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}\{x, y\}$ for each $k \geq 0$ and $\beta_0 \in (x^2, y)$. Under the assumptions of Theorem 3.6,

(i) $\beta_k = \frac{k-1}{k}\alpha_{k-1}$, for $k \geq 1$.

(ii) Moreover,

$$\gamma_0 = \frac{\partial \beta_0}{\partial x}, \qquad \gamma_1 = \frac{\partial \beta_0}{\partial y} - \frac{\partial \alpha_0}{\partial x},$$
$$\gamma_k = -\frac{1}{k}\frac{\partial \alpha_{k-1}}{\partial x} - \frac{1}{k-1}\frac{\partial \alpha_{k-2}}{\partial y}, \qquad k \geq 2.$$

Since
$$\frac{\partial}{\partial y}\gamma_0 = \frac{\partial}{\partial x}\Big(\frac{\partial \alpha_0}{\partial x} + \gamma_1\Big),$$
β_0 is the solution of the Cauchy problem
$$\frac{\partial \beta_0}{\partial x} = \gamma_0, \qquad \frac{\partial \beta_0}{\partial y} = \frac{\partial \alpha_0}{\partial x} + \gamma_1, \qquad \beta_0 \in (x^2, y).$$

4. Categories of deformations

A category $\mathfrak{C}$ is said to be a *groupoid* if all morphisms of $\mathfrak{C}$ are isomorphisms.

Let $p : \mathfrak{F} \to \mathfrak{C}$ be a functor. Let S be an object of $\mathfrak{C}$. We will denote by $\mathfrak{F}(S)$ the subcategory of $\mathfrak{F}$ given by the following conditions:

- Ψ is an object of $\mathfrak{F}(S)$ if $p(\Psi) = S$.
- χ is a morphism of $\mathfrak{F}(S)$ if $p(\chi) = \mathrm{id}_S$.

Let χ [Ψ] be a morphism [an object] of $\mathfrak{F}$. Let f, [S] be a morphism [an object] of $\mathfrak{C}$. We say that χ [Ψ] is a morphism [an object] of $\mathfrak{F}$ over f [S] if $p(\chi) = f$ [$p(\Psi) = S$].

A morphism $\chi' : \Psi' \to \Psi$ of $\mathfrak{F}$ over $f : S' \to S$ is said to be *cartesian* if for each morphism $\chi'' : \Psi'' \to \Psi$ of $\mathfrak{F}$ over f there is exactly one morphism $\chi : \Psi'' \to \Psi'$ over $\mathrm{id}_{S'}$ such that $\chi' \circ \chi = \chi''$. If the morphism $\chi' : \Psi' \to \Psi$ over f is cartesian, Ψ' is well defined up to a unique isomorphism. We will denote Ψ' by $f^*\Psi$ or $\Psi \times_S S'$.

We say that $\mathfrak{F}$ is a *fibered category* over $\mathfrak{C}$ if

(1) For each morphism $f : S' \to S$ in $\mathfrak{C}$ and each object Ψ of $\mathfrak{F}$ over S there is a morphism $\chi' : \Psi' \to \Psi$ over f that is cartesian.

(2) The composition of cartesian morphisms is cartesian.

A fibered groupoid is a fibered category such that $\mathfrak{F}(S)$ is a groupoid for each $S \in \mathfrak{C}$.

Remark 4.1. If $p : \mathfrak{F} \to \mathfrak{C}$ satisfies (1) and $\mathfrak{F}(S)$ is a groupoid for each object S of $\mathfrak{C}$, then $\mathfrak{F}$ is a fibered groupoid over $\mathfrak{C}$.

Let $\mathfrak{A}n$ be the category of analytic complex space germs. Let 0 denote the complex vector space of dimension 0. Let $p : \mathfrak{F} \to \mathfrak{A}n$ be a fibered category.

Definition 4.2. Let T be an analytic complex space germ. Let ψ $[\Psi]$ be an object of $\mathfrak{F}(0)$ $[\mathfrak{F}(T)]$. We say that Ψ is a *versal deformation* of ψ if given

- a closed embedding $f : T'' \hookrightarrow T'$,
- a morphism of complex analytic space germs $g : T'' \to T$,
- an object Ψ' of $\mathfrak{F}(T')$ such that $f^*\Psi' \cong g^*\Psi$.

There is a morphism of complex analytic space germs $h : T' \to T$ such that

$$h \circ f = g \qquad \text{and} \qquad h^*\Psi \cong \Psi'.$$

If Ψ is versal and for each Ψ' the tangent map $T(h) : T_{T'} \to T_T$ is determined by Ψ', then Ψ is called a *semiuniversal deformation* of ψ.

Let T be a germ of a complex analytic space. Let A be the local ring of T and let $\mathfrak{m}$ be the maximal ideal of A. Let T_n be the complex analytic space with local ring $A/\mathfrak{m}^n$ for each positive integer n. The canonical morphisms

$$A \to A/\mathfrak{m}^n \qquad \text{and} \qquad A/\mathfrak{m}^n \to A/\mathfrak{m}^{n+1}$$

induce morphisms $\alpha_n : T_n \to T$ and $\beta_n : T_{n+1} \to T_n$.

A morphism $f : T'' \to T'$ induces morphisms $f_n : T''_n \to T'_n$ such that the diagram

$$\begin{tikzcd}
T'' \arrow[r, "f"] & T' \\
T''_n \arrow[u, hook, "\alpha''_n"] \arrow[r, "f_n"] & T'_n \arrow[u, hook, "\alpha'_n"'] \\
T''_{n+1} \arrow[u, hook, "\beta''_n"] \arrow[r, "f_{n+1}"] & T'_{n+1} \arrow[u, hook, "\beta'_n"']
\end{tikzcd}$$

commutes.

Definition 4.3. We will follow the terminology of Definition 4.2. Let $g_n = g \circ \alpha''_n$. We say that Ψ is a *formally versal deformation* of ψ if there are morphisms $h_n : T'_n \to T$ such that

$$h_n \circ f_n = g_n, \qquad h_n \circ \beta'_n = h_{n+1} \qquad \text{and} \qquad h_n^*\Psi \cong {\alpha'_n}^*\Psi'.$$

If Ψ is formally versal and for each Ψ' the tangent maps $T(h_n) : T_{T'_n} \to T_T$ are determined by ${\alpha'_n}^*\Psi'$, then Ψ is called a *formally semiuniversal deformation* of ψ.

Theorem 4.4 ([4, Theorem 5.2]). *Let $\mathfrak{F} \to \mathfrak{C}$ be a fibered groupoid. Let $\psi \in \mathfrak{F}(0)$. If there is a versal deformation of ψ, every formally versal [semiuniversal] deformation of ψ is versal [semiuniversal].*

Let Z be a curve of $\mathbb{C}^n$ with irreducible components $Z_1, \dots, Z_r$. Set $\overline{\mathbb{C}} = \bigsqcup_{i=1}^r \overline{C}_i$ where each $\overline{C}_i$ is a copy of $\mathbb{C}$. Let φ_i be a parametrization of Z_i, $1 \le i \le r$. Let $\varphi : \overline{\mathbb{C}} \to \mathbb{C}^n$ be the map such that $\varphi|_{\overline{C}_i} = \varphi_i$, $1 \le i \le r$. We say that φ is the *parametrization* of Z. All the results of this section should be read locally at $0 \in \overline{C}_i$.

Let T be an analytic space. A morphism of analytic spaces $\Phi : \overline{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ is called a *deformation of φ over T* if the diagram

$$\begin{tikzcd}
\overline{\mathbb{C}} \arrow[r, "\varphi"] \arrow[d, hook] & \mathbb{C}^n \arrow[d, hook] \\
\overline{\mathbb{C}} \times T \arrow[r, "\Phi"] \arrow[d] & \mathbb{C}^n \times T \arrow[d] \\
T \arrow[r, "\mathrm{id}_T"] & T
\end{tikzcd}$$

commutes. The analytic space T is called the *base space* of the deformation.

We will denote by Φ_i the composition

$$\overline{C}_i \times T \hookrightarrow \overline{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n, \qquad 1 \le i \le r.$$

The maps Φ_i, $1 \le i \le r$, determine Φ.

Let Φ be a deformation of φ over T. Let $f : T' \to T$ be a morphism of analytic spaces. We will denote by $f^*\Phi$ the deformation of φ over T' given by

$$(f^*\Phi)_i = \Phi_i \circ (\mathrm{id}_{\overline{C}_i} \times f).$$

We say that $f^*\Phi$ is the *pullback* of Φ by f.

Let $\Phi' : \overline{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ be another deformation of φ over T. A morphism from Φ' into Φ is a pair (χ, ξ) where $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ and $\xi : \overline{\mathbb{C}} \times T \to \overline{\mathbb{C}} \times T$ are isomorphisms of analytic spaces such that the diagram

$$\begin{tikzcd}
T & \overline{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi"] & \mathbb{C}^n \times T \arrow[r] & T \\
 & \overline{\mathbb{C}} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\varphi"] & \mathbb{C}^n \times \{0\} \arrow[u, hook] \arrow[d, hook] & \\
T \arrow[uu, "\mathrm{id}_T"] & \overline{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi'"] \arrow[uu, bend left=60, "\xi"] & \mathbb{C}^n \times T \arrow[r] \arrow[uu, bend right=60, "\chi"'] & T \arrow[uu, "\mathrm{id}_T"']
\end{tikzcd}$$

commutes.

Let Φ' be a deformation of φ over S and $f : S \to T$ a morphism of analytic spaces. A *morphism of* Φ' *into* Φ *over* f is a morphism from Φ' into $f^*\Phi$. There is a functor p that associates T to a deformation Ψ over T and f to a morphism of deformations over f.

Given $s \in T$ let Z_s be the curve parametrized by the composition

$$\overline{\mathbb{C}} \times \{s\} \hookrightarrow \overline{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n.$$

We say that Z_s is the *fibre of the deformation* Φ *at the point* s.

Assume $\Phi_i(t_i, \mathbf{s}) = (X_{1,i}(t_i, \mathbf{s}), \dots, X_{n,i}(t_i, \mathbf{s}))$, $1 \le i \le r$. Assume Z_i has multiplicity m_i. We say that Φ_i is equimultiple if $X_{j,i} \in (t^{m_i})$ for each $1 \le i \le r$, $1 \le j \le n$. We say that Φ is *equimultiple* if each Φ_i is equimultiple.

Assume Z is a plane curve. Set

$$\Phi_i(t_i, \mathbf{s}) = \big(X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s})\big), \qquad 1 \le i \le r. \tag{4.1}$$

We will denote by $\mathcal{D}\mathrm{ef}_\varphi$ $[\mathcal{D}\mathrm{ef}_\varphi^{\mathrm{em}}]$ the *category of deformations [equimultiple deformations] of* φ. We say that Φ is an object of $\overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi$ $[\overrightarrow{\overrightarrow{\mathcal{D}\mathrm{ef}}}_\varphi]$ if Φ is equimultiple and $Y_i \in (t_i X_i)$ $[Y_i \in (X_i^2)]$, $1 \le i \le r$.

If T is reduced, $\Phi \in \mathcal{D}\mathrm{ef}_\varphi^{\mathrm{em}}$ $[\overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi, \overrightarrow{\overrightarrow{\mathcal{D}\mathrm{ef}}}_\varphi]$ if and only if all fibres of Φ are equimultiple [have tangent cone $\{y = 0\}$, have tangent cone $\{y = 0\}$ and are in generic position].

Consider in $\mathbb{C}^3$ the contact structure given by the differential form $\omega = dy - p\,dx$. Assume Z is a Legendrian curve parametrized by $\psi : \overline{\mathbb{C}} \to \mathbb{C}^3$. Let Ψ be a deformation of ψ given by

$$\Psi_i(t_i, \mathbf{s}) = \big(X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s}), P_i(t_i, \mathbf{s})\big), \qquad 1 \le i \le r. \tag{4.2}$$

We say that Ψ is a *Legendrian deformation of* ψ if $\Psi_i^*(\rho^*\omega) = 0$ for $1 \le i \le r$. We say that (χ, ξ) is an isomorphism of Legendrian deformations if χ is a relative contact transformation. We will denote by $\widehat{\mathcal{D}\mathrm{ef}}_\psi$ $[\widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}]$ the category of Legendrian [equimultiple Legendrian] deformations of ψ. All deformations are assumed to have trivial section.

Assume that $\psi = \mathcal{C}\mathrm{on}\,\varphi$ parametrizes a germ of a Legendrian curve L, in generic position. If (4.1) defines an object of $\overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi$, setting

$$P_i(t_i, \mathbf{s}) := \partial_{t_i} Y_i(t_i, \mathbf{s}) / \partial_{t_i} X_i(t_i, \mathbf{s}), \qquad 1 \le i \le r,$$

the deformation Ψ given by (4.2) is a Legendrian deformation of ψ. We say that Ψ is the *conormal of* Φ and denote Ψ by $\mathcal{C}\mathrm{on}\,\Phi$. If $\Psi \in \widehat{\mathcal{D}\mathrm{ef}}_\psi$ is given by (4.2), the deformation Φ of φ given by (4.1) is said to be the *plane projection of* Ψ. We will denote Φ by Ψ^π.

We define in this way the functors

$$\mathcal{C}\mathrm{on} : \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi \to \widehat{\mathcal{D}\mathrm{ef}}_\psi, \qquad \pi : \widehat{\mathcal{D}\mathrm{ef}}_\psi \to \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi.$$

Notice that the conormal of the plane projection of a Legendrian deformation always exists and we have that $\mathcal{C}\mathrm{on}(\Psi^\pi) = \Psi$ for each $\Psi \in \widehat{\mathcal{D}\mathrm{ef}}_\psi$ and $(\mathcal{C}\mathrm{on}\,\Phi)^\pi = \Phi$ where $\Phi \in \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi$.

Example 4.5. Set $\varphi(t) = (t, 0)$, $\psi = \mathcal{C}\mathrm{on}\,\varphi$ and $X(t,s) = t$, $Y(t,s) = st$. Then we get $P(t,s) = s$ and although X, Y define an object of $\mathcal{D}\mathrm{ef}_\varphi^{\mathrm{em}}$, its conormal Ψ is not an element of $\widehat{\mathcal{D}\mathrm{ef}}_\psi$, because Ψ is a deformation with section $s \mapsto (0, 0, s, s)$.

Example 4.6. Set $\varphi(t) = (t^2, t^5)$, $X(t,s) = t^2$, $Y(t,s) = t^5 + st^3$. Then we get $2P(t,s) = 5t^3 + 3st$. Although X, Y defines an object of $\overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi$, its conormal is not equimultiple.

Remark 4.7. Under the assumptions above,

$$\mathcal{C}\mathrm{on}(\overset{\twoheadrightarrow}{\mathcal{D}\mathrm{ef}}_\varphi) \subset \widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}} \qquad \text{and} \qquad (\widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}})^\pi \subset \overset{\twoheadrightarrow}{\mathcal{D}\mathrm{ef}}_\varphi.$$

Remark 4.8. If $\mathfrak{C}$ is one of the categories $\widehat{\mathcal{D}\mathrm{ef}}_\psi$, $\widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}$, then $p : \mathfrak{C} \to \mathfrak{A}n$ is a fibered groupoid.

5. Equimultiple versal deformations

For Sophus Lie a contact transformation was a transformation that takes curves into curves, instead of points into points. We can recover the initial point of view. Given a plane curve Z at the origin, with tangent cone $\{y = 0\}$, and a contact transformation χ from a neighbourhood of $(0; dy)$ into itself, χ acts on Z in the following way: $\chi \cdot Z$ is the plane projection of the image by χ of the conormal of Z. We can define in a similar way the action of a relative contact transformation on a deformation of a plane curve Z, obtaining another deformation of Z.

We say that $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}\mathrm{ef}}_\varphi(T)$ is *trivial* (relative to the action of the group of relative contact transformations over T) if there is χ such that $\chi \cdot \Phi := \pi \circ \chi \circ \mathcal{C}\mathrm{on}\,\Phi$ is the constant deformation of φ over T, given by

$$(t_i, \mathbf{s}) \mapsto \varphi_i(t_i), \qquad i = 1, \dots, r.$$

Let Z be the germ of a plane curve parametrized by $\varphi : \overline{\mathbb{C}} \to \mathbb{C}^2$. In the following, we will identify each ideal of $\mathcal{O}_Z$ with its image by $\varphi^* : \mathcal{O}_Z \to \mathcal{O}_{\overline{\mathbb{C}}}$. Hence,

$$\mathcal{O}_Z = \mathbb{C}\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \right\} \subset \bigoplus_{i=1}^r \mathbb{C}\{t_i\} = \mathcal{O}_{\overline{\mathbb{C}}}.$$

Set $\dot{\mathbf{x}} = [\dot{x}_1, \dots, \dot{x}_r]^t$, where $\dot{x}_i$ is the derivative of x_i with respect to t_i, $1 \leq i \leq r$. Let

$$\dot{\varphi} := \dot{\mathbf{x}} \frac{\partial}{\partial x} + \dot{\mathbf{y}} \frac{\partial}{\partial y}$$

be an element of the free $\mathcal{O}_{\overline{\mathbb{C}}}$-module

$$\mathcal{O}_{\overline{\mathbb{C}}} \frac{\partial}{\partial x} \oplus \mathcal{O}_{\overline{\mathbb{C}}} \frac{\partial}{\partial y}. \tag{5.1}$$

Notice that (5.1) has a structure of $\mathcal{O}_Z$-module induced by φ^*.

Let m_i be the multiplicity of Z_i, $1 \leq i \leq r$. Consider the $\mathcal{O}_{\overline{\mathbb{C}}}$-module

$$\left(\bigoplus_{i=1}^{r} t_i^{m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial x} \right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial y} \right). \tag{5.2}$$

Let $\mathfrak{m}_{\overline{\mathbb{C}}} \dot{\varphi}$ be the sub $\mathcal{O}_{\overline{\mathbb{C}}}$-module of (5.2) generated by

$$(a_1, \dots, a_r) \left(\dot{\mathbf{x}} \frac{\partial}{\partial x} + \dot{\mathbf{y}} \frac{\partial}{\partial y} \right),$$

where $a_i \in t_i \mathbb{C}\{t_i\}$, $1 \leq i \leq r$. For $i = 1, \dots, r$ set $p_i = \dot{y}_i / \dot{x}_i$. For each $k \geq 0$ set

$$\mathbf{p}^k = [p_1^k, \dots, p_r^k]^t.$$

Let $\widehat{I}$ be the sub $\mathcal{O}_Z$-module of (5.2) generated by

$$\mathbf{p}^k \frac{\partial}{\partial x} + \frac{k}{k+1} \mathbf{p}^{k+1} \frac{\partial}{\partial y}, \qquad k \geq 1.$$

Set

$$\widehat{M}_\varphi = \frac{\left(\bigoplus_{i=1}^{r} t_i^{m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial x} \right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial y} \right)}{\mathfrak{m}_{\overline{\mathbb{C}}} \dot{\varphi} + (x, y) \frac{\partial}{\partial x} \oplus (x^2, y) \frac{\partial}{\partial y} + \widehat{I}}.$$

Given a category $\mathfrak{C}$ we will denote by $\underline{\mathfrak{C}}$ the set of isomorphism classes of elements of $\mathfrak{C}$.

Theorem 5.1. *Let ψ be the parametrization of a germ of a Legendrian curve L of a contact manifold X. Let $\chi : X \to \mathbb{C}^3$ be a contact transformation such that $\chi(L)$ is in generic position. Let φ be the plane projection of $\chi \circ \psi$. Then there is a canonical isomorphism*

$$\underline{\widehat{\mathfrak{Def}}_\psi^{\mathrm{em}}}(T_\varepsilon) \xrightarrow{\sim} \widehat{M}_\varphi.$$

Proof. Let $\Psi \in \widehat{\mathfrak{Def}}_\psi^{\mathrm{em}}(T_\varepsilon)$. Then, Ψ is the conormal of its projection $\Phi \in \overrightarrow{\mathfrak{Def}}_\varphi(T_\varepsilon)$ (see Remark 4.7). Moreover, Ψ is given by

$$\Psi_i(t_i, \varepsilon) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}$, $\operatorname{ord} a_i \geq m_i$, $\operatorname{ord} b_i \geq 2m_i$, $i = 1, \dots, r$. The deformation Ψ is trivial if and only if Φ is trivial for the action of the relative contact transformations. Moreover, Φ is trivial if and only if there are

$$\xi_i(t_i) = \tilde{t}_i = t_i + \varepsilon h_i,$$
$$\chi(x, y, p, \varepsilon) = (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon),$$

such that χ is a relative contact transformation, where $\alpha, \beta, \gamma \in (x, y, p)\mathbb{C}\{x, y, p\}$, ξ_i is an isomorphism, where $h_i \in t_i\mathbb{C}\{t_i\}$, $1 \leq i \leq r$, and

$$x_i(t_i) + \varepsilon a_i(t_i) = x_i(\tilde{t}_i) + \varepsilon\alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$
$$y_i(t_i) + \varepsilon b_i(t_i) = y_i(\tilde{t}_i) + \varepsilon\beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$

for $i = 1, \dots, r$. By Taylor's formula $x_i(\tilde{t}_i) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$, $y_i(\tilde{t}_i) = y_i(t_i) + \varepsilon\dot{y}_i(t_i)h_i(t_i)$ and

$$\varepsilon\alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon\alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$
$$\varepsilon\beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon\beta(x_i(t_i), y_i(t_i), p_i(t_i)),$$

for $i = 1, \dots, r$. Hence Φ is trivialized by χ if and only if

$$a_i(t_i) = \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{5.3}$$
$$b_i(t_i) = \dot{y}_i(t_i)h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{5.4}$$

for $i = 1, \dots, r$. By Remark 3.7 (i), (5.3) and (5.4) are equivalent to the condition

$$\mathbf{a}\frac{\partial}{\partial x} + \mathbf{b}\frac{\partial}{\partial y} \in \mathfrak{m}_{\overline{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x^2, y)\frac{\partial}{\partial y} + \widehat{I}. \qquad \blacksquare$$

Set

$$M_\varphi = \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\overline{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x, y)\frac{\partial}{\partial y}},$$
$$\overrightarrow{M}_\varphi = \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\overline{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x^2, y)\frac{\partial}{\partial y}}.$$

By [5, Proposition 2.27],

$$\underline{\mathfrak{Def}}^{\mathrm{em}}_\varphi(T_\varepsilon) \cong M_\varphi.$$

A similar argument shows that

$$\underline{\overrightarrow{\mathfrak{Def}}}_\varphi(T_\varepsilon) \cong \overrightarrow{M}_\varphi.$$

We have linear maps

$$M_\varphi \overset{\iota}{\hookrightarrow} \overrightarrow{M}_\varphi \twoheadrightarrow \widehat{M}_\varphi. \tag{5.5}$$

Theorem 5.2 ([5, II, Proposition 2.27 (3)]). *Set* $k = \dim M_\varphi$. *Let*

$$\mathbf{a}^j, \mathbf{b}^j \in \bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}, \qquad 1 \le j \le k.$$

If

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{b}^j \frac{\partial}{\partial y} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b_1^j \\ \vdots \\ b_r^j \end{bmatrix} \frac{\partial}{\partial y}, \qquad 1 \le j \le k, \tag{5.6}$$

represents a basis of M_φ, *the deformation* $\Phi : \overline{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ *given by*

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j, \qquad Y_i(t_i, \mathbf{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j, \tag{5.7}$$

$i = 1, \ldots, r$, *is a semiuniversal deformation of* φ *in* $\mathcal{D}\mathrm{ef}_\varphi^{\mathrm{em}}$.

Lemma 5.3. *Set* $\overrightarrow{k} = \dim \overrightarrow{M}_\varphi$. *Let* $\mathbf{a}^j \in \bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \le j \le \overrightarrow{k}$. *If* (5.6) *represents a basis of* $\overrightarrow{M}_\varphi$, *the deformation* $\overrightarrow{\Phi}$ *given by* (5.7), $1 \le i \le r$, *is a semiuniversal deformation of* φ *in* $\overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi$. *Moreover,* $\mathcal{C}\mathrm{on}\, \overrightarrow{\Phi}$ *is a versal deformation of* $\psi = \mathcal{C}\mathrm{on}\, \varphi$ *in* $\widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}$.

Proof. We only show the completeness of $\overrightarrow{\Phi}$ and $\mathcal{C}\mathrm{on}\, \overrightarrow{\Phi}$. Since the linear inclusion map $\imath$ referred in (5.5) is injective, the deformation $\overrightarrow{\Phi}$ is the restriction to $\overrightarrow{M}_\varphi$ of the deformation Φ introduced in Theorem 5.2. Let $\Phi_0 \in \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi(T)$. Since $\Phi_0 \in \mathcal{D}\mathrm{ef}_\varphi^{\mathrm{em}}(T)$, there is a morphism of analytic spaces $f : T \to M_\varphi$ such that $\Phi_0 \cong f^*\Phi$. Since $\Phi_0 \in \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi(T)$, $f(T) \subset \overrightarrow{M}_\varphi$. Hence $f^*\overrightarrow{\Phi} = f^*\Phi$.

If $\Psi \in \widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}(T)$, then $\Psi^\pi \in \overrightarrow{\mathcal{D}\mathrm{ef}}_\varphi(T)$. Hence there is $f : T \to \overrightarrow{M}_\varphi$ such that $\Psi^\pi \cong f^*\overrightarrow{\Phi}$. Therefore, $\Psi = \mathcal{C}\mathrm{on}\, \Psi^\pi \cong \mathcal{C}\mathrm{on}\, f^*\overrightarrow{\Phi} = f^*\mathcal{C}\mathrm{on}\, \overrightarrow{\Phi}$. ∎

Theorem 5.4. *Let* $\mathbf{a}^j \in \bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \le j \le \ell$. *Assume that* (5.6) *represents a basis [a system of generators] of* $\widehat{M}_\varphi$. *Let* Φ *be the deformation given by* (5.7), $1 \le i \le r$. *Then* $\mathcal{C}\mathrm{on}\, \Phi$ *is a semiuniversal [versal] deformation of* $\psi = \mathcal{C}\mathrm{on}\, \varphi$ *in* $\widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}$.

Proof. By Theorem 4.4 and Lemma 5.3 it is enough to show that $\mathcal{C}\mathrm{on}\, \Phi$ is formally semiuniversal [versal].

Let $\imath : T' \hookrightarrow T$ be a small extension. Let $\Psi \in \widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}(T)$. Set $\Psi' = \imath^*\Psi$. Let $\eta' : T' \to \mathbb{C}^\ell$ be a morphism of complex analytic spaces. Assume that (χ', ξ') define an isomorphism

$$\eta'^* \mathcal{C}\mathrm{on}\, \Phi \cong \Psi'.$$

We need to find $\eta : T \to \mathbb{C}^\ell$ and χ, ξ such that $\eta' = \eta \circ \imath$ and χ, ξ define an isomorphism

$$\eta^* \mathcal{C}\mathrm{on}\, \Phi \cong \Psi$$

that extends (χ', ξ'). Let A $[A']$ be the local ring of T $[T']$. Let δ be the generator of $\mathrm{Ker}(A \twoheadrightarrow A')$. We can assume $A' \cong \mathbb{C}\{\mathbf{z}\}/I$, where $\mathbf{z} = (z_1, \ldots, z_m)$. Set

$$\widetilde{A}' = \mathbb{C}\{\mathbf{z}\} \quad \text{and} \quad \widetilde{A} = \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).$$

Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\mathfrak{m}_A \delta = 0$ and $\delta \in \mathfrak{m}_A$, there is a morphism of local analytic algebras from $\widetilde{A}$ onto A that takes ε into δ such that the diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \widetilde{A}' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd} \tag{5.8}$$

commutes. Assume $\tilde{t}$ $[\tilde{t}']$ has local ring $\widetilde{A}$ $[\widetilde{A}']$. We also denote by $\imath$ the morphism $\tilde{t}' \hookrightarrow \tilde{t}$. We denote by κ the morphisms $T \hookrightarrow \tilde{t}$ and $T' \hookrightarrow \tilde{t}'$. Let $\widetilde{\Psi} \in \widehat{\mathcal{D}\mathrm{ef}}^{\mathrm{em}}_{\psi}(\tilde{t})$ be a lifting of Ψ.

We fix a linear map $\sigma : A' \hookrightarrow \widetilde{A}'$ such that $\kappa^* \sigma = \mathrm{id}_{A'}$. Set $\widetilde{\chi}' = \chi_{\sigma(\alpha), \sigma(\beta_0)}$, where $\chi' = \chi_{\alpha, \beta_0}$. Define $\widetilde{\eta}'$ by $\widetilde{\eta}'^* s_i = \sigma(\eta'^* s_i)$, $i = 1, \ldots, \ell$. Let $\widetilde{\xi}'$ be the lifting of ξ' determined by σ. Then

$$\widetilde{\Psi}' := \widetilde{\chi}'^{-1} \circ \widetilde{\eta}'^* \mathcal{C}\mathrm{on}\, \Phi \circ \widetilde{\xi}'^{-1}$$

is a lifting of Ψ' and

$$\widetilde{\chi}' \circ \widetilde{\Psi}' \circ \widetilde{\xi}' = \widetilde{\eta}'^* \mathcal{C}\mathrm{on}\, \Phi. \tag{5.9}$$

By Theorem 3.4 it is enough to find liftings $\widetilde{\chi}, \widetilde{\xi}, \widetilde{\eta}$ of $\widetilde{\chi}', \widetilde{\xi}', \widetilde{\eta}'$ such that

$$\widetilde{\chi} \cdot \widetilde{\Psi}^{\pi} \circ \widetilde{\xi} = \widetilde{\eta}^* \Phi$$

in order to prove the theorem.

Consider the commutative diagram of full arrows

$$\begin{tikzcd} \overline{\mathbb{C}} \times \tilde{t}' \arrow[r, hook] \arrow[d, "\widetilde{\Psi}'"] & \overline{\mathbb{C}} \times \tilde{t} \arrow[r, dotted] \arrow[d, "\widetilde{\Psi}"] & \overline{\mathbb{C}} \times \mathbb{C}^\ell \arrow[d, "\mathcal{C}\mathrm{on}\, \Phi"] \\ \mathbb{C}^3 \times \tilde{t}' \arrow[r, hook] \arrow[d, "pr"] & \mathbb{C}^3 \times \tilde{t} \arrow[r, dotted] \arrow[d, "pr"] & \mathbb{C}^3 \times \mathbb{C}^\ell \arrow[d] \\ \tilde{t}' \arrow[r, hook] \arrow[rr, bend right, "\widetilde{\eta}'"'] & \tilde{t} \arrow[r, dotted, "\widetilde{\eta}"] & \mathbb{C}^\ell . \end{tikzcd}$$

If $\mathcal{C}$on Φ is given by

$$X_i(t_i, \mathbf{s}),\ Y_i(t_i, \mathbf{s}),\ P_i(t_i, \mathbf{s})\ \in \mathbb{C}\{\mathbf{s}, t_i\},$$

then $\widetilde{\eta}'^* \mathcal{C}$on Φ is given by

$$X_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ Y_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ P_i(t_i, \widetilde{\eta}'(\mathbf{z}))\ \in \widetilde{A}'\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\}$$

for $i = 1, \ldots, r$. Suppose that $\widetilde{\Psi}'$ is given by

$$U_i'(t_i, \mathbf{z}),\ V_i'(t_i, \mathbf{z}),\ W_i'(t_i, \mathbf{z})\ \in \mathbb{C}\{\mathbf{z}, t_i\}.$$

Then, $\widetilde{\Psi}$ must be given by

$$U_i = U_i' + \varepsilon u_i,\ V_i = V_i' + \varepsilon v_i,\ W_i = W_i' + \varepsilon w_i\ \in \widetilde{A}\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\} \oplus \varepsilon\mathbb{C}\{t_i\}$$

with $u_i, v_i, w_i \in \mathbb{C}\{t_i\}$ and $i = 1, \ldots, r$. By definition of deformation we have that, for each i,

$$(U_i, V_i, W_i) = (x_i(t_i), y_i(t_i), p_i(t_i)) \bmod \mathfrak{m}_{\widetilde{A}}.$$

Suppose $\widetilde{\eta}' : \widetilde{t}' \to \mathbb{C}^\ell$ is given by $(\widetilde{\eta}'_1, \ldots, \widetilde{\eta}'_\ell)$, with $\widetilde{\eta}'_i \in \mathbb{C}\{\mathbf{z}\}$. Then $\widetilde{\eta}$ must be given by $\widetilde{\eta} = \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0$ for some $\widetilde{\eta}^0 = (\widetilde{\eta}^0_1, \ldots, \widetilde{\eta}^0_\ell) \in \mathbb{C}^\ell$. Suppose that $\widetilde{\chi}' : \mathbb{C}^3 \times \widetilde{t}' \to \mathbb{C}^3 \times \widetilde{T}'$ is given at the ring level by

$$(x, y, p) \mapsto (H_1', H_2', H_3'),$$

such that $H' = \mathrm{id} \bmod \mathfrak{m}_{\widetilde{A}'}$ with $H_i' \in (x, y, p)A'\{x, y, p\}$. Let $\widetilde{\xi}' : \overline{\mathbb{C}} \times \widetilde{t}' \to \overline{\mathbb{C}} \times \widetilde{t}'$ be an automorphism given at the ring level by

$$t_i \mapsto h_i',$$

such that $h' = \mathrm{id} \bmod \mathfrak{m}_{\widetilde{A}'}$ with $h_i' \in (t_i)\mathbb{C}\{\mathbf{z}, t_i\}$.

Then, it follows from (5.9) that

$$\begin{aligned} X_i(t_i, \widetilde{\eta}') &= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\ Y_i(t_i, \widetilde{\eta}') &= H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\ P_i(t_i, \widetilde{\eta}') &= H_3'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')). \end{aligned} \tag{5.10}$$

Now, $\widetilde{\eta}'$ must be extended to $\widetilde{\eta}$ such that the first two previous equations extend as well. That is, we must have

$$\begin{aligned} X_i(t_i, \widetilde{\eta}) &= (H_1' + \varepsilon\alpha)(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0), W_i(h_i' + \varepsilon h_i^0)), \\ Y_i(t_i, \widetilde{\eta}) &= (H_2' + \varepsilon\beta)(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0), W_i(h_i' + \varepsilon h_i^0)), \end{aligned} \tag{5.11}$$

with $\alpha, \beta \in (x, y, p)\mathbb{C}\{x, y, p\}$, and $h_i^0 \in (t_i)\mathbb{C}\{t_i\}$ such that

$$(x, y, p) \mapsto (H_1' + \varepsilon\alpha, H_2' + \varepsilon\beta, H_3' + \varepsilon\gamma)$$

gives a relative contact transformation over $\tilde{t}$ for some $\gamma \in (x, y, p)\mathbb{C}\{x, y, p\}$. The existence of this extended relative contact transformation is guaranteed by Theorem 3.6. Moreover, again by Theorem 3.6, this extension depends only on the choices of α and β_0. So, we need only to find α, β_0, $\widetilde{\eta}^0$ and h_i^0 such that (5.11) holds. Using Taylor's formula and $\varepsilon^2 = 0$ we see that

$$\begin{aligned} X_i(t_i, \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0) &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, \widetilde{\eta}')\widetilde{\eta}_j^0 \\ (\varepsilon\mathfrak{m}_{\widetilde{A}} = 0) \qquad &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, 0)\widetilde{\eta}_j^0, \\ Y_i(t_i, \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0) &= Y_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial Y_i}{\partial s_j}(t_i, 0)\widetilde{\eta}_j^0. \end{aligned} \tag{5.12}$$

Again, by Taylor's formula and noticing that $\varepsilon\mathfrak{m}_{\widetilde{A}} = 0$, $\varepsilon\mathfrak{m}_{\widetilde{A}'} = 0$ in $\widetilde{A}$, $h' = \mathrm{id} \bmod \mathfrak{m}_{\widetilde{A}'}$ and $(U_i, V_i) = (x_i(t_i), y_i(t_i)) \bmod \mathfrak{m}_{\widetilde{A}}$ we see that

$$\begin{aligned} U_i(h_i' + \varepsilon h_i^0) &= U_i(h_i') + \varepsilon \dot{U}_i(h_i')h_i^0 \\ &= U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), \\ V_i(h_i' + \varepsilon h_i^0) &= V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), \end{aligned} \tag{5.13}$$

where U_i, V_i were defined in the previous page. Now, $H' = \mathrm{id} \bmod \mathfrak{m}_{\widetilde{A}'}$, so

$$\frac{\partial H_1'}{\partial x} = 1 \bmod \mathfrak{m}_{\widetilde{A}'}, \qquad \frac{\partial H_1'}{\partial y}, \frac{\partial H_1'}{\partial p} \in \mathfrak{m}_{\widetilde{A}'}\widetilde{A}'\{x, y, p\}.$$

In particular,

$$\varepsilon\frac{\partial H_1'}{\partial y} = \varepsilon\frac{\partial H_1'}{\partial p} = 0.$$

By this and arguing as in (5.12) and (5.13) we see that

$$\begin{aligned} &(H_1' + \varepsilon\alpha)\big(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), \\ &\qquad\qquad W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)\big) \\ &\quad = H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) \\ &\qquad + \varepsilon\big(\alpha(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + 1(\dot{x}_i h_i^0 + u_i)\big) \\ &\quad = H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\alpha(x_i, y_i, p_i) + \dot{x}_i h_i^0 + u_i), (H_2' + \varepsilon\beta) \\ &\qquad \cdot \big(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)\big) \\ &\quad = H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\beta(x_i, y_i, p_i) + \dot{y}_i h_i^0 + v_i). \end{aligned}$$

Substituting this in (5.11) and using (5.10) and (5.12) we see that we have to find $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \dots, \widetilde{\eta}_\ell^0) \in \mathbb{C}^\ell$ and h_i^0 such that

$$(u_i(t_i), v_i(t_i)) = \sum_{j=1}^{\ell} \widetilde{\eta}_j^0 \left(\frac{\partial X_i}{\partial s_j}(t_i, 0), \frac{\partial Y_i}{\partial s_j}(t_i, 0) \right) - h_i^0(t_i)(\dot{x}_i(t_i), \dot{y}_i(t_i))$$
$$- \big(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))\big). \qquad (5.14)$$

Note that, because of Remark 3.7 (i),

$$\big(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))\big) \in \widehat{I}$$

for each i. Also note that $\widetilde{\Psi} \in \widehat{\mathcal{D}\mathrm{ef}}_\psi^{\mathrm{em}}(\tilde{t})$ means that $u_i \in t_i^{m_i}\mathbb{C}\{t_i\}$, $v_i \in t_i^{2m_i}\mathbb{C}\{t_i\}$. Then, if the vectors

$$\left(\frac{\partial X_1}{\partial s_j}(t_1, 0), \dots, \frac{\partial X_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial x} + \left(\frac{\partial Y_1}{\partial s_j}(t_1, 0), \dots, \frac{\partial Y_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial y}$$
$$= (a_1^j(t_1), \dots, a_r^j(t_r)) \frac{\partial}{\partial x} + (b_1^j(t_1), \dots, b_r^j(t_r)) \frac{\partial}{\partial y}, \qquad j = 1, \dots, \ell$$

form a basis of [generate] $\widehat{M}_\varphi$, we can solve (5.14) with unique $\widetilde{\eta}_1^0, \dots, \widetilde{\eta}_\ell^0$ [respectively, solve] for all $i = 1, \dots, r$. This implies that the conormal of Φ is a formally semiuniversal [respectively, versal] equimultiple deformation of ψ over $\mathbb{C}^\ell$. ■

6. Versal deformations

Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$. We will denote by $\int f dx_i$ the solution of the Cauchy problem

$$\frac{\partial g}{\partial x_i} = f, \qquad g \in (x_i).$$

Let ψ be a Legendrian curve with parametrization given by

$$t_i \mapsto (x_i(t_i), y_i(t_i), p_i(t_i)), \qquad i = 1, \dots, r. \qquad (6.1)$$

We will say that the *fake plane projection* of (6.1) is the plane curve σ with parametrization given by

$$t_i \mapsto (x_i(t_i), p_i(t_i)), \qquad i = 1, \dots, r. \qquad (6.2)$$

We will denote σ by ψ^{π_f}.

Given a plane curve σ with parametrization (6.2), we will say that the *fake conormal* of σ is the Legendrian curve ψ with parametrization (6.1), where

$$y_i(t_i) = \int p_i(t_i)\dot{x}_i(t_i)dt_i.$$

We will denote ψ by $\mathscr{C}\mathrm{on}_f\,\sigma$. Applying the construction above to each fibre of a deformation we obtain functors

$$\pi_f : \widehat{\mathscr{D}\mathrm{ef}}_\psi \to \mathscr{D}\mathrm{ef}_\sigma, \qquad \mathscr{C}\mathrm{on}_f : \mathscr{D}\mathrm{ef}_\sigma \to \widehat{\mathscr{D}\mathrm{ef}}_\psi.$$

Notice that

$$\mathscr{C}\mathrm{on}_f(\Psi^{\pi_f}) = \Psi, \qquad (\mathscr{C}\mathrm{on}_f(\Sigma))^{\pi_f} = \Sigma \tag{6.3}$$

for each $\Psi \in \widehat{\mathscr{D}\mathrm{ef}}_\psi$ and each $\Sigma \in \mathscr{D}\mathrm{ef}_\sigma$.

Let ψ be the parametrization of a Legendrian curve given by (6.1). Let σ be the fake plane projection of ψ. Set $\dot\sigma := \dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{p}}\frac{\partial}{\partial p}$. Let I^f be the linear subspace of

$$\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p} = \left(\bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial p}\right)$$

generated by

$$\alpha_0\frac{\partial}{\partial x} - \left(\frac{\partial\alpha_0}{\partial x} + \frac{\partial\alpha_0}{\partial y}\mathbf{p}\right)\mathbf{p}\frac{\partial}{\partial p}, \qquad \left(\frac{\partial\beta_0}{\partial x} + \frac{\partial\beta_0}{\partial y}\mathbf{p}\right)\frac{\partial}{\partial p},$$

and

$$\alpha_k\mathbf{p}^k\frac{\partial}{\partial x} - \frac{1}{k+1}\left(\frac{\partial\alpha_k}{\partial x}\mathbf{p}^{k+1} + \frac{\partial\alpha_k}{\partial y}\mathbf{p}^{k+2}\right)\frac{\partial}{\partial p}, \qquad k \geq 1,$$

where $\alpha_k \in (x, y)$, $\beta_0 \in (x^2, y)$ for each $k \geq 0$. Set

$$M_\sigma^f = \frac{\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p}}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot\sigma + I^f}.$$

Theorem 6.1. *Assuming the notations above,* $\underline{\widehat{\mathscr{D}\mathrm{ef}}}_\psi(T_\varepsilon) \cong M_\sigma^f$.

Proof. Let $\Psi \in \widehat{\mathscr{D}\mathrm{ef}}_\psi(T_\varepsilon)$ be given by

$$\Psi_i(t_i, \varepsilon) = (X_i, Y_i, P_i) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where x_i, y_i, p_i define the parametrization ψ_i, as well as $a_i, b_i, c_i \in \mathbb{C}\{t_i\}t_i$ and $Y_i = \int P_i\partial_{t_i}X_i dt_i$, $i = 1, \dots, r$. Hence

$$b_i = \int(\dot x_i c_i + \dot a_i p_i)dt_i, \qquad i = 1, \dots, r.$$

By (6.3), Ψ is trivial if and only if there an isomorphism $\xi : \bar{\mathbb{C}} \times T_\varepsilon \to \bar{\mathbb{C}} \times T_\varepsilon$ given by

$$t_i \to \tilde t_i = t_i + \varepsilon h_i, \qquad h_i \in \mathbb{C}\{t_i\}t_i, \qquad i = 1, \dots, r,$$

and a relative contact transformation $\chi : \mathbb{C}^3 \times T_\varepsilon \to \mathbb{C}^3 \times T_\varepsilon$ given by

$$(x, y, p, \varepsilon) \mapsto (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon)$$

such that

$$\begin{aligned} X_i &= x_i(\tilde{t}_i) + \varepsilon\alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)), \\ P_i &= p_i(\tilde{t}_i) + \varepsilon\gamma(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)), \end{aligned}$$

$i = 1, \dots, r$. Following the argument of the proof of Theorem 5.1, Ψ^{π_f} is trivial if and only if

$$\begin{aligned} a_i(t_i) &= \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \\ c_i(t_i) &= \dot{p}_i(t_i)h_i(t_i) + \gamma(x_i(t_i), y_i(t_i), p_i(t_i)), \end{aligned}$$

$i = 1, \dots, r$. The result follows from Remark 3.7 (ii). ■

Lemma 6.2. *Let ψ be the parametrization of a Legendrian curve. Let Φ be a semiuniversal deformation in $\mathfrak{D}\mathrm{ef}_\sigma$ of the fake plane projection σ of ψ. Then $\mathfrak{C}\mathrm{on}_f\ \Phi$ is a versal deformation of ψ in $\widehat{\mathfrak{D}\mathrm{ef}}_\psi$.*

Proof. It follows from the argument of Lemma 5.3. ■

Theorem 6.3. *Let $\mathbf{a}^j, \mathbf{c}^j \in \mathfrak{m}_{\bar{\mathbb{C}}}$ such that*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{c}^j \frac{\partial}{\partial p} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} c_1^j \\ \vdots \\ c_r^j \end{bmatrix} \frac{\partial}{\partial p}, \tag{6.4}$$

$1 \le j \le \ell$, represents a basis [a system of generators] of M_σ^f. Let $\Phi \in \mathfrak{D}\mathrm{ef}_\sigma$ be given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{\ell} a_i^j(t_i)s_j, \qquad P_i(t_i, \mathbf{s}) = p_i(t_i) + \sum_{j=1}^{\ell} c_i^j(t_i)s_j, \tag{6.5}$$

$i = 1, \dots, r$. Then $\mathfrak{C}\mathrm{on}_f\ \Phi$ is a semiuniversal [versal] deformation of ψ in $\widehat{\mathfrak{D}\mathrm{ef}}_\psi$.

Proof. It follows from the argument of Theorem 5.4 and Remark 3.7 (ii). ■

Remark 6.4. The category of [equimultiple] deformations of parametrizations of Legendrian curves is unobstructed. In particular, the base space of any versal deformation is smooth.

7. Examples

Example 7.1. Let $\varphi(t) = (t^3, t^{10})$, $\psi(t) = (t^3, t^{10}, \frac{10}{3}t^7)$, $\sigma(t) = (t^3, \frac{10}{3}t^7)$. The deformations given by

$$\begin{aligned} &X(t,\mathbf{s}) = t^3, \\ &Y(t,\mathbf{s}) = s_1t^4 + s_2t^5 + s_3t^7 + s_4t^8 + t^{10} + s_5t^{11} + s_6t^{14} \end{aligned} \tag{7.1}$$

$$\begin{aligned} &X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3, \\ &Y(t,\mathbf{s}) = s_3t + s_4t^2 + s_5t^4 + s_6t^5 + s_7t^7 + s_8t^8 + t^{10} + s_9t^{11} + s_{10}t^{14} \end{aligned} \tag{7.2}$$

are respectively

- an equimultiple semiuniversal deformation, see (7.1);
- a semiuniversal deformation, see (7.2),

of φ. The conormal of the deformation given by

$$X(t,\mathbf{s}) = t^3, \qquad Y(t,\mathbf{s}) = s_1t^7 + s_2t^8 + t^{10} + s_3t^{11}$$

is an equimultiple semiuniversal deformation of ψ. The fake conormal of the deformation given by

$$X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3, \qquad P(t,\mathbf{s}) = s_3t + s_4t^2 + s_5t^4 + s_6t^5 + \frac{10}{3}t^7 + s_7t^8$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$\begin{aligned} X(t,\mathbf{s}) &= s_1t + s_2t^2 + t^3, \\ Y(t,\mathbf{s}) &= \alpha_2t^2 + \alpha_3t^3 + \alpha_4t^4 + \alpha_5t^5 + \alpha_6t^6 \\ &\quad + \alpha_7t^7 + \alpha_8t^8 + \alpha_9t^9 + \alpha_{10}t^{10} + \alpha_{11}t^{11} \end{aligned}$$

with

$$\begin{aligned} &\alpha_2 = \frac{s_1s_3}{2}, && \alpha_3 = \frac{s_1s_4 + 2s_2s_3}{3}, && \alpha_4 = \frac{3s_3 + 2s_2s_4}{4}, \\ &\alpha_5 = \frac{3s_4 + s_1s_5}{5}, && \alpha_6 = \frac{2s_2s_5 + s_1s_6}{6}, && \alpha_7 = \frac{3s_5 + 2s_2s_6}{7}, \\ &\alpha_8 = \frac{10s_1 + 9s_6}{24}, && \alpha_9 = \frac{3s_1s_7 + 20s_2}{27}, && \alpha_{10} = 1 + \frac{s_2s_7}{5}, \\ &\alpha_{11} = \frac{3s_7}{11}, \end{aligned}$$

is a semiuniversal deformation of ψ.

Example 7.2. Let $Z = \{(x,y) \in \mathbb{C}^2 : (y^2 - x^5)(y^2 - x^7) = 0\}$. Consider the parametrization φ of Z given by

$$x_1(t_1) = t_1^2, \quad y_1(t_1) = t_1^5, \qquad x_2(t_2) = t_2^2, \quad y_2(t_2) = t_2^7.$$

Let σ be the fake projection of the conormal of φ given by

$$x_1(t_1) = t_1^2, \quad p_1(t_1) = \frac{5}{2}t_1^3, \qquad x_2(t_2) = t_2^2, \quad p_2(t_2) = \frac{7}{2}t_2^5.$$

The deformations given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= t_1^2, & Y_1(t_1,\mathbf{s}) &= s_1t_1^3 + t_1^5, \\ X_2(t_2,\mathbf{s}) &= t_2^2, & Y_2(t_2,\mathbf{s}) &= s_2t_2^2 + s_3t_2^3 + s_4t_2^4 + s_5t_2^5 \\ & & &\quad + s_6t_2^6 + t_2^7 + s_7t_2^8 + s_8t_2^{10} + s_9t_2^{12}; \end{aligned} \tag{7.3}$$

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= s_1t_1 + t_1^2, & Y_1(t_1,\mathbf{s}) &= s_3t_1 + s_4t_1^3 + t_1^5, \\ X_2(t_2,\mathbf{s}) &= s_2t_2 + t_2^2, & Y_2(t_2,\mathbf{s}) &= s_5t_2 + s_6t_2^2 + s_7t_2^3 + s_8t_2^4 \\ & & &\quad + s_9t_2^5 + s_{10}t_2^6 + t_2^7 + s_{11}t_2^8 \\ & & &\quad + s_{12}t_2^{10} + s_{13}t_2^{12}; \end{aligned} \tag{7.4}$$

are respectively

- an equimultiple semiuniversal deformation, see (7.3);
- a semiuniversal deformation, see (7.4),

of φ. The conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= t_1^2, & Y_1(t_1,\mathbf{s}) &= t_1^5, \\ X_2(t_2,\mathbf{s}) &= t_2^2, & Y_2(t_2,\mathbf{s}) &= s_1t_2^4 + s_2t_2^5 + s_3t_2^6 + t_2^7 + s_4t_2^8; \end{aligned}$$

is an equimultiple semiuniversal deformation of the conormal of φ. The fake conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= s_1t_1 + t_1^2, & P_1(t_1,\mathbf{s}) &= s_3t_1 + \frac{5}{2}t_1^3, \\ X_2(t_2,\mathbf{s}) &= s_2t_2 + t_2^2, & P_2(t_2,\mathbf{s}) &= s_4t_2 + s_5t_2^2 + s_6t_2^3 + s_7t_2^4 + \frac{7}{2}t_2^5 + s_8t_2^6; \end{aligned}$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= s_1t_1 + t_1^2, & Y_1(t_1,\mathbf{s}) &= \alpha_2t_1^2 + \alpha_3t_1^3 + \alpha_4t_1^4 + t_1^5, \\ X_2(t_2,\mathbf{s}) &= s_2t_2 + t_2^2, & Y_2(t_2,\mathbf{s}) &= \beta_2t_2^2 + \beta_3t_2^3 + \beta_4t_2^4 + \beta_5t_2^5 \\ & & &\quad + \beta_6t_2^6 + \beta_7t_2^7 + \beta_8t_2^8; \end{aligned}$$

with

$$\alpha_2 = \frac{s_1 s_3}{2}, \qquad \alpha_3 = \frac{2s_3}{3}, \qquad \alpha_4 = \frac{5s_1}{8},$$
$$\beta_2 = \frac{s_2 s_4}{2}, \qquad \beta_3 = \frac{2s_4 + s_2 s_5}{3}, \qquad \beta_4 = \frac{2s_5 + s_2 s_6}{4},$$
$$\beta_5 = \frac{2s_6 + s_2 s_7}{5}, \qquad \beta_6 = \frac{4s_7 + 7s_2}{12}, \qquad \beta_7 = 1 + \frac{s_2 s_8}{7},$$
$$\beta_8 = \frac{2s_8}{8},$$

is a semiuniversal deformation of the conormal of φ.

During the preparation of this paper all non trivial calculations were made with the help of the Computer Algebra System [3].

References

[1] A. Araújo and O. Neto, Moduli of germs of Legendrian curves. *Ann. Fac. Sci. Toulouse Math. (6)* **18** (2009), no. 4, 797–809 Zbl 1187.14004 MR 2590388

[2] J. Cabral and O. Neto, Microlocal versal deformations of the plane curves $y^k = x^n$. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 23–24, 1409–1414 Zbl 1256.32012 MR 2588791

[3] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 4-3-0 — A computer algebra system for polynomial computations. https://www.singular.uni-kl.de (2022)

[4] H. Flenner, Ein Kriterium für die Offenheit der Versalität. *Math. Z.* **178** (1981), no. 4, 449–473 Zbl 0453.14002 MR 638811

[5] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to singularities and deformations*. Springer Monogr. Math., Springer, Berlin, 2007 Zbl 1125.32013 MR 2290112

[6] M. Kashiwara, *Systems of microdifferential equations*. Prog. Math. 34, Birkhäuser Boston, Inc., Boston, MA, 1983 Zbl 0521.58057 MR 725502

[7] M. Kashiwara and T. Kawai, On holonomic systems of microdifferential equations. III. Systems with regular singularities. *Publ. Res. Inst. Math. Sci.* **17** (1981), no. 3, 813–979 Zbl 0505.58033 MR 650216

[8] M. Kashiwara and P. Schapira, *Sheaves on manifolds*. Grundlehren Math. Wiss. 292, Springer, Berlin, 1994 Zbl 0709.18001 MR 1299726

[9] A. R. Martins and O. Neto, On the group of germs of contact transformations. *Port. Math.* **72** (2015), no. 4, 393–406 Zbl 1333.53118 MR 3450733

[10] D. Nadler, Non-characteristic expansions of Legendrian singularities. 2016, arXiv:1507.01513v2

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