Einstein–Klein–Gordon spacetimes in the harmonic near-Minkowski regime

Philippe G. LeFloch and Yue Ma

Abstract. We study the initial value problem for the Einstein–Klein–Gordon system and establish the global nonlinear stability of massive matter in the near-Minkowski regime when the initial geometry is a perturbation of an asymptotically flat, spacelike hypersurface in Minkowski spacetime and the metric enjoys the harmonic decay $1/r$ (in terms of a suitable distance function r at spatial infinity). Our analysis encompasses matter fields that have small energy norm and solely enjoys a slow decay at spacelike infinity. Our proof is based on the Euclidean-hyperboloidal foliation method recently introduced by the authors, and distinguishes between the decay along asymptotically hyperbolic slices and the decay along asymptotically Euclidean slices. We carefully analyze the decay of metric components at the harmonic level $1/r$, especially the metric component in the direction of the light cone. In presence of such a slow-decaying matter field, we establish a global existence theory for the Einstein equations expressed as a coupled system of nonlinear wave and Klein–Gordon equations.

1. Introduction

1.1. Global evolution problem for the Einstein-matter equations

Main purpose. We consider solutions to the Einstein equations in the near-Minkowski regime and study the global nonlinear evolution problem when a suitable initial data set is prescribed, consisting of a Riemannian metric, a symmetric two-tensor, and matter data. These data are assumed to be close to a sufficiently flat and asymptotically Schwarzschild-like perturbation of a hypersurface in Minkowski spacetime. For the *vacuum* Einstein equations, this problem was solved first by Christodoulou and Klainerman [9], while an alternative proof in wave coordinates was given later on by Lindblad and Rodnianski [35, 36]. Solutions with lower decay at spacelike infinity

---

were constructed by Bieri [1] (and [2]), while most recent contributions are due to Hintz and Vasy [15, 16]. All of these results easily extend to *massless* matter fields.

Our aim in the present paper is to solve the global evolution problem for the Einstein-matter system when the matter field under consideration is *massive.* We are going to describe the problem and first state a simplified version of our main result (Theorem 1.1) when the data have Schwarzschild decay. Later in this paper (cf. Theorem 3.1) we state a more general result, when the metric at spatial infinity enjoys harmonic-type decay.

A recent literature on the global dynamics of self-gravitating massive fields is now available. The present paper is a companion to our work [30] and, on the one hand, provides the arguments that are necessary in order to establish the Schwarzschild-type decay of solutions to the Einstein equations and, on the other hand, can also be considered as an overview of the more general proof in [30] (see also [31]). These results grew from earlier work by the authors in [26, 29].

While our project came under completion we learned that Ionescu and Pausader simultaneously solved the same problem by introducing a completely different methodology, which is based on the notion of spacetime resonances; see [21]. A somewhat different class of initial data sets is covered therein, as far as the functional norms and the spatial decay of solutions are concerned (specially since the regularity in [21] is stated in weighted Fourier spaces). We will not attempt to review here the vast literature on the subject, and we refer the reader to our detailed overview in [30]. Recent work includes, among others, the major contributions by Bigorgne [3, 4], Fajman et al. [11, 13], Lindblad et al. [37], Smulevici [42], and Wang [44]. This field is extremely active and, for additional results and references, the reader is referred to [5, 6, 10, 12, 14, 17–20, 22–25, 27, 28, 32–34, 38–41, 43, 45].

Einstein equations and massive fields. We consider four-dimensional manifolds $(\mathcal{M}, g)$ in which $\mathcal{M} \simeq [0, +\infty) \times \mathbb{R}^3$ and g is a Lorentzian metric with signature $(-, +, +, +)$ and Levi-Civita connection $\nabla_g = \nabla$. Greek indices $\alpha, \beta, \ldots$ describe $0, 1, 2, 3$, and we denote by $R_{\alpha\beta}$ the Ricci curvature of g and by $R = g^{\alpha\beta} R_{\alpha\beta}$ its scalar curvature. Throughout, we use implicit summation over repeated indices, as well as raising and lowering indices with respect to the metric $g_{\alpha\beta}$ and its inverse denoted by $g^{\alpha\beta}$.

We impose that Einstein-matter equations hold, namely

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1.1}$$

in which the left-hand side is Einstein's curvature tensor defined (in abstract indices) as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta}. \tag{1.2}$$

The right-hand side of (1.1) is the energy-momentum tensor, which depends upon the matter content and, for instance, is taken to vanish identically for vacuum Einstein spacetimes, so that (1.1) then reduces to the Ricci-flat condition $R_{\alpha\beta} = 0$.

While our method of analysis should apply to many massive matter models, we are interested here in Klein–Gordon scalar fields $\phi : \mathcal{M} \to \mathbb{R}$ with energy-momentum tensor

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \left(\frac{1}{2} \nabla_\gamma \phi \nabla^\gamma \phi + U(\phi) \right) g_{\alpha\beta}. \tag{1.3}$$

Here, the potential function $U = U(\phi)$ depends on the nature of the matter under consideration and, throughout, we assume that

$$U(\phi) = \frac{1}{2} c^2 \phi^2 + \mathcal{O}(\phi^3) \tag{1.4}$$

for some (mass-like) constant $c > 0$. Recalling the Bianchi identity $\nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \nabla_\beta R$, we have $\nabla^\alpha G_{\alpha\beta} = 0$ and, consequently,

$$\nabla^\alpha T_{\alpha\beta} = 0. \tag{1.5}$$

In turn, we deduce that ϕ satisfies the nonlinear Klein–Gordon equation

$$\Box_g \phi - U'(\phi) = 0. \tag{1.6}$$

For instance, when $U(\phi) = \frac{c^2}{2} \phi^2$, this is $\Box_g \phi - c^2 \phi = 0$, which is linear in ϕ but also involves the unknown metric.

Initial value problem. For suitable initial data, equation (1.6) is expected to uniquely determine the evolution of the matter. Our challenge is precisely to study the nonlinear coupling problem when the metric g itself is one of the unknowns and solves the Einstein equations with suitably prescribed initial data. When a foliation by spacelike hypersurfaces is chosen and a suitable gauge choice is made, we use Latin indices varying between 1 and 3 and the Einstein equations decompose into evolution equations

$$G_{ab} = 8\pi T_{ab}, \qquad a, b = 1, 2, 3, \tag{1.7}$$

and constraint equations

$$G_{00} = 8\pi T_{00}, \qquad G_{0a} = 8\pi T_{0a}, \qquad a = 1, 2, 3. \tag{1.8}$$

The initial value problem is then formulated by specifying a Riemannian metric g_0 and a symmetric two-tensor field k_0, defined on a manifold with topology $\mathbb{R}^3$, and satisfying the constraints (1.8), namely

$$\begin{aligned}
R_{g_0} + (\mathrm{Tr}_{g_0} k_0)^2 - |k_0|_{g_0}^2 &= 16\pi T_{00}, \\
\mathrm{div}_{g_0} \left(k_0 - (\mathrm{Tr}_{g_0} k_0) g_0 \right) &= -8\pi T_{0\bullet},
\end{aligned} \tag{1.9}$$

where R_{g_0} denotes the scalar curvature, Tr_{g_0} the trace, and div_{g_0} the divergence of the metric g_0. We then seek a globally hyperbolic Cauchy development [7, 8] consisting of a spacetime metric g together with a matter field ϕ satisfying Einstein's evolution equations and defining a manifold whose induced geometry (first- and second-fundamental forms) on an initial hypersurface is given by the pair (g_0, k_0).

1.2. Self-gravitating fields in the asymptotically Schwarzschild regime

Wave-Klein–Gordon system of interest. We work on the manifold $\mathcal{M} \simeq \mathbb{R}^{1+3}_+ := \{(t, x) \in \mathbb{R}^{1+3}, t \geq 1\}$ covered by a single coordinate chart $(t, x) = (t, x^a)$ with $t \geq 1$ and $x \in \mathbb{R}^3$. We also write $r^2 := |x|^2 = \sum_{a=1}^3 |x^a|^2$. We introduce the Minkowski metric $g_{\mathrm{Mink}} := -dt^2 + \sum_a (dx^a)^2$, and observe that $\mathcal{M}$ is the future of the initial hypersurface $\{t = 1\}$. We also introduce the *outgoing light cone*

$$\mathcal{L} := \{r = t - 1\} \subset \mathbb{R}^{3+1}_+ \tag{1.10}$$

and its constant-t slices are denoted by $\mathcal{L}_t$.

More precisely, we rely on global coordinate functions $x^\alpha : \mathcal{M} \to \mathbb{R}$ (with $\alpha = 0, 1, 2, 3$) satisfying the wave gauge conditions

$$\Box_g x^\alpha = 0, \tag{1.11}$$

and we express the Einstein equations as a nonlinear system of second-order partial differential equations, supplemented with second-order constraints. The unknowns are the metric coefficients $g_{\alpha\beta}$ and the matter field ϕ. Specifically, we find

$$\begin{aligned}
\widetilde{\Box}_g g_{\alpha\beta} &= \mathbb{F}_{\alpha\beta}(g, g; \partial g, \partial g) - 16\pi\big(\partial_\alpha\phi\partial_\beta\phi + U(\phi)g_{\alpha\beta}\big), \\
\widetilde{\Box}_g \phi - U'(\phi) &= 0,
\end{aligned} \tag{1.12}$$

where $\widetilde{\Box}_g := g^{\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}$ denotes a modified wave operator, and the wave gauge constraints take the form

$$\Gamma^\alpha = g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0, \qquad \Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\lambda'}\big(\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'} - \partial_{\lambda'}g_{\alpha\beta}\big). \tag{1.13}$$

The nonlinearities $\mathbb{F} = \mathbb{P} + \mathbb{Q}$ have an important structure especially in connection with the Euclidian-hyperboloidal foliation, which is described and plays a central role in [30].

Merging the Minkowski and the Schwarzschild solutions. In wave coordinates the Schwarzschild metric g_{Sch} takes the form (with $\omega_a := x_a/r$)

$$\begin{aligned}
g_{\mathrm{Sch},00} &= -\frac{r - m}{r + m}, \qquad g_{\mathrm{Sch},0a} = 0, \\
g_{\mathrm{Sch},ab} &= \frac{r + m}{r - m}\omega_a\omega_b + \frac{(r + m)^2}{r^2}(\delta_{ab} - \omega_a\omega_b).
\end{aligned} \tag{1.14}$$

Consider a (regular) cut-off function $\chi^\star(r)$ that vanishes for $r \leq 1/2$ and is identically 1 for all $r \geq 3/4$. Given a (small) mass coefficient $m > 0$ we introduce the reference metric

$$g^\star = g_{\mathrm{Mink}} + \chi^\star(r)\chi^\star(r/(t-1))(g_{\mathrm{Sch}} - g_{\mathrm{Mink}}), \qquad t \geq 2. \tag{1.15}$$

This metric coincides with g_{Mink} in the cone $\{r/(t-1) < 1/2\}$ and coincides with g_{Sch} in the exterior $\{r/(t-1) \geq 3/4\}$ which contains the light cone $\mathcal{L}$. This metric satisfies the so-called light-bending property, saying by definition that the light cone coefficient

$$rg^\star(\mathbf{l}, \mathbf{l}) = 4m + \mathcal{O}(1/r) \qquad \text{for the metric } g^\star, \tag{1.16}$$

is positive, where the *light cone direction* is defined by

$$\mathbf{l} := \partial_t - (x^a/r)\partial_a. \tag{1.17}$$

This is essentially the same construction as in [35].

Class of initial data sets. We are interested in solutions to the Einstein-matter system in the harmonic regime (corresponding to $\lambda = 1$ in [30]) and we establish a sharp decay property in the case where the reference metric is constructed from the Schwarzschild metric. The initial metric g_0 is assumed to be sufficiently close to the Euclidean metric while the initial second fundamental form k_0 is sufficiently small. Let us introduce the following decomposition (for $a, b = 1, 2, 3$):

$$g_{0ab} = g_{0ab}^\star + u_{0ab} = \delta_{ab} + h_{0ab}^\star + u_{0ab}, \qquad k_{0ab} = k_{0ab}^\star + l_{0ab}. \tag{1.18}$$

We aim at covering a variety of asymptotic behaviors and, at this juncture, it is convenient to introduce the following terminology.

- The part $h_0^\star$ is referred to as the *initial reference* and will be assumed to be small in a (weighted, high-order) pointwise norm.

- The part u_0 is referred to as the *initial perturbation* and will be assumed to be small in (weighted, high-order) energy norm.

An example of a such decomposition is provided by the construction in Lindblad and Rodnianski [35], where the initial data is decomposed as the sum of a finite-energy perturbation plus an (asymptotically) Schwarzschild metric outside of a compact set (with sufficiently small and positive mass). In our theory, the two parts are treated differently. Indeed, $h_0^\star$ is the initial trace of $h^\star$ while u_0 is propagated.

For the metric perturbation, we introduce the energy norms

$$
\begin{aligned}
\mathbf{E}_{\kappa,N}^{\mathrm{metric}}(g_0, k_0) &:= \sum_{|I| \leq N} \big\| \langle r \rangle^{\kappa+|I|} \big(|\partial_x^I \partial_x u_0| + |\partial_x^I u_1| \big) \big\|_{L^2(\mathbb{R}^3)}, \\
\mathbf{E}_{\mu,N}^{\mathrm{matter}}(\phi_0, \phi_1) &:= \sum_{|I| \leq N} \big\| \langle r \rangle^{\mu+|I|} \big(|\partial_x^I \partial_x \phi_0| + |\partial_x^I \phi_0| + |\partial_x^I \phi_1| \big) \big\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
\tag{1.19}
$$

While our proof below will provide a more general result as stated below, we find it convenient to state first our result for the perturbation of the Schwarzschild solution.

Given an initial data set we decompose it according to (1.18) and we introduce the *linear development* denoted by u_{init} of the initial data set $(u_{0\alpha\beta}, u_{1\alpha\beta})$, that is, we introduce the solution to the (free, linear) wave equation with this initial data. It was established in [30, Proposition 10.6] that, under the assumption that the norm above is finite,

$$|u_{\text{init}}| \lesssim C_0 \varepsilon (t + r + 1)^{-1}. \tag{1.20}$$

Our main assumption beyond the smallness on the norms (1.19) is the following sign condition which we referred to as the *light-bending condition*

$$\inf_{\mathcal{M}_\ell^{\text{near}}} \left(4m + r u_{\text{init}}(\mathbf{l}, \mathbf{l}) \right) \geq \varepsilon_\star. \tag{1.21}$$

Here, a parameter $\ell \in (0, 1/2]$ being fixed once for all, we have defined the *near-light cone* domain to be

$$\mathcal{M}_\ell^{\text{near}} := \left\{ t \geq 2, t - 1 \leq r \leq \frac{t}{1 - \ell} \right\}, \tag{1.22}$$

where in agreement with [30] it is convenient to restrict attention to $t \geq 2$. Interestingly, this condition can be easily satisfied when

- either ε is small with respect to $\varepsilon_\star$, so that the contribution m from the Schwarzschild metric dominates,

- both initial data $u_0 \geq 0$ and $u_1 \geq 0$ are non-negative (as is clear from the fact that the fundamental solution to the wave equation is a non-negative measure),

- or yet a combination of the above two extreme examples, namely, the negative contribution of the perturbation is small with respect to the Schwarzschild mass.

Main statement. We are in a position to state our main result. In fact, a more general statement concerning perturbations of reference metrics with harmonic decay is actually established in this paper; see Theorem 3.1.

Theorem 1.1 (Nonlinear stability of self-gravitating Klein–Gordon fields. Perturbations of the Schwarzschild solution). *A constant $C_\star > 0$ being fixed, the following result holds for all sufficiently small $\varepsilon, \varepsilon_\star$ satisfying $\varepsilon \leq C_\star \varepsilon_\star$. Consider the reference metric $g^\star$ defined in* (1.15) *by merging together the Minkowski metric g_{Mink} and the Schwarzschild metric with mass $\varepsilon_\star$ along a light cone. Consider a set of constraint-satisfying initial data $(g_0, k_0, \phi_0, \phi_1)$, a large integer N, and some exponents $(\kappa, \mu, \varepsilon)$ satisfying*

$$\kappa \in (1/2, 1), \qquad \mu \in (3/4, 1), \qquad \kappa \leq \mu. \tag{1.23}$$

Then, provided the initial data satisfies the light-bending condition (1.21) *together with the smallness condition*

$$\mathbf{E}^{\text{metric}}_{\kappa,N}(g_0, k_0) + \mathbf{E}^{\text{matter}}_{\mu,N}(\phi_0, \phi_1) \leq \varepsilon, \qquad (1.24)$$

the maximal globally hyperbolic Cauchy development of $(g_0, k_0, \phi_0, \phi_1)$ *associated with the Einstein-massive field system* (1.1) *and* (1.3) *is future causally geodesically complete, and asymptotically approaches Minkowski spacetime in all (timelike, null, spacetime) directions. Moreover, the component* $g(\mathbf{l}, \mathbf{l})$ *has a harmonic decay and enjoys the light-bending condition, namely*

$$|g(\mathbf{l}, \mathbf{l})| \lesssim \frac{\varepsilon_\star + \varepsilon}{t + r + 1}, \qquad \inf_{\mathcal{M}^{\text{near}}_\ell} r g(\mathbf{l}, \mathbf{l}) \geq \varepsilon_\star/2. \qquad (1.25)$$

Harmonic decay. The first inequality in (1.25) is the main novelty of this paper, and establishing it requires significant new observation beyond our previous strategy. Indeed, our proof of Theorems 1.1 and 3.1 follows from the general method in [30] in which we now take the parameter λ therein to be the critical value $\lambda = 1$. We would like to point out here several significant differences and several new ingredients that are required for the proof in the present paper. First of all, the use of a hierarchy property for commutators was not necessary for the range $\lambda < 1$, but turns out to be essential now. On the other hand, we revisit [30, Section 8] concerning the pointwise decay of metric components (and their derivatives) and we use a decomposition of the wave operators in which the component $H^{\mathcal{N}00}$ is directly controlled. In turn, in the analysis in [30, Section 13] we now decouple the contributions of the Hessian and of the commutators. Interestingly, this leads to significant simplifications in comparison with the arguments in [30], while simultaneously new estimates are required. For instance, in the analysis of the (now harmonic) decay of the null metric component in [30, Section 14], we no longer introduce the "loss" exponent θ. In order to recover the desired $1/r$ behavior for the metric, the application of the Kirchhoff formula must now be done at the $1/r$ level of decay, and this requires a sharp control of the source-terms in the wave equation. Such terms are due to the Ricci curvature of the reference metric (assumed to have sufficient decay, and even to vanish for the Schwarzschild metric), the commutators (discussed earlier) and the quasi-linear terms (also discussed earlier). Observe that it is also important to distinguish between zero-order estimates (i.e. without differentiation) and high-order estimates. Finally, we point out that, interestingly, our results also apply in the *vacuum spacetimes* by taking ϕ to vanish identically: our proof then is somewhat simpler than the one in [36], thanks to the fact that it takes advantage of the light-bending condition—a consequence of the assumed positivity of the mass of the Schwarzschild metric.

2. Euclidean-hyperboloidal foliation and functional inequalities

2.1. Spacetime foliation and vector fields

Preliminary. We provide here an overview of the technical tools introduced by the authors in [30] when developing the Euclidean-hyperboloidal foliation method. This method generally applies to establish the global existence of solutions with small amplitude for coupled systems of wave and Klein–Gordon equations. The method relies on the following key ingredients.

- Construction of a spacetime foliation consisting of asymptotically hyperboloidal slices and asymptotically flat slices.

- Functional analysis tools on such a foliation: admissible vector fields, Sobolev inequalities, and energy estimates.

- Sup-norm estimates for wave equations and Klein–Gordon equations.

For the full set of notions and results, we refer to [30].

Foliation of interest. In order to foliate $\mathcal{M}$ by spacelike hypersurfaces, we introduce a cut-off function $\chi : \mathbb{R} \to [0, 1]$ satisfying

$$\chi(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 1. \end{cases} \tag{2.1a}$$

We also introduce the *hyperboloidal and Euclidean radii* at a time s

$$\mathbf{r}^{\mathcal{H}}(s) := \frac{1}{2}(s^2 - 1), \qquad \mathbf{r}^{\mathcal{E}}(s) := \frac{1}{2}(s^2 + 1), \tag{2.1b}$$

Then, a function ξ referred to as the *foliation coefficient* is defined by

$$\xi(s, r) := 1 - \chi(r - \mathbf{r}^{\mathcal{H}}(s)) = \begin{cases} 1, & r < \mathbf{r}^{\mathcal{H}}(s), \\ 0, & r > \mathbf{r}^{\mathcal{E}}(s). \end{cases} \tag{2.1c}$$

We next define the *Euclidean–hyperboloidal time function* $T = \mathbf{T}(s, r)$ by solving the ordinary differential equation

$$\partial_r \mathbf{T}(s, r) = \frac{r \xi(s, r)}{(s^2 + r^2)^{1/2}}, \qquad \mathbf{T}(s, 0) = s. \tag{2.1d}$$

In turn, our foliation $\mathcal{M}_s := \{(t, x^a) \in \mathcal{M} \,/\, t = \mathbf{T}(s, r)\}$ is a one-parameter family of asymptotically Euclidean, spacelike hypersurfaces. The future of the initial surface $\{t = 1\}$ is decomposed as

$$\{t \ge 1\} = \mathcal{M}^{\text{init}} \cup \bigcup_{s \ge 2} \mathcal{M}_s, \qquad \mathcal{M}_s = \mathcal{M}_s^{\mathcal{H}} \cup \mathcal{M}_s^{\mathcal{M}} \cup \mathcal{M}_s^{\mathcal{E}}, \tag{2.2a}$$

with $\mathcal{M}^{\mathrm{init}} := \{(t, x) \, / \, 1 \leq t \leq T(2, r)\}$ and

$$
\begin{array}{lll}
\mathcal{M}_s^{\mathcal{H}} & := \{t = \mathbf{T}(s, |x|), |x| \leq \mathbf{r}^{\mathcal{H}}(s)\} & \text{asymptotically hyperboloidal,} \\
\mathcal{M}_s^{\mathcal{M}} & := \{t = \mathbf{T}(s, |x|), \mathbf{r}^{\mathcal{H}}(s) \leq |x| \leq \mathbf{r}^{\mathcal{E}}(s)\} & \text{merging (or transition),} \\
\mathcal{M}_s^{\mathcal{E}} & := \{t = \mathbf{T}(s), \mathbf{r}^{\mathcal{E}}(s) \leq |x|\} & \text{asymptotically Euclidean.}
\end{array}
$$
(2.2b)

We also write $\mathcal{M}_s^{\mathcal{E}\mathcal{M}} := \mathcal{M}^{\mathcal{E}} \cup \mathcal{M}^{\mathcal{M}}$.

Frames of interest. The following terminology will be used.

- *The semi-hyperboloidal frame* (SHF)

$$
\partial_0^{\mathcal{H}} := \partial_t, \qquad \partial_a^{\mathcal{H}} = \mathring{\partial}_a^{\mathcal{H}} := \frac{x^a}{t} \partial_t + \partial_a
$$
(2.3a)

  is defined in $\mathcal{M}_s$ and is the appropriate frame in the hyperboloidal domain in order to establish the relevant *decay properties in timelike and null directions.* Some of our arguments will involve radial integration based on $\mathring{\partial}_r^{\mathcal{H}} := (x^a/r)\mathring{\partial}_a^{\mathcal{H}}$.

- *The semi-null frame* (SNF)

$$
\partial_0^{\mathcal{N}} := \partial_t, \qquad \partial_a^{\mathcal{N}} = \mathring{\partial}_a^{\mathcal{N}} := \frac{x^a}{r} \partial_t + \partial_a
$$
(2.3b)

  is defined in $\mathcal{M}_s$ *except* on the center line $r = 0$, and is the appropriate frame within the Euclidean-merging domain in order to exhibit the structure of the non-linearities of the field equations and, in turn, to establish the relevant *decay properties in spatial and null directions.*

- *The Euclidean-hyperboloidal frame* (EHF) defined as

$$
\begin{aligned}
\partial_0^{\mathcal{E}\mathcal{H}} &:= \partial_t, \\
\partial_a^{\mathcal{E}\mathcal{H}} = \mathring{\partial}_a^{\mathcal{E}\mathcal{H}} &:= \partial_a + (x^a/r)\partial_r \mathbf{T} \partial_t = \partial_a + x^a \xi(s, r)(s^2 + r^2)^{-1/2}\partial_t
\end{aligned}
$$
(2.3c)

  consists of tangent vectors $\mathring{\partial}_a^{\mathcal{E}\mathcal{H}}$ to the slices $\mathcal{M}_s$. Observe that $\partial_a^{\mathcal{E}\mathcal{H}} = \partial_a^{\mathcal{H}}$ in $\mathcal{M}_s^{\mathcal{H}}$, while $\partial_a^{\mathcal{E}\mathcal{H}} = \partial_a$ in $\mathcal{M}_s^{\mathcal{E}}$. The expressions of the vectors $\partial_a^{\mathcal{E}\mathcal{H}}$ are more involved in the merging $\mathcal{M}_s^{\mathcal{M}}$, where the vectors $\partial_a^{\mathcal{E}\mathcal{H}}$ interpolate between $\partial_a^{\mathcal{H}}$ and ∂_a. Some of our arguments will involve radial integration based on $\mathring{\partial}_r^{\mathcal{E}\mathcal{H}} := (x^a/r)\mathring{\partial}_a^{\mathcal{E}\mathcal{H}}$.

Admissible vector fields. Minkowski spacetime admits three sets of *Killing fields.*

- The spacetime translations generated by the vector fields ∂_α ($\alpha = 0, 1, 2, 3$).
- The Lorentz boosts generated by the vector fields $L_a := x_a \partial_t + t \partial_a$ ($a = 1, 2, 3$).
- The spatial rotations generated by the vector fields $\Omega_{ab} := x_a \partial_b - x_b \partial_a$ ($a, b = 1, 2, 3$).

We refer to ∂_α, L_a, Ω_{ab} as the *admissible fields* which commute with the wave and Klein–Gordon operators in Minkowski spacetime. In defining high-order norms, we combine admissible vector fields together. An operator $Z = \partial^I L^J \Omega^K$ is called an *ordered admissible operator*. To such an operator $Z = \partial^I L^J \Omega^K$, we associate its *order, degree,* and *rank* by

$$\text{ord}(Z) = |I| + |J| + |K|, \qquad \deg(Z) = |I|, \qquad \text{rank}(Z) = |J| + |K|. \quad (2.4)$$

2.2. Energy and Sobolev inequalities

Weighted energy inequality. For the foliation under consideration, the fundamental energy functional associated with the wave equation and, more generally, the Klein–Gordon equations involve the weight $\zeta = \zeta(t, x)$ defined by

$$\zeta(s, r)^2 := 1 - \frac{r^2 \xi^2(s, r)}{s^2 + r^2} = \begin{cases} s^2/t^2, & r < \mathbf{r}^{\mathcal{H}}(s), \\ 1, & r > \mathbf{r}^{\mathcal{E}}(s). \end{cases} \quad (2.5)$$

In addition, we introduce a weight which reduces to a constant in the interior of the light cone and is essentially the distance to the light cone in the exterior domain. Further, we are given a smooth and non-decreasing function $\aleph$ satisfying $\aleph(y) = 0$ for $y \leq -1$ and $\aleph(y) = y + 1$ for $y \geq 0$, and we set

$$\mathbf{X} := 1 + \aleph(r - t). \quad (2.6)$$

Consider the wave or Klein–Gordon equation (with $c \geq 0$) with unknown u, namely $g^{\alpha\beta} \partial_\alpha \partial_\beta u - c^2 u = f$ associated with a metric $g^{\alpha\beta} =: g^{\alpha\beta}_{\text{Mink}} + H^{\alpha\beta}$ and a right-hand side f. The energy-flux vector

$$V_{g,\eta,c}[u] := -\mathbf{X}^{2\eta}\big(g^{00}|\partial_t u|^2 - g^{ab}\partial_a u \partial_b u - c^2 u^2, \ 2g^{a\beta}\partial_t u \partial_\beta u\big) \quad (2.7a)$$

depends upon the metric g as well as the weight $\mathbf{X}^\eta$. By setting

$$\begin{aligned}
\Omega_{g,\eta,c}[u] := & -2\eta \mathbf{X}^{-1}\aleph'(r - t)(-1, x^a/r) \cdot V_{g,\eta,c}[u] \\
= & \ 2\eta \mathbf{X}^{2\eta-1}\aleph'(r - t)\big(g^{\mathcal{N}ab}\partial^{\mathcal{N}}_a u \partial^{\mathcal{N}}_b u - H^{\mathcal{N}00}|\partial_t u|^2 + c^2 u^2\big), \quad (2.7b)
\end{aligned}$$
$$-G_{g,\eta}[u] := \partial_t H^{00}|\mathbf{X}^\eta \partial_t u|^2 - \partial_t H^{ab}\mathbf{X}^{2\eta}\partial_a u \partial_b u + 2\mathbf{X}^{2\eta}\partial_a H^{a\beta}\partial_t u \partial_\beta u,$$

we find

$$-2\mathbf{X}^{2\eta}\partial_t u f = \text{div}\, V_{g,\eta,c}[u] + \Omega_{g,\eta,c}[u] - G_{g,\eta}[u]. \quad (2.7c)$$

Next, we consider the integral

$$\mathcal{E}_{g,\eta,c}(s, u) := \int_{\mathcal{M}_s} V_{g,\eta,c}[u] \cdot n_s \, d\sigma_s. \quad (2.8a)$$

By integrating the energy identity (2.7c) over the domain limited by a slice of the foliation and the initial slice and using Stokes' formula, we arrive at the *energy identity*

$$\mathcal{E}_{g,\eta,c}(s_1, u) - \mathcal{E}_{g,\eta,c}(s_0, u) + \int_{\mathcal{M}_{[s_0,s_1]}} \left(\Omega_{g,\eta,c}[u] - G_{g,\eta}[u] \right) dx dt$$

$$= -2 \int_{\mathcal{M}_{[s_0,s_1]}} \partial_t u f \mathbf{X}^{2\eta} \, dx dt. \tag{2.8b}$$

The Jacobian J of our parameterization $(t, x) \mapsto (s, x)$, as established in [30, Lemma 3.5], satisfies

$$J \leq \begin{cases} s/t & \text{in } \mathcal{M}_s^{\mathcal{H}}, \\ \xi s(s^2 + r^2)^{-1/2} + (1 - \xi) 2s & \text{in } \mathcal{M}_s^{\mathcal{M}}, \\ 2s & \text{in } \mathcal{M}_s^{\mathcal{E}}, \end{cases} \tag{2.9}$$

and

$$\zeta^2 s \lesssim J \lesssim \zeta^2 s \qquad \text{in } \mathcal{M}^{\mathcal{M}}. \tag{2.10}$$

After a change of variable and differentiation we can rewrite (2.8b) in the equivalent form

$$\frac{d}{ds} \mathcal{E}_{g,\eta,c}(s, u) + 2\eta \int_{\mathcal{M}_s} \left(g^{\mathcal{N}ab} \partial_a^{\mathcal{N}} u \partial_b^{\mathcal{N}} u + c^2 u^2 \right) \aleph'(r - t) \mathbf{X}^{2\eta-1} J dx$$

$$= \int_{\mathcal{M}_s} \left(G_{g,\eta}[u] + \eta \mathbf{X}^{2\eta-1} \aleph'(r - t) H^{\mathcal{N}00} |\partial_t u|^2 \right) J dx \tag{2.11a}$$

$$+ \int_{\mathcal{M}_s} |\partial_t u f| \mathbf{X}^{2\eta} J dx,$$

in which the latter integral is controlled by

$$\int_{\mathcal{M}_s} |\partial_t u f| \mathbf{X}^{2\eta} J dx \lesssim \int_{s_0}^{s_1} \mathcal{E}_{\eta,c}(s, u)^{1/2} \| J \zeta^{-1} \mathbf{X}^{\eta} f \|_{L^2(\mathcal{M}_s)} ds. \tag{2.11b}$$

We will rely on the following weighted energy estimate in the Euclidean-merging domain concerning any solution $u : \mathcal{M}_{[s_0,s_1]}^{\mathcal{E}\mathcal{M}} \to \mathbb{R}$ to the wave or Klein–Gordon equation $g^{\alpha\beta} \partial_\alpha \partial_\beta u - c^2 u = f$ with right-hand side $f : \mathcal{M}_{[s_0,s_1]}^{\mathcal{E}\mathcal{M}} \to \mathbb{R}$:

$$\frac{d}{ds} \mathcal{E}_{g,\eta,c}^{\mathcal{E}\mathcal{M}}(s, u) + \frac{d}{ds} \mathcal{E}_{g,c}^{\mathcal{L}}(s, u; s_0)$$

$$+ 2\eta \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \left(g^{\mathcal{N}ab} \partial_a^{\mathcal{N}} u \partial_b^{\mathcal{N}} u + c^2 u^2 \right) \mathbf{X}^{2\eta-1} \aleph'(r - t) J dx$$

$$= \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \left(G_{g,\eta}[u] + \eta \mathbf{X}^{2\eta-1} \aleph'(r - t) H^{\mathcal{N}00} |\partial_t u|^2 \right) J dx ds \tag{2.12}$$

$$+ \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \mathbf{X}^{2\eta} \partial_t u f J dx ds,$$

in which the latter integral is bounded by

$$\int_{s_0}^{s_1} (\mathcal{E}_{\eta,c}^{\mathcal{EM}}(s,u))^{1/2} \| J\zeta^{-1} \mathbf{X}^\eta f \|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \, ds,$$

while the second term in the left-hand side is defined as

$$\frac{d}{ds} \mathcal{E}_{g,c}^{\mathcal{L}}(s,u;s_0) = s \int \left(-H^{\mathcal{N}00} |\partial_t u|^2 + g^{\mathcal{N}ab} \slashed{\partial}_a^{\mathcal{N}} u \slashed{\partial}_b^{\mathcal{N}} u + c^2 u^2 \right) d\sigma, \quad (2.13)$$

where the domain of integration is defined by $t = r + 1$ and $\mathbf{r}^{\mathcal{H}}(s_0) \le r \le \mathbf{r}^{\mathcal{H}}(s_1)$.

Commutator estimates. We also introduce the Japanese bracket $\langle y \rangle := \sqrt{1 + |y|^2}$ for all real y. Then, for any function $u = u(t,x)$ we define $|u|_N := \max_{\mathrm{ord}(Z) \le N} |Zu|$ and

$$|u|_{p,k} := \max_{\substack{\mathrm{ord}(Z) \le N \\ \mathrm{rank}(Z) \le k}} |Zu|,$$

where the first maximum is over all ordered admissible operators. Various calculus rules based on this notation were established in [30], which we will not repeat here. Let us only extract an important consequence of [30, Proposition 6.10]. We also decompose each slice $\mathcal{M}_s^{\mathcal{EM}}$ into near and far regions so that

$$\mathcal{M}_s^{\mathrm{near}} := \mathcal{M}_s^{\mathcal{EM}} \cap \{t - 1 \le r \le 2t\}, \qquad \mathcal{M}_s^{\mathrm{far}} := \mathcal{M}_s^{\mathcal{EM}} \cap \{r \ge 2t\}. \quad (2.14)$$

Proposition 2.1. (Hierarchy property for quasi-linear commutators. Euclidean-merging domain).

1. Estimate in the near-light cone domain. For any function u defined in the near-light cone domain and for any operator Z with $\mathrm{ord}(Z) = p$ and $\mathrm{rank}(Z) = k$, one has

$$\left| [Z, H^{\alpha\beta} \partial_\alpha \partial_\beta] u \right| \lesssim \left(|H^{\mathcal{N}00}| + t^{-1} |r - t| |H| \right) |\partial\partial u|_{p-1,k-1} + T_{p,k}^{\mathrm{hier}}[H,u]$$
$$+ T_{p,k}^{\mathrm{easy}}[H,u] + T_{p,k}^{\mathrm{super}}[H,u] \qquad in \ \mathcal{M}_{[s_0,s_1]}^{\mathrm{near}}, \quad (2.15a)$$

with

$$T_{p,k}^{\mathrm{hier}}[H,u] := \sum_{\substack{p_1+p_2=p \\ p_1+k_2=k}} \left(|LH^{\mathcal{N}00}|_{p_1-1} + t^{-1} |r-t| |LH|_{p_1-1} \right) |\partial\partial u|_{p_2,k_2},$$

$$T_{p,k}^{\mathrm{easy}}[H,u] := \sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} \left(|\partial H^{\mathcal{N}00}|_{p_1-1,k_1} + t^{-1} |r-t| |\partial H|_{p_1-1,k_1} \right) |\partial\partial u|_{p_2,k_2},$$

$$T_{p,k}^{\mathrm{super}}[H,u] := t^{-1} |H| |\partial u|_p + t^{-1} \sum_{0 \le p_1 \le p-1} |H|_{p_1+1} |\partial u|_{p-p_1}. \quad (2.15b)$$

2. Estimate in the Euclidean-merging domain. *For any function u defined in the Euclidean-merging domain, one has (with $Y_{\rm rot} = L_a$ or Ω_{ab})*

$$\big|[Z, H^{\alpha\beta}\partial_\alpha\partial_\beta]u\big| \lesssim |H||\partial\partial u|_{p-1,k-1} + \sum_{\substack{p_1+p_2=p \\ p_1+k_2=k \\ \text{with } k_1=p_1}} |Y_{\rm rot}H|_{p_1-1}|\partial\partial u|_{p_2,k_2}$$

$$+ \sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} |\partial H|_{p_1-1,k_1}|\partial\partial u|_{p_2,k_2} \qquad in \ \mathcal{M}^{\mathcal{E}\mathcal{M}}_{[s_0,s_1]}. \qquad (2.16)$$

The above estimates will be applied to study the evolution of our high-order energy functionals. In (2.15), the term $T^{\rm hier}_{p,k}[H, u]$ is the most challenging contribution to the change of the energy functional but involves terms at a *lower rank*, that is, contains *strictly fewer boosts or rotations*; this structure will allow us to formulate an induction argument on the rank. The term $T^{\rm easy}$ is easier since it contains the factor ∂H which enjoys (comparatively) good L^2 and pointwise decay, while $T^{\rm super}$ contains a favorable $1/t$ factor.

Functional inequalities. It is necessary to revisit the standard Sobolev inequalities and formulate them along the foliation of interest. We only select here two results. In the inequalities below, recall that $\partial^{\mathcal{E}\mathcal{M}\,I}$ denotes any $|I|$-order operator determined from the fields $\{\partial^{\mathcal{E}\mathcal{M}}_a\}_{a=1,2,3}$, while $\partial^{\mathcal{E}\,I}$ denotes any $|I|$-order operator determined from the fields $\{\partial_a\}_{a=1,2,3}$.

Proposition 2.2 (Sobolev inequality in the Euclidean-merging domain). *Fix an exponent $\eta \geq 0$ and set $C(\eta) := 1 + \eta + \eta^2$. For all sufficiently regular functions defined in $\mathcal{M}_{[s_0,s_1]}$ with $2 \leq s_0 \leq s \leq s_1$, one has*

$$r\mathbf{X}^\eta|u(t,x)| \lesssim C(\eta) \sum_{|I|+|J|\leq 2} \|\mathbf{X}^\eta\partial^{\mathcal{E}\mathcal{M}\,I}\Omega^J u\|_{L^2(\mathcal{M}^{\mathcal{E}\mathcal{M}}_s)}, \quad (t,x) \in \mathcal{M}^{\mathcal{E}\mathcal{M}}_s, \ (2.17\text{a})$$

$$r\mathbf{X}^\eta|u(t,x)| \lesssim C(\eta) \sum_{|I|+|J|\leq 2} \|\mathbf{X}^\eta\partial^{\mathcal{E}\,I}\Omega^J u\|_{L^2(\mathcal{M}^{\mathcal{E}}_s)}, \quad (t,x) \in \mathcal{M}^{\mathcal{E}}_s. \qquad (2.17\text{b})$$

Next, we rely on the boosts $L_a = x^a\partial_t + t\partial_a$ which are tangent to the hyperboloidal slices.

Proposition 2.3 (Sobolev inequality in the hyperboloidal domain). *For any function defined on a hypersurface $\mathcal{M}^{\mathcal{H}}_s$, the following estimate holds (where $t^2 = s^2 + |x|^2$):*

$$\sup_{\mathcal{M}^{\mathcal{H}}_s} t^{3/2}|u(t,x)| \lesssim \sum_{|J|\leq 2} \|L^J u\|_{L^2(\mathcal{M}^{\mathcal{H}}_s)} \simeq \sum_{m=0,1,2} \|t^m(\partial^{\mathcal{H}})^m u\|_{L^2(\mathcal{M}^{\mathcal{H}}_s)}. \quad (2.18)$$

Estimates based on the energy functional. We also have the following decay properties for the wave or Klein–Gordon equation; cf. [30, Propositions 7.2, 7.4, and 9.5], respectively, together with [30, Proposition 7.3] (namely (7.9) therein).

Proposition 2.4 (Hardy–Poincaré inequality for high-order derivatives). *For any* $\eta = 1/2 + \delta$ *with* $\delta > 0$ *and any sufficiently decaying function* u *defined in* $\mathcal{M}_{[s_0,s_1]}$ *and for all* $s \in [s_0, s_1]$ *one has*

$$\|\mathbf{X}^{-1+\eta}|u|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \lesssim (1 + \delta^{-1}) \mathcal{F}_\eta^{\mathcal{EM},p,k}(s,u) + \mathcal{F}_\eta^0(s,u), \quad (2.19)$$

$$\|\mathbf{X}^{-1+\eta}\zeta|Y_{\text{rot}}u|_{k-1}\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \lesssim (1 + \delta^{-1}) \mathcal{F}_\eta^{\mathcal{EM},k}(s,u). \quad (2.20)$$

Proposition 2.5 (Sobolev decay for wave fields). *For all* $\eta \in [0,1)$ *and all functions* u, *one has (with* $k \leq p$)

$$\begin{aligned}
\|r\mathbf{X}^\eta |\partial u|_{p,k}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} &+ \|r^{1+\eta}|\partial^{\mathcal{N}}u|_{p,k}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \\
&\lesssim (1-\eta)^{-1} \mathcal{F}_\eta^{\mathcal{EM},p+3,k+3}(s,u)
\end{aligned} \quad (2.21a)$$

and, for $1/2 < \eta = 1/2 + \delta < 1$,

$$\|r\mathbf{X}^{-1+\eta}|u|_{N-2}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \lesssim \delta^{-1} \mathcal{F}_\eta^{\mathcal{EM},N}(s,u) + \mathcal{F}_\eta^0(s,u). \quad (2.21b)$$

Proposition 2.6 (Pointwise decay of Klein–Gordon fields). *Given any exponents* $\eta \in (0,1)$, *for any solution* v *to* $-\Box v + c^2 v = f$ *defined in* $\mathcal{M}_{[s_0,s_1]}^{\mathcal{EM}}$ *one has*

$$c^2|v|_{p,k} \lesssim \begin{cases} r^{-2}\mathbf{X}^{1-\eta}\mathcal{F}_{\eta,c}^{\mathcal{EM},p+4,k+4}(s,v) + |f|_{p,k} & \text{in } \mathcal{M}_{[s_0,s_1]}^{\text{near}}, \\ r^{-1-\eta}\mathcal{F}_{\eta,c}^{\mathcal{EM},p+2,k+2}(s,v) & \text{in } \mathcal{M}_{[s_0,s_1]}^{\text{far}}. \end{cases}$$

3. Nonlinear stability of Einstein–Klein–Gordon spacetimes

3.1. The class of reference metrics

We introduce a class of reference spacetime metrics $(\mathbb{R}_+^{3+1}, g^\star)$ which represent "approximate solutions" to Einstein's vacuum equations .

- The metric $g^\star = g_{\text{Mink}} + h^\star$ is *asymptotically Minkowski* in the sense that

$$|h^\star|_{N+2} + \langle r + t \rangle |\partial h^\star|_{N+1} + \langle r + t \rangle^2 |\partial\partial h^\star|_N \leq \varepsilon_\star \langle r + t \rangle^{-1}. \quad (3.1a)$$

- This metric is *almost Ricci flat* in the sense that

$$|{}^{(w)}R^\star|_N + \langle r - t \rangle |\partial^{(w)}R^\star|_{N-1} \leq \begin{cases} \varepsilon_\star^2 \langle r + t \rangle^{-4} & \text{in } \mathcal{M}_{[s_0,+\infty)}^{\mathcal{EM}}, \\ \varepsilon_\star \langle r + t \rangle^{-3} & \text{in } \mathcal{M}_{[s_0,+\infty)}^{\mathcal{H}}, \end{cases} \quad (3.1b)$$

where ${}^{(w)}R^\star$ denotes the "reduced" Ricci curvature classically defined by removing suitable contractions of the Christoffel symbols.

- This metric satisfies the *light-bending property,* that is,

$$\inf_{\mathcal{M}_\ell^{\mathrm{near}}} \left(r h^\star(\mathbf{l}, \mathbf{l}) \right) \geq \varepsilon_\star, \tag{3.1c}$$

  in which $g^\star(\mathbf{l}, \mathbf{l})$ is (linearly) equivalent to $-h_\star^{\mathcal{N} 00}$ and we recall that $\mathbf{l} = \partial_t - (x^a/r)\partial_a$ (as stated earlier).

As a consequence of (3.1b) and (2.9) and after introducing a small parameter $\delta > 0$ and some decay exponent

$$\kappa \in (1/2, 1), \tag{3.2}$$

the following rough bound follows by integration:

$$\left\| \mathbf{X}^\kappa J \zeta^{-1} |^{(w)} R^\star |_N \right\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \leq R_{\mathrm{err}}^\star(s) = C_{R\star} \varepsilon_\star^2 \delta^{-1/2} s^{-1-\delta}, \tag{3.3}$$

in which $C_{R\star} > 0$ is a constant determined by N and

$$\int_{s_0}^s R_{\mathrm{err}}^\star(s')\, ds' \leq C_{R\star} \varepsilon_\star^2 \delta^{-3/2} s_0^{-\delta}. \tag{3.4}$$

For further motivations and results we also refer the reader to the more general theory in [30, Section 12]. Furthermore, it is straightforward to check that the Minkowski–Schwarzschild reference metric $g^\star$ in the introduction satisfies all of the conditions above.

3.2. The class of initial data sets

In the following $\delta \in (0, 1)$ denotes a fixed exponent which should be small in comparison to a small multiple of $\kappa - 1/2$, which arises as a critical value for the decay of the metric perturbation. It will be interesting to keep track, in our estimates, of the most relevant parameters and constants. For the initial data we assume the following smallness conditions.

- In the hyperboloidal domain, we assume

$$\mathcal{F}^{\mathcal{H}, N-5}(s_0, u) + s_0^{-1/2} \mathcal{F}_c^{\mathcal{H}, N-5}(s_0, \phi) \leq C_0 \varepsilon s_0^\delta, \tag{3.5a}$$
$$\mathcal{F}^{\mathcal{H}, N-7}(s_0, u) + \mathcal{F}_c^{\mathcal{H}, N-7}(s_0, \phi) \leq C_0 \varepsilon s_0^\delta. \tag{3.5b}$$

- In the Euclidean-merging domain, we assume

$$\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M}, N}(s_0, u) + s_0^{-1} \mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M}, N}(s_0, \phi) \leq C_0 \varepsilon s_0^\delta, \tag{3.6a}$$
$$\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M}, N-5}(s_0, u) + \mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M}, N-5}(s_0, \phi) \leq C_0 \varepsilon s_0^\delta. \tag{3.6b}$$

- Finally, we impose the linear light-bending condition on the *total* initial data (defined as the sum of the reference data and the perturbation) (with $\ell \in (0, 1/2]$ being fixed),

$$\inf_{\mathcal{M}^{\text{near}}_{\ell,[s_0,+\infty)}} \left(rg^\star(\mathbf{l}, \mathbf{l}) + ru_{\text{init}}(\mathbf{l}, \mathbf{l}) \right) \geq \varepsilon_\star. \tag{3.7}$$

Concerning (3.7), we point out that the contributions due to the nonlinearities of the Einstein-matter equations on the key metric component $h^{\mathcal{N}00}$ will turn out to be negligible in comparison to the background and initial linear contributions which are of amplitude $\varepsilon_\star/r$. As a consequence of our energy bounds, recalling [30, (10.38)] and applying the Klainerman–Sobolev inequality on the initial slice, we obtain

$$|\partial_t u(1, x)| + |\partial_x u(1, x)| \lesssim C_0 \varepsilon \langle r \rangle^{-3/2-\kappa}.$$

Integrating with the (spatial) radial derivative from spatial infinity, we obtain

$$\langle r \rangle |\partial_t u(1, x)| + |u(1, x)| \lesssim C_0 \varepsilon \langle r \rangle^{-1/2-\kappa}.$$

Recalling that $\kappa + 1/2 > 1$, we apply [30, Proposition F.2] (based on Kirchkoff's formula) and obtain

$$|u_{\text{init}}|_{N-4} \lesssim C_0 \varepsilon (t + r + 1)^{-1} \qquad \text{in } \mathcal{M}_{[s_0,+\infty)}. \tag{3.8}$$

Here, the linear development u_{init} of the initial data $(u_{0\alpha\beta}, u_{1\alpha\beta})$ is the solution of the (free, linear) wave equation with this initial data. (See also [30, Proposition 10.6].) The passage from geometric statements to statements in coordinates was already discussed in an appendix of [30] to which the reader is referred.

3.3. Nonlinear stability statement

We distinguish between estimates at low- or high-order of differentiation, estimates within the hyperboloidal and Euclidean-merging domains, and a positivity condition near the light cone. The following energy bounds will be established at each time $s \geq s_0$.

- In the hyperboloidal domain, we have

$$\mathscr{F}^{\mathcal{H},N-5}(s, u) + s^{-1/2} \mathscr{F}^{\mathcal{H},N-5}_c(s, \phi) \leq (\varepsilon_\star + C_1 \varepsilon) s^\delta, \tag{3.9a}$$

$$\mathscr{F}^{\mathcal{H},N-7}(s, u) + \mathscr{F}^{\mathcal{H},N-7}_c(s, \phi) \leq (\varepsilon_\star + C_1 \varepsilon) s^\delta. \tag{3.9b}$$

- In the Euclidean-merging domain, we have

$$\mathscr{F}^{\mathcal{EM},N}_\kappa(s, u) + s^{-1} \mathscr{F}^{\mathcal{EM},N}_{\mu,c}(s, \phi) \leq (\varepsilon_\star + C_1 \varepsilon) s^\delta, \tag{3.10a}$$

$$\mathscr{F}^{\mathcal{EM},N-5}_\kappa(s, u) + \mathscr{F}^{\mathcal{EM},N-5}_{\mu,c}(s, \phi) \leq (\varepsilon_\star + C_1 \varepsilon) s^\delta. \tag{3.10b}$$

- Near the light cone, we have the light-bending condition

$$\inf_{\mathcal{M}^{near}_{\ell,s}} (-rh^{\mathcal{N}\,00}) \geq 0. \tag{3.11}$$

Theorem 3.1 (Nonlinear stability of self-gravitating Klein–Gordon fields. Perturbations of metrics with harmonic decay). *A constant $C_\star > 0$ being fixed, the following result holds for all sufficiently small $\varepsilon, \varepsilon_\star$ satisfying $\varepsilon \leq C_\star \varepsilon_\star$. Consider a reference metric $g^\star$, defined as a perturbation of the Minkowski metric satisfying* (3.1) *for some decay exponent*

$$\kappa \in (1/2, 1). \tag{3.12}$$

Consider an initial data set $(u_0, u_1, \phi_0, \phi_1)$ satisfying the decay and regularity conditions (3.5)–(3.7) *at the initial time s_0 for some parameters (N, μ, ε) for a sufficiently large integer N with*

$$\mu \in (3/4, 1), \qquad \kappa \leq \mu. \tag{3.13}$$

Then, the initial value problem for the Einstein–Klein–Gordon system in wave gauge (1.12) *admits a global in time solution (g, ϕ) defined for all $t \geq 1$ and $x \in \mathbb{R}^3$. This solution satisfies the decay and regularity conditions* (3.9)–(3.11) *for all times $s \geq s_0$ and remains close to the reference spacetime $(\mathbb{R}^{3+1}_+, g^\star)$ (in the norms under consideration) and enjoys the following decay estimate*

$$\inf_{\mathcal{M}^{near}_{\ell,s}} (-rh^{\mathcal{N}\,00}) \geq \varepsilon_\star/2, \qquad |h(\mathbf{l}, \mathbf{l})| \lesssim \frac{\varepsilon_\star}{t+r+1}. \tag{3.14}$$

4. Consequences of the energy estimates

4.1. Bootstrap assumptions and basic estimates

Bootstrap assumptions. This result will be established in the succeeding sections by following the strategy already outlined in the introduction. From now on, we assume that the local in time solution in fact extends over an interval $[s_0, s_1]$, and we assume that $[s_0, s_1]$ is the maximal interval of time within which (3.9), (3.10) and (3.11) hold for all $s \in [s_0, s_1]$ so that, by continuity, one of these conditions is an equality at the end time s_1. Our objective is to establish the following *improved estimates* for all $s \in [s_0, s_1]$:

$$\mathcal{F}^{\mathcal{H},N-5}(s, u) + s^{-1/2}\mathcal{F}^{\mathcal{H},N-5}_c(s, \phi) \leq \frac{1}{2}(\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.1a}$$

$$\mathcal{F}^{\mathcal{H},N-7}(s, u) + \mathcal{F}^{\mathcal{H},N-7}_c(s, \phi) \leq \frac{1}{2}(\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.1a}$$

$$\mathcal{F}^{\mathcal{EM},N}_\kappa(s, u) + s^{-1}\mathcal{F}^{\mathcal{EM},N}_{\mu,c}(s, \phi) \leq \frac{1}{2}(\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.2a}$$

$$\mathscr{F}_\kappa^{\mathcal{EM},N-5}(s,u) + \mathscr{F}_{\mu,c}^{\mathcal{EM},N-5}(s,\phi) \leq \frac{1}{2}(\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.2b}$$

$$\inf_{\mathcal{M}_{\ell,[s_0,s_1]}^{\mathrm{near}}} (-rh^{\mathcal{N}00}) \geq \frac{1}{2}\varepsilon_\star. \tag{4.3}$$

Observe that the factor $s^{-1/2}$ arises in (4.1a), while s^{-1} arises in (4.2a). In the following, our statements will be established under the assumptions on the reference metric and initial data and the bootstrap assumptions, as stated above.

Energy-based estimates. The following L^2 estimates for the metric perturbation and the matter field are immediate from the bootstrap assumption (3.10a) and (3.10b) in the Euclidean-merging domain:

$$\|\mathbf{X}^\kappa \zeta|\partial u|_N\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} + \|\mathbf{X}^\kappa|\not\partial^{\mathcal{N}} u|_N\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \lesssim (\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.4a}$$

$$\|\mathbf{X}^\mu \zeta|\partial\phi|_p\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} + \|\mathbf{X}^\mu|\not\partial^{\mathcal{N}}\phi|_p\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} + \|\mathbf{X}^\mu|\phi|_p\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})}$$
$$\lesssim (\varepsilon_\star + C_1\varepsilon) \begin{cases} s^{1+\delta}, & p = N, \\ s^\delta, & p = N - 5. \end{cases} \tag{4.4b}$$

As a consequence of the weighted Poincaré inequality in Proposition 2.4 we find

$$\|\mathbf{X}^{-1+\kappa}|u|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \lesssim \delta^{-1}\mathscr{F}_\kappa^{\mathcal{EM},p,k}(s,u) + \mathscr{F}_\kappa^0(s,u)$$
$$\lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^\delta, \tag{4.5}$$

which provides us with a control of the metric components possibly without partial derivatives.

Pointwise decay of the metric. Basic sup-norm estimates are then derived by applying the generalized Sobolev estimate[1] in Proposition 2.5, and we thus control the metric perturbation ∂u at order $N - 3$, as

$$\|r\mathbf{X}^\kappa|\partial u|_{N-3}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} + \|r^{1+\kappa}|\not\partial^{\mathcal{N}} u|_{N-3}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \lesssim (\varepsilon_\star + C_1\varepsilon)s^\delta. \tag{4.6}$$

In view of $h_{\alpha\beta} = h_{\alpha\beta}^\star + u_{\alpha\beta}$, by combining this result with the asymptotically Minkowski condition (3.1a) on the reference metric, for the metric unknown ∂h we obtain

$$\|r\mathbf{X}^\kappa|\partial h|_{N-3}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} + \|r^{1+\kappa}|\not\partial^{\mathcal{N}} h|_{N-3}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \lesssim (\varepsilon_\star + C_1\varepsilon)s^\delta. \tag{4.7}$$

We apply the Sobolev decay for wave fields (2.21b), together with (3.9a) and (3.10a), and obtain

$$\|r\mathbf{X}^{\kappa-1}|u|_{N-2}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^\delta \tag{4.8}$$

and, using again (3.1a) enjoyed by the reference,

$$\|r^\kappa|h|_{N-2}\|_{L^\infty(\mathcal{M}_s^{\mathcal{EM}})} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^\delta. \tag{4.9}$$

---

[1]We neglect a factor $1/(1-\kappa)$ since we have fixed $\kappa < 1$.

Pointwise decay of the matter field. The Sobolev decay inequality in [30, Proposition 7.4] and the bootstrap assumptions (3.10) provide us with sup-norm estimates

$$\|r\mathbf{X}^{\mu}|\partial\phi|_{p-3}\|_{L^{\infty}(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} + \|r^{1+\mu}|\partial^{\mathcal{N}}\phi|_{p-3}\|_{L^{\infty}(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})}$$
$$\lesssim (\varepsilon_{\star} + C_1\varepsilon) \begin{cases} s^{1+\delta}, & p = N, \\ s^{\delta}, & p = N - 5, \end{cases} \tag{4.10}$$

and, thanks to the consequence in [30, (7.20)] of our generalized Sobolev inequality,

$$\|r\mathbf{X}^{\mu}|\phi|_{p-2}\|_{L^{\infty}(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \lesssim (\varepsilon_{\star} + C_1\varepsilon) \begin{cases} s^{1+\delta}, & p = N, \\ s^{\delta}, & p = N - 5. \end{cases} \tag{4.11}$$

However, within $\mathcal{M}_s^{\text{near}}$ this is not sufficient for our purpose below and we establish a stronger decay, as follows.

Lemma 4.1. *The matter field satisfies the pointwise bound*

$$\|r^2\mathbf{X}^{\mu-1}|\phi|_{p-4}\|_{L^{\infty}(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \lesssim (\varepsilon_{\star} + C_1\varepsilon) \begin{cases} s^{1+2\delta}, & p = N, \\ s^{2\delta}, & p = N - 5. \end{cases}$$

Proof. In view of inequality (4.11), we want to "trade" a factor $\mathbf{X}$ for a factor r, and we only need to deal with the domain $\mathcal{M}_s^{\text{near}}$. We need the decay property near the light cone stated in Proposition 2.6. We consider the Klein–Gordon equation $g^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\phi - c^2\phi = 0$ and with the notation therein, we set

$$f = h^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi = h^{\star\mu\nu}\partial_{\mu}\partial_{\nu}\phi + u^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi.$$

For the first term $h^{\star\mu\nu}\partial_{\mu}\partial_{\nu}\phi$ above, the decay condition on $h^{\star}$ in (3.1a) yields us

$$|h^{\star\mu\nu}\partial_{\mu}\partial_{\nu}\phi|_{p-4} \lesssim \varepsilon_{\star}(\varepsilon_{\star} + C_1\varepsilon)r^{-2}\mathbf{X}^{-\mu} \begin{cases} s^{1+\delta}, & p = N, \\ s^{\delta}, & p = N - 5. \end{cases}$$

For the second term $u^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi$, by [30, Lemma 10.1] we have $|u^{\mu\nu}|_p \lesssim |u|_p$ and, by recalling the Sobolev decay (4.8) and (4.10), we find

$$|u^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi|_{p-4} \lesssim |u|_{p-4}|\partial\phi|_{p-3}$$
$$\lesssim \delta^{-1}(\varepsilon_{\star} + C_1\varepsilon)^2 r^{-2}\mathbf{X}^{1-\kappa-\mu} \begin{cases} s^{1+2\delta}, & p = N, \\ s^{2\delta}, & p = N - 5. \end{cases}$$

On the other hand, recalling (3.10) we have

$$r^{-2}\mathbf{X}^{1-\mu}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,\phi) \lesssim (\varepsilon_{\star} + C_1\varepsilon)r^{-2}\mathbf{X}^{1-\mu} \begin{cases} s^{1+\delta}, & p = N, \\ s^{\delta}, & p = N - 5. \end{cases}$$

We are thus in a position to use the general pointwise decay enjoyed by Klein–Gordon fields, as stated in [30, Proposition 9.5] and we arrive at the desired conclusion. ∎

4.2. Basic estimates for nonlinearities: energy norm

Improving the energy estimates. We differentiate the metric and matter evolution equations (1.12) with respect to $Z = \partial^I L^J \Omega^K$ (with $\operatorname{ord}(Z) = |I| + |J| + |K| \leq N$ or $\leq N - 5$) and obtain

$$\widetilde{\Box}_g Z u = -[Z, h^{\mu\nu}\partial_\mu\partial_\nu]u_{\alpha\beta} + Z\big(\mathbb{P}^\star_{\alpha\beta}[u]\big) + Z\big(\mathbb{Q}^\star_{\alpha\beta}[u]\big)$$
$$+ Z\big(\mathbb{I}^\star_{\alpha\beta}[u] + 2^{(w)}R^\star{}_{\alpha\beta} - u^{\mu\nu}\partial_\mu\partial_\nu g^\star_{\alpha\beta}\big)$$
$$- 8\pi Z\big(2T_{\alpha\beta} - T g_{\alpha\beta}\big), \tag{4.12a}$$

and

$$\widetilde{\Box}_g Z\phi - c^2 Z\phi = -[Z, h^{\mu\nu}\partial_\mu\partial_\nu]\phi, \tag{4.12b}$$

in which $\mathbb{I}^\star[u]$ represent the reference-perturbation interaction terms presented in [30, (10.8)]. Our main task is to control $\|J\zeta^{-1}\mathbf{X}^\kappa|T|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})}$, where T represents any of the terms of the right-hand sides of (4.12). Thanks to (2.10) we have

$$\|J\zeta^{-1}\mathbf{X}^\kappa|T|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} \lesssim \|s\mathbf{X}^\kappa\zeta|T|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})}. \tag{4.13}$$

Linear-critical and super-critical nonlinearities. We treat first the comparatively easier terms, that is, the reference-perturbation interaction terms $\mathbb{I}^\star[u]$, the term $u^{\mu\nu}\partial_\mu\partial_\nu h^\star_{\alpha\beta}$, and the source terms associated with the scalar field. On the other hand, the null terms, the quasi-null term and the commutators require different arguments and will be the subject of later sections. We set

$$W^{\text{linear}} := \mathbb{F}_{\alpha\beta}(g^\star, g^\star; \partial g^\star, \partial u) + \mathbb{F}_{\alpha\beta}(g^\star, g^\star; \partial u, \partial g^\star),$$
$$W^{\text{super}} := \mathbb{F}_{\alpha\beta}(u, g^\star; \partial g^\star, \partial g^\star) + \mathbb{F}_{\alpha\beta}(g^\star, u; \partial g^\star, \partial g^\star) \tag{4.14}$$
$$+ \mathbb{B}^\star_{\alpha\beta}[u] + \mathbb{C}^\star_{\alpha\beta}[u] - 8\pi(2T_{\alpha\beta} - T g_{\alpha\beta}) + 2^{(w)}R^\star{}_{\alpha\beta}.$$

It is a simple matter to check that

$$\|J\zeta^{-1}\mathbf{X}^\kappa|W^{\text{linear}}|_{p,k}\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-1-\delta}, \tag{4.15}$$

while

$$\|J\zeta^{-1}\mathbf{X}^\kappa|W^{\text{super}}|_{p,k}\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-1-\delta}. \tag{4.16}$$

On the other hand, for super-critical terms involving the reference metric we find

$$\|s\mathbf{X}^\kappa\zeta|\mathbb{F}_{\alpha\beta}(u, g^\star; \partial g^\star, \partial g^\star)|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} + \|s\mathbf{X}^\kappa\zeta|\mathbb{B}^\star_{\alpha\beta}[u]|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})}$$
$$+ \|s\mathbf{X}^\kappa\zeta|\mathbb{C}^\star_{\alpha\beta}[u]|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} \lesssim (\varepsilon_\star + C_1\varepsilon)^2 s^{-1-\delta}, \tag{4.17}$$

as well as for super-critical terms involving the matter field

$$\|s\mathbf{X}^\kappa\zeta|2T_{\alpha\beta} - T g_{\alpha\beta}|_N\|_{L^2(\mathcal{M}_s^{\varepsilon\mathcal{M}})} \lesssim (\varepsilon_\star + C_1\varepsilon)^2 s^{-1-\delta}.$$

We can now apply the Hardy–Poincaré inequality in Proposition 2.4 and obtain

$$\|s\mathbf{X}^\kappa \zeta|u\partial\partial h^\star|_N\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})} \lesssim \delta^{-1}\varepsilon_\star(\varepsilon_\star + C_1\varepsilon)s^{-3+\delta} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}. \quad (4.18)$$

4.3. Basic estimates for nonlinearities: pointwise norm

Estimates for the metric. We use our linear bounds and obtain

$$|\mathbb{I}_{\alpha\beta}^\star[u]|_{N-4} + |u^{\mu\nu}\partial_\mu\partial_\nu g^\star|_{N-4} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-3}\mathbf{X}^{-\kappa}s^{2\delta} \quad (4.19)$$

and

$$|u^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta}^\star|_{N-3} \lesssim \delta^{-1}\varepsilon_\star(\varepsilon_\star + C_1\varepsilon)r^{-4}\mathbf{X}^{1-\kappa}s^\delta. \quad (4.20)$$

Estimates for the matter field. For the matter interaction terms, thanks to (4.10), (4.11), and Lemma 4.1 we have

$$\sum_{\alpha,\beta}|2T_{\alpha\beta} - (T_{\gamma\gamma}g^{\gamma\gamma})g_{\alpha\beta}|_{N-3} =: |\mathbb{T}(\phi)|_{N-3}$$

$$\lesssim (\varepsilon_\star + C_1\varepsilon)^2 r^{-3}\mathbf{X}^{1-2\mu}s^{1+3\delta}. \quad (4.21)$$

Here and from now on, we use the short-hand notation $\mathbb{T}(\phi)$ for the matter term contributions $2T_{\alpha\beta} - (T_{\gamma\gamma}g^{\gamma\gamma})g_{\alpha\beta}$. For the proof, we recall the expression of $T_{\alpha\beta}$ and, using the pointwise decay of Klein–Gordon fields in Proposition 2.6, obtain (see [30, (12.35) and Lemma 12.1])

$$|\partial\phi\partial\phi|_{N-3} \lesssim |\partial\phi|_{N-3}|\partial\phi|_{[(N-3)/2]} \lesssim (\varepsilon_\star + C_1\varepsilon)^2 r^{-3}\mathbf{X}^{1-2\mu}s^{1+3\delta}. \quad (4.22)$$

The bound for $|\phi^2|_{N-3}$ is similar and we omit the details. Using the fact that $|h^{\alpha\beta}|_{N-3} \lesssim 1$, we obtain (4.21).

Bounds on null metric component. The null component $g^{\mathcal{N}00}$ of the metric plays a special role in our analysis, and the decay of its gradient is considered now. Thanks to the wave gauge condition as derived in [30, Lemma 11.3], we have the decay property

$$|\partial g^{\mathcal{N}00}|_{N-3} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1-\kappa}s^\delta \qquad \text{in } \mathcal{M}_{[s_0,s_1]}^{\mathcal{EM}}. \quad (4.23)$$

Indeed, consider the right-hand side the inequality in [30, Lemma 11.3] with $p = N - 3$, namely

$$|\partial g^{\mathcal{N}00}|_{N-3} \lesssim |\partial^{\mathcal{N}} h|_{N-3} + r^{-1}|h|_{N-3} + \sum_{p_1+p_2=N-3}|h|_{p_1}|\partial h|_{p_2}.$$

We need to substitute the Sobolev bounds (4.7) on $|\partial h|_{N-3}$ and $|\partial^{\mathcal{N}} h|_{N-3}$, together with the bound for $|h|_{N-2}$ from (4.9).

5. Estimates based on the structure of the Einstein equations

5.1. Commutator and Hessian estimates for the metric

Basic pointwise estimates. Throughout, the bootstrap assumptions (3.10) and (3.11) are assumed, and $(\varepsilon_\star + C_1 \varepsilon)$ is taken to be sufficiently small. So far all the estimates are similar to the ones we had derived in the general regime (but by taking $\lambda = 1$) in [30]. Now, we focus on the *harmonic decay* estimates. From [30, Sections 6 and 8], we recall the following result (extracted from Propositions 6.10 and 8.5 therein). Observe that the inequalities now are simpler in comparison to the general λ regime, due to the fact that now we have a "good" control of the component $H^{\mathcal{N}00}$. We first write [30, Proposition 6.10] in a different form that will be more appropriate in the present paper, and next we state our improved estimate for the Hessian.

Proposition 5.1 (Hessian for the wave equation near the light cone). *Consider the wave operator $\widetilde{\Box}_g u = g^{\alpha\beta} \partial_\alpha \partial_\beta u$ in which $g^{\alpha\beta} = g^{\alpha\beta}_{\text{Mink}} + H^{\alpha\beta}$, and assume that, for some $\varepsilon_1 \ll 1$,*

$$|H| \leq \varepsilon_1 \qquad in \; \mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]}, \tag{5.1}$$

$$\left| H^{\mathcal{N}00} \right| \leq \varepsilon_1 \frac{1 + |r - t|}{r} \qquad in \; \mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]}. \tag{5.2}$$

Then, for any function u one has

$$\frac{1 + |r - t|}{r} |\partial\partial u| \lesssim |\widetilde{\Box}_g u| + t^{-1} |\partial u|_{1,1} \quad in \; \mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]} \tag{5.3}$$

and, more generally at arbitrary order (p, k),

$$\frac{1 + |r - t|}{r} |\partial\partial u|_{p,k} \lesssim |\widetilde{\Box}_g u|_{p,k} + t^{-1} |\partial u|_{p+1,k+1} + T^{\text{hier}}_{p,k}[H, u]$$
$$+ T^{\text{easy}}_{p,k}[H, u] + T^{\text{super}}_{p,k}[H, u] \qquad in \; \mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]}, \tag{5.4}$$

where the notation of (2.15) is used.

Sketch of the proof of Proposition 5.1. From [30, Proposition 8.5] (and by recalling [30, (8.18)]) we find

$$\frac{1 + |r - t|}{r} |\partial\partial u|_{p,k}$$
$$\lesssim |\widetilde{\Box}_g u|_{p,k} + t^{-1} |\partial u|_{p+1,k+1} + \sum_{\substack{\text{ord}(Z) \leq p \\ \text{rank}(Z) \leq k}} \left| [Z, H^{\alpha\beta} \partial_\alpha \partial_\beta] u \right|, \tag{5.5}$$

in which we control the last term thanks to the hierarchy property for commutators (2.15), namely

$$\frac{1 + |r - t|}{r} |\partial\partial u|_{p,k} \lesssim |\widetilde{\Box}_g u|_{p,k} + t^{-1} |\partial u|_{p+1,k+1}$$
$$+ \left(|H^{\mathcal{N}00}| + t^{-1}|r - t||H|\right)|\partial\partial u|_{p-1,k-1}$$
$$+ T^{\mathrm{hier}}_{p,k}[H, u] + T^{\mathrm{easy}}_{p,k}[H, u] + T^{\mathrm{super}}_{p,k}[H, u].$$

Thanks to our smallness assumptions (5.1) and (5.2) (for a sufficiently small ε_1), the third term in the right-hand side above is absorbed on the left-hand side, and (5.4) is established.   ∎

We will also use the second part of [30, Proposition 8.5], restated in the following form.

Proposition 5.2 (Hessian for the wave equation away from the light cone). *Suppose that, for some $\varepsilon_1 \ll \ell$,*

$$|H| \leq \varepsilon_1 \qquad in \ \mathcal{M}^{\mathrm{far}}_{\ell,[s_0,s_1]}. \tag{5.6}$$

Then, for any function u the following pointwise inequality holds (where the commutator is bounded by (2.16)):

$$|\partial\partial u|_{p,k} \lesssim (1 + t\mathbf{X}^{-1})\left(|\widetilde{\Box}_g u|_{p,k} + t^{-1}|\partial u|_{p+1,k+1}\right)$$
$$+ \sum_{\substack{\mathrm{ord}(Z)\leq p \\ \mathrm{rank}(Z)\leq k}} |[Z, H^{\alpha\beta}\partial_\alpha\partial_\beta]u| \qquad in \ \mathcal{M}^{\mathrm{far}}_{\ell,[s_0,s_1]}.$$

Estimates on the Hessian of the metric perturbation. The arguments in [30, Section 13] somewhat simplify since the coupling between the commutator and the Hessian contributions is no longer required at the leading order. For clarity in the presentation, we treat first the remaining terms in (5.4).

Lemma 5.3. *Using the notation (2.15b), for all $p \leq N - 4$ the metric perturbation satisfies*

$$|T^{\mathrm{hier}}[H, u]|_{p,k} + |T^{\mathrm{easy}}[H, u]|_{p,k} + |T^{\mathrm{super}}[H, u]|_{p,k}$$
$$\lesssim \left((\varepsilon_\star + C_1\varepsilon) + r\mathbf{X}^{-1}|Lh^{\mathcal{N}00}|_{k-1}\right)\frac{\mathbf{X}}{r}|\partial\partial u|_{p-1,k-1}$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 \langle r\rangle^{-2}\mathbf{X}^{-\kappa}s^{2\delta} \qquad in \ \mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s_1]}, \tag{5.7}$$

while

$$|[Z, H^{\alpha\beta}\partial_\alpha\partial_\beta]u| \lesssim (\varepsilon_\star + C_1\varepsilon)|\partial\partial u|_{p,k},$$
$$\mathrm{ord}(Z) \leq p, \quad \mathrm{rank}(Z) \leq k \qquad in \ \mathcal{M}^{\mathrm{far}}_{\ell,[s_0,s_1]}. \tag{5.8}$$

Proof. The result is analogous to the one in [30, Proposition 13.1]. In view of (3.1a), (4.7), (4.8) and (4.23), we have

$$
\begin{aligned}
|H^{\mathcal{N}00}|_{N-2} &\lesssim |h^{\mathcal{N}00}|_{N-2}, \\
|\partial H^{\mathcal{N}00}|_{N-3} &\lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1-\kappa}s^\delta, \\
|H|_{N-3} &\lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-\kappa}s^\delta, \\
|\partial H|_{N-3} &\lesssim (\varepsilon_\star + C_1\varepsilon)r^{-1}\mathbf{X}^{-\kappa}s^\delta.
\end{aligned}
\tag{5.9}
$$

Observing that

$$
|\partial\partial u|_{N-4} \lesssim |\partial u|_{N-3} \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-1}\mathbf{X}^{-\kappa}s^\delta
$$

and substituting these bounds in (2.15b), we obtain the desired result (5.7). The estimate (5.8) is established in the same manner and we omit the details. ∎

Proposition 5.4. *There exists a constant $\varepsilon_s > 0$, determined by N, such that, provided*

$$
|h^{\mathcal{N}00}|_{N-4} \le \varepsilon_s r^{-1} \qquad in\ \mathcal{M}^{\mathcal{EM}}_{[s_0,s_1]},
\tag{5.10}
$$

the metric perturbation satisfies

$$
\frac{\mathbf{X}}{r}|\partial\partial u|_{N-4} + |\partial\partial^{\mathcal{N}} u|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-2}\mathbf{X}^{-\kappa}s^{2\delta} \qquad in\ \mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s_1]},
\tag{5.11}
$$

together with

$$
|\partial\partial u|_{N-4} \lesssim \ell^{-1}(\varepsilon_\star + C_1\varepsilon)t^{-1}r^{-1}\mathbf{X}^{-\kappa}s^{2\delta} \qquad in\ \mathcal{M}^{\mathrm{far}}_{\ell,[s_0,s_1]}.
\tag{5.12}
$$

Proof. We recall [30, Lemma 13.3], which we apply with the choice λ replaced by κ which is possible since $\kappa < 1$. We obtain

$$
|\widetilde{\Box}_g u|_{p,k} \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-2}\mathbf{X}^{-\kappa}s^{2\delta} + \varepsilon_\star r^{-3} \qquad in\ \mathcal{M}^{\mathcal{EM}}_{[s_0,s_1]},\ p \le N-4.
\tag{5.13}
$$

Then we substitute (5.7) and (4.6) (applied in $t^{-1}|\partial u|_{p+1,k+1}$) into (5.4), and we obtain the bound for $|\partial\partial u|_{p,k}$, provided $(\varepsilon_\star + C_1\varepsilon) + r\mathbf{X}^{-1}|Lh^{\mathcal{N}00}|_{k-1}$ is sufficiently small. The bound for $|\partial\partial^{\mathcal{N}} u|_{p,k}$ is obtained thanks to [30, (6.4a)]. For (5.12), we apply (5.8) and Proposition 5.2 with $(\varepsilon_\star + C_1\varepsilon)$ sufficiently small. ∎

Hessian of the null component. We also need the following pointwise estimate which is a direct application of the wave gauge condition (1.11).

Lemma 5.5. *As a consequence of* (5.11) *and the wave gauge condition one has*

$$
|\partial\partial h^{\mathcal{N}00}|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-1-\kappa}\mathbf{X}^{-\kappa}s^{2\delta} \qquad in\ \mathcal{M}^{\mathrm{near}}_{[s_0,s_1]}.
$$

Proof. We only give an outline of the argument. Recalling [30, (11.22b)], we have

$$|\partial_t \partial_t h^{\mathcal{N}00}|_{p,k} \lesssim |\partial \partial^{\mathcal{N}} h|_{p,k} + r^{-1}|\partial h|_{p,k}$$
$$+ \sum_{\substack{p_1+p_2=p\\k_1+k_2=k}} \left(|h|_{p_1,k_1}|\partial\partial h|_{p_2,k_2} + |\partial h|_{p_1,k_1}|\partial h|_{p_2,k_2} \right)$$
$$+ r^{-1} \sum_{\substack{p_1+p_2=p\\k_1+k_2=k}} |\partial h|_{p_1,k_1}|h|_{p_2,k_2}.$$

By substituting (4.6), (4.9), and (5.11), the desired result holds under the condition $\delta^{-1}(\varepsilon_\star + C_1\varepsilon) \lesssim 1$. ∎

5.2. Asymptotically harmonic decay of the null metric component

Objective. We now establish the following result which is similar to [30, Proposition 14.1]. Importantly, our bound here is *sharper* than the one derived in [30], which was only concerned with a *sub-harmonic decay*.

Proposition 5.6. *Under the bootstrap assumptions within* $[s_0, s_1]$, *one has*

$$|h^{\mathcal{N}00}|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)\langle r \rangle^{-1} \qquad in\ \mathcal{M}_{[s_0,s_1]}^{\mathcal{EM}}, \tag{5.14}$$

$$\inf_{\mathcal{M}_{\ell,[s_0,s_1]}^{near}} (-r\, h^{\mathcal{N}00}) \geq \frac{1}{2}\varepsilon_\star. \tag{5.15}$$

Pointwise estimates. An important ingredient of the method is the derivation of pointwise estimates, as now stated. Given a triple of data (f, u_0, u_1) we consider the solution $u = u(t, x)$ to

$$\Box u = f, \qquad u(1, x) = u_0(x), \qquad \partial_t u(1, x) = u_1(x), \qquad x \in \mathbb{R}^3, \tag{5.16}$$

and we use the notation

$$u = \Box^{-1}[u_0, u_1, f],$$
$$\Box_{init}^{-1}[u_0, u_1] := \Box^{-1}[u_0, u_1, 0], \tag{5.17}$$
$$\Box_{sour}^{-1}[f] := \Box^{-1}[0, 0, f].$$

We consider the effect of a decaying source, represented by the operator $\Box_{sour}^{-1}$. We denote by $\Lambda_{t,x} := \{(\tau, y) / t - \tau = |x - y|, 1 \leq \tau \leq t\}$ the past light cone associated with a point (t, x). In the present paper, we will rely especially on the critical case below (and refer to [30] for additional details).

Proposition 5.7 (Sup-norm estimates for the wave equation). *Consider a function* f *satisfying, for some* $\alpha_1, \alpha_2, \alpha_3$,

$$|f(\tau, y)| \lesssim C_1 \tau^{\alpha_1}(\tau + |y|)^{\alpha_2}(1 + |\tau - |y||)^{\alpha_3}, \qquad (\tau, y) \in \Lambda_{t,x}. \tag{5.18}$$

Case 1 (typical).   *When $\alpha_1 = -1 + \upsilon$ and $\alpha_2 = -1 - \nu$ and $\alpha_3 = -1 + \mu$ for some $\upsilon + \mu < \nu$ and $0 < \mu, \nu, \upsilon \leq 1/2$, one has*

$$\left|\square_{\mathrm{sour}}^{-1}[f](t,x)\right| \lesssim C_1\left(\upsilon^{-1} + \mu^{-1} + |\mu - \nu|^{-1}\right)|\upsilon + \mu - \nu|^{-1}(t+r)^{-1}. \quad (5.19\mathrm{a})$$

Case 2 (sub-critical).   *When $\alpha_1 = 0$ and $\alpha_2 = -2 - \nu$ and $\alpha_3 = -1 + \mu$ for some $0 < \nu, \mu \leq 1/2$, one has*

$$\left|\square_{\mathrm{sour}}^{-1}[f](t,x)\right| \lesssim C_1 \begin{cases} \mu^{-1}|\mu - \nu|^{-1}(t+r)^{-1}t^{\mu - \nu}, & \mu > \nu, \\ \mu^{-1}(t+r)^{-1}\ln(t+1), & \mu = \nu, \\ \mu^{-1}|\mu - \nu|^{-1}(t+r)^{-1}, & \mu < \nu. \end{cases} \quad (5.19\mathrm{b})$$

Case 3 (critical).   *When $\alpha_1 = 0$ and $\alpha_2 = -2$ and $\alpha_3 = -1 - \mu$ for some $\mu \in (0, 1/2)$, one has*

$$\left|\square_{\mathrm{sour}}^{-1}[f](t,x)\right| \lesssim C_1\mu^{-1}(t+r)^{-1}\left(1 + \mathbf{X}^{-\mu}\ln\left(\frac{t}{\mathbf{X}}\right)\right). \quad (5.19\mathrm{c})$$

Case 4 (super-critical).   *When $\alpha_1 = 0$ and $\alpha_2 = -2 + \nu$ and $\alpha_3 = -1 - \mu$ for some $0 < \nu < \mu < 1/2$, one has*

$$|\square_{\mathrm{sour}}^{-1}[f](t,x)| \lesssim C_1\left(|\mu - \nu|^{-1} + \mu^{-1}\nu^{-1}\mathbf{X}^{-\mu}t^{\nu}\right)(t+r)^{-1}. \quad (5.19\mathrm{d})$$

Controlling $|\square u|_{p,k}$. We recall the following decomposition introduced in [30, Section 14]:

$$\begin{aligned} \mathcal{M}_{[s_0,s_1]}^{\mathrm{good}} &:= \left\{r \geq t - 1 + (\varepsilon_\star + C_1\varepsilon)t^{1/2}\right\} \cap \mathcal{M}_{[s_0,s_1]}^{\mathcal{EM}}, \\ \mathcal{M}_{[s_0,s_1]}^{\mathrm{bad}} &:= \left\{t - 1 \leq r \leq t - 1 + (\varepsilon_\star + C_1\varepsilon)t^{1/2}\right\} \cap \mathcal{M}_{[s_0,s_1]}^{\mathcal{EM}}. \end{aligned} \quad (5.20)$$

The formulation [30, (14.1)–(14.4)], and specifically [30, (14.2)], can be restated as

$$|\mathbb{A}^{\alpha\beta}[h]|_{N-2} \lesssim \delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2\kappa}s^{2\delta}. \quad (5.21)$$

Then the statement in [30, (14.4)] still holds. Furthermore, thanks to (3.1a) and (3.8), we have

$$r|\Xi^{\mathcal{N}00}|_{N-4} + r|u_{\mathrm{init}}^{\mathcal{N}00}|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon).$$

Then we find

$$(1 + r + t)|h^{\mathcal{N}00}|_{p,k} \lesssim \langle r\rangle|u_{\mathrm{sour}}|_{p,k} + (\varepsilon_\star + C_1\varepsilon), \qquad p \leq N - 4. \quad (5.22)$$

The only issue is to estimate $r|u_{\mathrm{sour}}|_{p,k}$ and, for this purpose, we need to bound $|\square u|_{p,k}$. The following result is analogous to [30, Lemma 14.3].

Lemma 5.8. *Under the condition* (5.10), *such that* (5.11) *and* (5.12) *hold, one has*

$$|\Box u|_{N-4} \lesssim \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta} r^{-1-\kappa} \mathbf{X}^{-1+(1-\kappa)}$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta} \qquad in \ \mathcal{M}^{\mathrm{far}}_{\ell,[s_0,s_1]}$$

and

$$|\Box u|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon) r^{-2+\delta} \mathbf{X}^{-1-\kappa} (r|h^{\mathcal{N}00}|_{N-4})$$
$$+ \delta^{-1}(\varepsilon + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta} \qquad in \ \mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s_1]}.$$

Proof. As in the proof of [30, Lemma 14.3], we need to establish pointwise bounds on the following quantities:

$$|h^{\mu\nu}\partial_\mu\partial_\nu u|_k, \qquad |u^{\mu\nu}\partial_\mu\partial_\nu h^\star|_k, \qquad |\mathbb{P}^\star[u]|_k, \qquad |\mathbb{Q}^\star[u]|_k,$$
$$|\mathbb{I}^\star[u]|_k, \qquad |\mathbb{T}[\phi]|_k, \qquad |^{(w)}R^\star_{\alpha\beta}|_k. \tag{5.23}$$

The estimates established in the general theory are sufficient for all but the first term and, for convenience, we recall the previous bounds as follows:

$$|\mathbb{P}^\star[u]|_{N-4} + |\mathbb{Q}^\star[u]|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)^{2-4\delta} r^{-2-3\delta} \mathbf{X}^{-1+\delta},$$
$$|\mathbb{I}^\star[u]|_{N-4} + |u^{\mu\nu}\partial_\mu\partial_\nu h^\star|_{N-4} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta}, \tag{5.24}$$
$$|\mathbb{T}[\phi]|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta},$$

and $|^{(w)}R^\star_{\alpha\beta}|_k$ is bounded by (3.1b), which is stronger than the inequality [30, (14.6)] applied in the proof of [30, Lemma 14.3]. We thus focus on the first term $|h^{\mu\nu}\partial_\mu\partial_\nu u|_k$ and, first of all, we have

$$|h^{\mu\nu}\partial_\mu\partial_\nu u_{\alpha\beta}|_k \lesssim |h^{\mathcal{N}00}\partial_t\partial_t u|_k + |H\partial\partial^{\mathcal{N}} u|_k + r^{-1}|H\partial u|_{N-4}. \tag{5.25}$$

Thanks to (4.8), (5.11) and provided $\kappa - 1/2 \geq (3/2)\delta$, the second term is bounded as

$$|H\partial\partial^{\mathcal{N}} u|_{N-4} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-3} \mathbf{X}^{1-2\kappa} s^{3\delta}$$
$$\lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta} \qquad in \ \mathcal{M}^{\mathrm{near}}_{[s_0,s_1]}.$$

On the other hand, to handle the "far" region we rely on (5.12) in combination with (5.9) which leads us to the relevant estimate that is stated in Lemma 5.8.

The last term of the right-hand side of (5.25) is trivial thanks to the decreasing factor r^{-1},

$$r^{-1}|H\partial u|_{N-4} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-\kappa} \mathbf{X}^{-\kappa} s^\delta \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta} \mathbf{X}^{-1+\delta},$$

provided $\kappa - 1/2 \geq (5/4)\delta$. For the first term of the right-hand side of (5.25), we first apply [30, (13.5)] and (4.9), so

$$|h^{\mathcal{N}00}\partial_t\partial_t u|_{N-4} \lesssim \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta} r^{-1-\kappa} \mathbf{X}^{-1+(1-\kappa)} \qquad in \ \mathcal{M}^{\mathrm{far}}_{[s_0,s_1]}.$$

We have arrived at the first inequality. On the other hand, we rely on (5.10) and apply (5.11) and obtain

$$|h^{\mathscr{N}00}\partial_t\partial_t u|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-2+\delta}\mathbf{X}^{-1-\kappa}\big(r|h^{\mathscr{N}00}|_{N-4}\big) \qquad \text{in } \mathscr{M}^{\text{near}}_{[s_0,s_1]},$$

and, in combination with the above inequalities, we arrive at the second inequality. ∎

Proof of Proposition 5.6. **Step 1.** We consider (5.14) first. We first establish the bound in $\mathscr{M}^{\text{far}}_{\ell,[s_0,s_1]}$, which is achieved by a direct application of Proposition 5.7 in combination with Lemma 5.8 and (5.22). We observe that $\mathscr{M}^{\text{far}}_{\ell,[s_0,s_1]}$ is *past complete*, in the sense that for any $(t,x) \in \mathscr{M}^{\text{far}}_{\ell,[s_0,s_1]}$ the past cone

$$\Lambda_{(t,x)} := \{(\tau,y) \in \mathscr{M}_{[s_0,s_1]} / |y-x| \le t-\tau\}$$

is entirely contained in $\mathscr{M}^{\text{far}}_{\ell,[s_0,s_1]}$. Then, by Proposition 5.7, Case 1 ($\upsilon = (3/2)\delta$, $\mu = 1 - \kappa$, $\nu = \kappa$) and Case 2 ($\mu = \delta$, $\nu = 3\delta$), we obtain (under the condition $\kappa - 1/2 \ge (5/4)\delta$)

$$|u_{\text{sour}}|_{N-4} \lesssim \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2(t+r)^{-1} + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2(t+r)^{-1},$$

which leads us to

$$|u_{\text{sour}}|_{N-4} \lesssim \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2\langle r\rangle^{-1}. \tag{5.26}$$

Then recalling (5.22), we obtain (5.14) in $\mathscr{M}^{\text{far}}_{\ell,[s_0,s_1]}$.

Then we consider the region $\mathscr{M}^{\text{near}}_{\ell,[s_0,s_1]}$ and define

$$s^* = \sup_{s' \in [s_0,s_1]} \{(5.10) \text{ is valid in } \mathscr{M}^{\text{near}}_{\ell,[s_0,s']}\}. \tag{5.27}$$

Provided $(\varepsilon_\star + C_1\varepsilon)$ is sufficiently small, we have $s^* > s_0$ and for all $(t,x) \in \mathscr{M}^{\text{good}}_{\ell,[s_0,s^*]}$,

$$\Lambda_{(t,x)} = \{(\tau,y) \in \mathscr{M}_{[s_0,s_1]} / |y-x| \le t-\tau\} \subset \mathscr{M}^{\text{good}}_{[s_0,s^*]}.$$

We also have

$$\mathbf{X}^{-8\delta} \lesssim (\varepsilon_\star + C_1\varepsilon)^{-8\delta}r^{-4\delta} \qquad \text{in } \mathscr{M}^{\text{good}}_{[s_0,s_1]} \cap \mathscr{M}^{\text{near}}_{\ell,[s_0,s_1]}.$$

Thus, by Lemma 5.8, in $\mathscr{M}^{\text{good}}_{[s_0,s^*]}$,

$$\begin{aligned}
|\Box u|_{N-4} \lesssim{}& \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta}r^{-1-\kappa}\mathbf{X}^{-1+(1-\kappa)} \\
&+ \big(\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 + (\varepsilon_\star + C_1\varepsilon)^{1-8\delta}r|h^{\mathscr{N}00}|_{N-4}\big)r^{-2-3\delta}\mathbf{X}^{-1+\delta}
\end{aligned} \tag{5.28}$$

provided $7\delta \le \kappa$. (Here C is a constant determined by N.) Then, by Proposition 5.7 (Case 1 and Case 2), we obtain

$$\begin{aligned}
|u_{\text{sour}}|_{N-4} \lesssim{}& \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2\langle r\rangle^{-1} \\
&+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^{1-8\delta}\langle r\rangle^{-1}\sup_{\mathscr{M}^{\text{good}}_{[s_0,s^*]}}\big(r|h^{\mathscr{N}00}|_{N-4}\big).
\end{aligned}$$

Recalling (5.22), we obtain

$$
\begin{aligned}
\langle r \rangle |h^{\mathcal{N}00}|_{N-4} \lesssim{} & (\varepsilon_\star + C_1 \varepsilon) + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 \\
& + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^{1-8\delta} \sup_{\mathcal{M}^{\mathrm{good}}_{[s_0,s^*]}} \left(r|h^{\mathcal{N}00}|_{N-4} \right),
\end{aligned}
$$

which leads us to (provided $(\varepsilon_\star + C_1\varepsilon)$ is sufficiently small, as determined by N)

$$
\langle r \rangle |h^{\mathcal{N}00}|_{N-4} \lesssim (\varepsilon_\star + C_1\varepsilon) + \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 \qquad \text{in } \mathcal{M}^{\mathrm{good}}_{[s_0,s^*]}. \tag{5.29}
$$

Next, we consider the region $\mathcal{M}^{\mathrm{bad}}_{[s_0,s^*]} \cap \mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s_1]}$ and we rely on a technique of integration towards the light cone. We observe that for $(t,x) \in \mathcal{M}^{\mathrm{bad}}_{[s_0,s^*]}$, we denote by $(t,\bar{x}) \in \mathcal{M}^{\mathrm{good}}_{[s_0,s^*]} \cap \mathcal{M}^{\mathrm{bad}}_{[s_0,s^*]}$ with $x/|x| = \bar{x}/|\bar{x}|$. Then, for $\mathrm{ord}(Z) \leq N-4$ we have (with $\partial_r^{\mathcal{E}\mathcal{H}} = (x^a/r)\partial_a^{\mathcal{E}\mathcal{H}}$)

$$
Zh^{\mathcal{N}00}(t,x) = Zh^{\mathcal{N}00}(t,\bar{x}) - \int_{|x|}^{|\bar{x}|} \partial_r^{\mathcal{E}\mathcal{H}} Zh^{\mathcal{N}00}(t,\rho x/|x|)\, d\rho.
$$

Then, in view of (5.29) and (4.23), we have

$$
\begin{aligned}
\langle r \rangle |Zh^{\mathcal{N}00}| \lesssim{} & (\varepsilon_\star + C_1\varepsilon) + \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 \\
& + \langle t \rangle \big||\bar{x}| - |x|\big| \sup_{\mathcal{M}^{\mathcal{E}\mathcal{M}}_{[s_0,s^*]}} \{|\partial Zh^{\mathcal{N}00}|_{N-4}\} \\
\lesssim{} & (\varepsilon_\star + C_1\varepsilon) + \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 + (\varepsilon_\star + C_1\varepsilon)^2 \langle t \rangle^{1/2-\kappa+\delta/2}.
\end{aligned}
$$

Combining with (5.29) we find

$$
\sup_{\mathcal{M}^{\mathcal{E}\mathcal{M}}_{[s_0,s^*]}} \{\langle r \rangle |h^{\mathcal{N}00}|_{N-4}\} \lesssim (\varepsilon_\star + C_1\varepsilon) + \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2. \tag{5.30}
$$

Recalling (5.27), we observe that when $(\varepsilon_\star + C_1\varepsilon)$ is sufficiently small, the above bound leads us to

$$
r|h^{\mathcal{N}00}|_{N-4} < \frac{1}{2}\varepsilon_s \qquad \text{in } \mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s^*]}. \tag{5.31}
$$

By continuity, we conclude that $s^* = s_1$ since, otherwise, $\sup_{\mathcal{M}^{\mathrm{good}}_{s^*}} (r|h^{\mathcal{N}00}|_{N-4}) = \varepsilon_s$ would contradict (5.31). Next, (5.14) follows from (5.30), provided $\ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon) \lesssim 1$. This completes the proof of (5.14).

Step 2. We now turn our attention to (5.15), which, in some sense, is a special case of (5.14) but also provides us with a key sign. We recall the following identity (established in [30, (14.3)])

$$
\begin{aligned}
rh^{\mathcal{N}00} &= rg^\star(\mathbf{l},\mathbf{l}) + ru_{\mathrm{init}}(\mathbf{l},\mathbf{l}) + rh^{\mathcal{N}00}_{\mathrm{pertur}} \\
&= r\big(g^\star(\mathbf{l},\mathbf{l}) + u_{\mathrm{init}}(\mathbf{l},\mathbf{l})\big) - r\sum_{\alpha,\beta} \Psi^{\mathcal{N}0}{}_\alpha \Psi^{\mathcal{N}0}{}_\beta u_{\mathrm{sour},\alpha\beta} + r\Psi^{\mathcal{N}0}{}_\alpha \Psi^{\mathcal{N}0}{}_\beta \mathbb{A}^{\alpha\beta}[h].
\end{aligned}
$$

Thanks to (3.7) and (5.21), we find

$$rh^{\mathcal{N}00} \leq -\varepsilon_\star + Cr|u_{\text{sour}}| + \delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 r^{1-2\kappa+\delta}.$$

Substituting (5.14) in (5.28) we obtain

$$
\begin{aligned}
|\Box u| \lesssim{} & \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta} r^{-1-\kappa}\mathbf{X}^{-1+(1-\kappa)} \\
& + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^{2-8\delta} r^{-2-3\delta}\mathbf{X}^{-1+\delta} \qquad \text{in } \mathcal{M}^{\text{good}}_{[s_0,s_1]}.
\end{aligned}
\tag{5.32}
$$

By Proposition 5.7 (Case 1 and Case 2), we obtain

$$\langle r\rangle|u_{\text{sour}}| \lesssim \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^{2-8\delta} \qquad \text{in } \mathcal{M}^{\text{good}}_{[s_0,s_1]},$$

which leads us to (for some constant $C > 0$)

$$rh^{\mathcal{N}00} \leq -\varepsilon_\star + C\ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^{2-8\delta} \qquad \text{in } \mathcal{M}^{\text{good}}_{[s_0,s_1]}.
\tag{5.33}$$

Again, we perform an integration towards the light cone in order to cover $\mathcal{M}^{\text{bad}}_{[s_0,s_1]}$

$$
\begin{aligned}
h^{\mathcal{N}00}(t,x) &= h^{\mathcal{N}00}(t,\overline{x}) - \int_{|x|}^{|\overline{x}|} \partial_r^{\mathcal{E}\mathcal{H}} h^{\mathcal{N}00}(t,\rho x/|x|)\,d\rho \\
&\leq h^{\mathcal{N}00}(t,\overline{x}) + C(\varepsilon_\star + C_1\varepsilon)^2 t^{-1} \\
&\leq -\varepsilon_\star|\overline{x}|^{-1} + C\ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^{2-8\delta}|\overline{x}|^{-1} + C(\varepsilon_\star + C_1\varepsilon)^2 t^{-1} \\
&\leq -\varepsilon_\star r^{-1} + C\big((\varepsilon_\star + C_1\varepsilon)^2 + \ell^{-1}\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^{2-8\delta}\big)r^{-1}
\end{aligned}
$$

in $\mathcal{M}^{\text{bad}}_{[s_0,s_1]}$. We recall that $\varepsilon \leq C_\star\varepsilon_\star$ with $C_\star$ a given constant, thus, when $\varepsilon_\star$ sufficiently small, the above inequality together with (5.33) leads us to (5.15). ∎

5.3. Sharp decay for the gradient of good metric components

Objective and strategy. We now turn our attention to the estimates for $\partial\!\!\!/\,u$ for the good components $u\!\!\!/$ of the metric perturbation.

Proposition 5.9. *Under the bootstrap assumption in $[s_0, s_1]$ and as a consequence of (5.14), one has in $\mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]}$*

$$
\begin{aligned}
|\partial\!\!\!/\,u\!\!\!/|_{N-4,k} \lesssim{} & \big(\ell^{-\delta/2} + \delta^{-2}\big)(\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1}\mathbf{X}^{-1/2-\delta/2} \\
& \cdot \big(1 + (\varepsilon_\star + C_1\varepsilon)(\ln\langle r\rangle)^k\big), \qquad 0 \leq k \leq N-4,
\end{aligned}
\tag{5.34}
$$

$$
\begin{aligned}
|\partial\partial\!\!\!/\,u\!\!\!/|_{N-5,k} \lesssim{} & (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1}\mathbf{X}^{-1-\delta} \\
& \cdot \big(1 + (\varepsilon_\star + C_1\varepsilon)(\ln\langle r\rangle)^k\big), \qquad 0 \leq k \leq N-5.
\end{aligned}
\tag{5.35}
$$

Our proof relies on the following result, which was pointed out first in [35] and formulated as follows in [30].

Proposition 5.10 (Weighted pointwise estimate in the Euclidean-merging domain). *Consider a metric $g^{\alpha\beta} = g_{\text{Mink}}^{\alpha\beta} + H^{\alpha\beta}$ defined in $\mathcal{M}_{\ell,[s_0,s_1]}^{\text{near}}$ and satisfying*

$$|H^{\mathcal{N} 00}| \ll 1, \qquad H^{\mathcal{N} 00} \le 0 \qquad in \ \mathcal{M}_{\ell,[s_0,s_1]}^{\text{near}}. \tag{5.36}$$

Given any $\rho \ge 0$, for any function u defined $\mathcal{M}_{\ell,[s_0,s_1]}^{\text{near}}$ one has

$$\mathbf{X}^{\rho}|(\partial_t - \partial_r)(ru)|(t,x) \lesssim \sup_{\Omega_{s_0,s_1}^{\ell}} (r - t + 2)^{\rho}\big(r|\partial u| + |u|\big)$$
$$+ \int_{t_0}^t \mathbf{X}(\tau,r)^{\rho} r \big(r^{-1}|\partial^{\mathcal{N}} u|_{1,1} + |H||\partial \partial^{\mathcal{N}} u|$$
$$+ r^{-1}|H||\partial u| + |\widetilde{\Box}_g u|\big)\big|_{\varphi_{t,x}(\tau)} \, d\tau,$$

in which the supremum is taken over the set $\Omega_{s_0,s_1}^{\ell} = \mathcal{L}_{\ell,[s_0,s_1]} \cup \mathcal{M}_{\ell,s_0}^{\text{near}}$.

Our strategy is to apply the above estimate to the wave equations satisfied by the components $Z u_{\alpha\beta}^{\mathcal{N}}$. More precisely, we observe that

$$\widetilde{\Box}_g u_{\alpha\beta}^{\mathcal{N}} = L_{1\alpha\beta} + L_{2\alpha\beta} + S_{1\alpha\beta}, \tag{5.37}$$

where $u_{\alpha\beta}^{\mathcal{N}} := \Psi^{\mathcal{N}\alpha'}{}_{\alpha} \Psi^{\mathcal{N}\beta'}{}_{\beta} u_{\alpha'\beta'}$ and

$$L_{1\alpha\beta} := u_{\alpha'\beta'} \widetilde{\Box}_g\big(\Phi^{\mathcal{N}\alpha'}{}_{\alpha} \Phi^{\mathcal{N}\beta'}{}_{\beta}\big),$$
$$L_{2\alpha\beta} := g^{\mu\nu} \partial_\mu\big(\Phi^{\mathcal{N}\alpha'}{}_{\alpha} \Phi^{\mathcal{N}\beta'}{}_{\beta}\big)\partial_\nu u_{\alpha'\beta'},$$
$$S_{1\alpha\beta} := \Phi^{\mathcal{N}\alpha'}{}_{\alpha} \Phi^{\mathcal{N}\beta'}{}_{\beta} \widetilde{\Box}_g u_{\alpha'\beta'}.$$

For any $Z = \partial^I L^J \Omega^K$ with $\text{ord}(Z) \le N - 4$, we then write

$$\widetilde{\Box}_g(Z u_{\alpha\beta}^{\mathcal{N}}) = Z L_{1\alpha\beta} + Z L_{2\alpha\beta} + Z S_{1\alpha\beta} + S_{2\alpha\beta},$$
$$S_{2\alpha\beta} = S_{2\alpha\beta}[u] := -[Z, h^{\mu\nu} \partial_\mu \partial_\nu] u_{\alpha\beta}^{\mathcal{N}}, \tag{5.38}$$
$$\widetilde{\Box}_g(Z \partial_t u_{\alpha\beta}^{\mathcal{N}}) = Z \partial_t(L_{1\alpha\beta} + L_{2\alpha\beta} + S_{1\alpha\beta}) + S_{2\alpha\beta}',$$
$$S_{2\alpha\beta}' = S_{2\alpha\beta}'[u] := -[Z \partial_t, h^{\mu\nu} \partial_\mu \partial_\nu] u_{\alpha\beta}^{\mathcal{N}}. \tag{5.39}$$

In order to eventually control $Z u_{\alpha\beta}^{\mathcal{N}}$, we need to estimate the source terms on the right-hand side, which is our main task in the rest of this Section 5.3.

Estimates on source terms. We are in a position to establish the following result.

Proposition 5.11. *As a consequence of (5.14), for all $\text{ord}(Z) \le N - 4$ and $\text{rank}(Z) = k \le N - 4$ one has*

$$\mathbf{X}^{1/2+\delta/2} r|\widetilde{\Box}_g Z u^{\mathcal{N}}| \lesssim (\varepsilon_\star + C_1 \varepsilon) r^{-1}\big(r \mathbf{X}^{1/2+\delta/2}|\partial u^{\mathcal{N}}|_{N-4,k-1}\big)$$
$$+ \delta^{-1}(\varepsilon_\star + C_1 \varepsilon) r^{-1-\delta}, \tag{5.40a}$$

while for all $\operatorname{ord}(Z) \leq N-5$ *and* $\operatorname{rank}(Z) = k \leq N-5$ *one has*

$$r\mathbf{X}^{1+\delta}|\widetilde{\Box}_g Z \partial_t \psi^{\mathcal{N}}| \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-1}\big(r\mathbf{X}^\kappa |\partial\partial\psi^{\mathcal{N}}|_{N-5,k-1}\big)$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1-\delta}. \tag{5.40b}$$

Proof. The proof is similar to that in [30, Proposition 15.2]. The only difference concerns the estimates for $S_{2\mu\nu}[u]$ and $S'_{2\mu\nu}[u]$, while the remaining terms are bounded by $\delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1-\delta}$. Recalling Proposition 2.1 together with (5.11) and (5.14) (which guarantees (5.10)), for $(\alpha, \beta) \neq (0,0)$ we find

$$|S_{2\alpha\beta}|_{p,k} \lesssim (\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1}|\partial\partial\psi^{\mathcal{N}}|_{p-1,k-1}$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2\langle r\rangle^{-2-\kappa}\mathbf{X}^{-\kappa}s^{3\delta}, \qquad k \leq p \leq N-4,$$
$$|S'_{2\alpha\beta}|_{p,k} \lesssim (\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1}|\partial\partial\partial_t\psi^{\mathcal{N}}|_{p-1,k-1}$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2\langle r\rangle^{-2-\kappa}\mathbf{X}^{-\kappa}s^{3\delta}, \qquad k \leq p \leq N-5. \tag{5.41}$$

Hence, (5.40a) and (5.40b) are established. ∎

Proof of Proposition 5.9. Relying on the notation in Proposition 5.10, we observe that by (4.6) and (5.11) expressed on the relevant cone $r = t(1-\ell)$ or $r - t \simeq \ell r$

$$\sup_{\Omega^\ell_{s_0,s_1}} \mathbf{X}^{1/2+\delta/2}\big(r|\partial Z\psi^{\mathcal{N}}| + |Z\psi^{\mathcal{N}}|\big) \lesssim (\ell^{-\delta/2} + \delta^{-1})(\varepsilon_\star + C_1\varepsilon),$$
$$\sup_{\Omega^\ell_{s_0,s_1}} \mathbf{X}^{1+\delta}\big(r|\partial Z\partial_t\psi^{\mathcal{N}}| + |Z\partial_t\psi^{\mathcal{N}}|\big) \lesssim \ell^{-\delta}(\varepsilon_\star + C_1\varepsilon). \tag{5.42}$$

Then, for $(t, x) \in \mathcal{M}^{\text{near}}_{\ell,[s_0,s_1]}$ and $\operatorname{ord}(Z) = N-4, \operatorname{rank}(Z) = k$, after observing that (5.36) are guaranteed by (3.11) and (5.9), we obtain

$$\mathbf{X}^{1/2+\delta/2}|(\partial_t - \partial_r)(rZ\psi^{\mathcal{N}})(t,x)|$$
$$\lesssim (\ell^{-\delta/2} + \delta^{-1})(\varepsilon_\star + C_1\varepsilon) + \int_{t_0}^t r\langle r-\tau\rangle^{1/2+\delta/2}|\widetilde{\Box}_g Z\psi^{\mathcal{N}}|\big|_{\varphi_{t,x}(\tau)}\, d\tau$$
$$+ \int_{t_0}^t \langle r-\tau\rangle^{1/2+\delta/2}\big(|\partial^{\mathcal{N}}\psi^{\mathcal{N}}|_{N-3} + r|H||\partial\partial^{\mathcal{N}}Z\psi^{\mathcal{N}}|$$
$$+ |H||\partial\psi^{\mathcal{N}}|_{N-4}\big)\big|_{\varphi_{t,x}(\tau)}\, d\tau$$
$$\lesssim (\ell^{-\delta/2} + \delta^{-1})(\varepsilon_\star + C_1\varepsilon)$$
$$+ \int_{t_0}^t (\varepsilon_\star + C_1\varepsilon)r^{-1}\mathbf{X}^{1/2+\delta/2}r|\partial\psi^{\mathcal{N}}|_{N-4,k-1}\big|_{\varphi_{t,x}(\tau)}\, d\tau$$
$$+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)\int_{t_0}^t \tau^{-1-\delta}\, d\tau$$

$$\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) + (\varepsilon_\star + C_1\varepsilon)$$
$$\cdot \int_{t_0}^t \tau^{-1} \mathbf{X}^{1/2+\delta/2} r |\partial \psi^{\mathcal{N}}|_{N-4,k-1}\big|_{\varphi_{t,x}(\tau)} \, d\tau,$$

where we used[2] (5.41), (4.6), and (4.4a). On the other hand, observe that

$$|Zu| \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1}\mathbf{X}^{1-\kappa}s^\delta \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{\delta/2-\kappa} \lesssim (\varepsilon_\star + C_1\varepsilon),$$

which we apply with u replaced by $\psi^{\mathcal{N}}$. This leads us to

$$r\big|(\partial_t - \partial_r)Z\psi^{\mathcal{N}}_{\alpha\beta}\big| \lesssim (\varepsilon_\star + C_1\varepsilon) + |(\partial_t - \partial_r)(rZ\psi^{\mathcal{N}})|. \tag{5.43}$$

Now recalling (4.6), for all $\mathrm{ord}(Z) \leq N - 4$ we have

$$r|\partial^{\mathcal{N}} Zu| \lesssim C_1\varepsilon r^{-\kappa+\delta/2} \lesssim (\varepsilon_\star + C_1\varepsilon).$$

Recalling the identities $2\partial_t = (x^a/r)\partial^{\mathcal{N}}_a + (\partial_t - \partial_r)$ and $\partial_a = \partial^{\mathcal{N}}_a - (x^a/r)\partial_t$, together with the ordering lemma [30, Lemma 5.8], we obtain the following bound in $\mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s_1]}$:

$$r\mathbf{X}^{1/2+\delta/2}|\partial\psi^{\mathcal{N}}|_{N-4,k} \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) + (\varepsilon_\star + C_1\varepsilon)$$
$$\cdot \int_{t_0}^t \tau^{-1} r\mathbf{X}^{1/2+\delta/2}|\partial\psi^{\mathcal{N}}|_{N-4,k-1}\big|_{\varphi_{t,x}(\tau)} \, d\tau.$$

Finally, with the notation $\mathbf{B}_k(t) := \sup_{\mathcal{M}^{\mathrm{near}}_{\ell,[s_0,s]}} (r\mathbf{X}^{1/2+\delta/2}|\partial\psi^{\mathcal{N}}|_{N-4,k})$ (with $t = T^{\mathcal{E}}(s)$), the above estimate reads

$$\mathbf{B}_k(t) \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) + (\varepsilon_\star + C_1\varepsilon)\int_{t_0}^t \tau^{-1}\mathbf{B}_{k-1}(\tau)\, d\tau, \tag{5.44}$$

in which the last term does not exist when $k = 0$. Next, by induction on k varying from $k = 0$ to $k = N - 4$, we conclude and arrive at (5.34). The estimate (5.35) is established in a similar way, and we omit the details.

Useful inequalities. For $k \geq 1$, we can relax (5.44) in the form

$$\mathbf{B}_k(t) \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) + (\varepsilon_\star + C_1\varepsilon)\int_{t_0}^t \tau^{-1}\mathbf{B}_k(\tau)\, d\tau. \tag{5.45}$$

By Gronwall's inequality, we obtain the slightly weaker version

$$|\partial\psi|_{N-4,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1+C(\varepsilon_\star + C_1\varepsilon)}\mathbf{X}^{-1/2-\delta/2}, & 1 \leq k \leq N-4, \\ (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)\langle r\rangle^{-1}\mathbf{X}^{-1/2-\delta/2}, & k = 0; \end{cases}$$
$$\tag{5.46a}$$

---

[2]The term $\psi^{\mathcal{N}}$ is a finite linear combination of u with homogeneous coefficients of degree zero, thus the bounds can be applied.

$$|\partial \partial \varkappa|_{N-5,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\langle r \rangle^{-1+C(\varepsilon_\star + C_1 \varepsilon)}\mathbf{X}^{-1-\delta}, & 1 \le k \le N-5, \\ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\langle r \rangle^{-1}\mathbf{X}^{-1-\delta}, & k = 0, \end{cases}$$
$$(5.46b)$$

5.4. Pointwise estimates for metric components at low order

Objective. We now collect the pointwise estimates obtained in the previous sections for the different metric components. The main result is stated as follows.

Proposition 5.12. *In* $\mathcal{M}_{\ell,[s_0,s_1]}^{\mathrm{near}}$, *the metric components satisfy*

$$|\partial h|_{N-4,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1/2-\delta/2}\langle r \rangle^{-1}, & k = 0, \\ (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1/2-\delta/2}\langle r \rangle^{-1+C(\varepsilon_\star + C_1 \varepsilon)}, & 1 \le k \le N-4, \end{cases}$$
$$(5.47)$$

$$|\partial \partial h|_{N-5,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1-\delta}\langle r \rangle^{-1}, & k = 0, \\ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1-\delta}\langle r \rangle^{-1+C(\varepsilon_\star + C_1 \varepsilon)}, & 1 \le k \le N-5. \end{cases}$$
$$(5.48)$$

Furthermore, the source terms $u_{\mathrm{sour},\alpha\beta}$ *defined in* [30, (14.1)] *enjoys the near-harmonic decay in the whole domain* $\mathcal{M}_{[s_0,s_1]}^{\mathcal{E}\mathcal{M}}$,

$$|u_{\mathrm{sour}}|_k \lesssim (\varepsilon_\star + C_1 \varepsilon) r^{-1+C(\varepsilon_\star + C_1 \varepsilon)^{1/2}}, \qquad 0 \le k \le N-5. \tag{5.49}$$

We also have estimates for the metric with upper indices, that is

$$|\partial H| := \max_{\alpha,\beta} |\partial h^{\alpha\beta}|.$$

Corollary 5.13. *In* $\mathcal{M}_{\ell,[s_0,s_1]}^{\mathrm{near}}$, *the metric perturbation satisfies*

$$|\partial u|_{N-4,k} + |\partial H|_{N-4,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1/2-\delta/2}\langle r \rangle^{-1}, & k = 0, \\ (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1/2-\delta/2}\langle r \rangle^{-1+C(\varepsilon_\star + C_1 \varepsilon)}, & 1 \le k \le N-4, \end{cases}$$

$$|\partial \partial u|_{N-5,k} + |\partial \partial H|_{N-5,k}$$
$$\lesssim \begin{cases} (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1-\delta}\langle r \rangle^{-1}, & k = 0, \\ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1 \varepsilon)\mathbf{X}^{-1-\delta}\langle r \rangle^{-1+C(\varepsilon_\star + C_1 \varepsilon)}, & 1 \le k \le N-4. \end{cases}$$
$$(5.50)$$

Furthermore, as a consequence of (3.1a), *in* $\mathcal{M}^{\mathcal{EM}}_{[s_0,s_1]}$, *one has*

$$|u|_{N-5} + |h_{\alpha\beta}|_{N-5} + |h^{\alpha\beta}|_{N-5} \lesssim (\varepsilon_\star + C_1\varepsilon)\langle r \rangle^{-1+C(\varepsilon_\star + C_1\varepsilon)^{1/2}}. \qquad (5.51)$$

Sharp decay bounds on gradient and Hessian. The following identities will be used.

Lemma 5.14. *When* $|h|_{[p/2]} \ll 1$, *one has* $h^{\mathcal{N}}_{00} = h_{00} = -\frac{1}{4}h^{\mathcal{N}00} + \mathbb{B}[h]$ *with*

$$\begin{aligned}
\left|\partial_t \mathbb{B}[h]\right|_{p,k} &\lesssim |\partial_t \not{h}^{\mathcal{N}}|_{p,k} + |\partial\mathbb{A}[h]|_{p,k}, \\
\left|\partial_t\partial_t \mathbb{B}[h]\right|_{p,k} &\lesssim |\partial_t\partial_t \not{h}^{\mathcal{N}}|_{p,k} + |\partial\partial\mathbb{A}[h]|_{p,k}
\end{aligned} \qquad (5.52)$$

with

$$|\partial\mathbb{A}[h]|_{p,k} \lesssim \sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} |\partial h|_{p_1,k_1}|h|_{p_2,k_2},$$

$$\begin{aligned}
|\partial\partial\mathbb{A}[h]|_{p,k} &\lesssim \sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} \big(|\partial\partial h|_{p_1,k_1}|h|_{p_2,k_2} + |\partial h|_{p_1,k_1}|\partial h|_{p_2,k_2}\big) \\
&\quad + \sum_{\substack{p_1+p_2+p_3=p \\ k_1+k_2+k_3=k}} |\partial h|_{p_1,k_1}|\partial h|_{p_2,k_2}|h|_{p_3,k_3}.
\end{aligned}$$

Proof. We only outline the proof and refer to [30] where we derived the algebraic identity

$$-h^{\mathcal{N}00} = h_{00} - \frac{2x^a}{r}h_{a0} + \frac{x^a x^b}{r^2}h_{ab} - \Psi^{\mathcal{N}0}{}_{\alpha}\Psi^{\mathcal{N}0}{}_{\beta}\mathbb{A}^{\alpha\beta}[h], \qquad (5.53a)$$

with

$$\begin{aligned}
h_{00} &= h^{\mathcal{N}}_{00}, \\
h_{a0} &= h^{\mathcal{N}}_{\alpha\beta}\Psi^{\mathcal{N}\alpha}{}_a\Psi^{\mathcal{N}\beta}{}_0 = -(x^a/r)h^{\mathcal{N}}_{00} + \sum_{(\alpha,\beta)\neq(0,0)} \Psi^{\mathcal{N}\alpha}{}_a\Psi^{\mathcal{N}\beta}{}_0 \not{h}^{\mathcal{N}}_{\alpha\beta}, \\
h_{ab} &= \frac{x^a x^b}{r^2}h^{\mathcal{N}}_{00} + \sum_{(\alpha,\beta)\neq(0,0)} \Psi^{\mathcal{N}\alpha}{}_a\Psi^{\mathcal{N}\beta}{}_b \not{h}^{\mathcal{N}}_{\alpha\beta}.
\end{aligned} \qquad (5.53b)$$

Substituting (5.53b) into (5.53a) and separating the linear terms from the nonlinear ones, we obtain the result. ∎

Proof of (5.47). Combining Lemma 5.14 with (4.7) and (4.9), we obtain

$$|\partial\mathbb{A}|_{N-4} + |\partial\partial\mathbb{A}|_{N-5} \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-1-\kappa}\mathbf{X}^{-\kappa}s^{2\delta}. \qquad (5.54)$$

Combining (4.23), Lemma 5.5, and the condition $\delta^{-1}(\varepsilon_\star + C_1\varepsilon) \lesssim 1$, we thus find

$$|\partial_t h_{00}^{\mathcal{N}}|_{N-4,k} \lesssim |\partial_t \hbar^{\mathcal{N}}|_{N-4,k} + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)r^{-1-\kappa}s^{2\delta},$$

$$|\partial_t \partial_t h_{00}^{\mathcal{N}}|_{N-5,k} \lesssim |\partial_t \partial_t \hbar^{\mathcal{N}}|_{N-5,k} + (\varepsilon_\star + C_1\varepsilon)r^{-1-\kappa}\mathbf{X}^{-\kappa}s^{2\delta},$$

together with

$$|\partial_t^m h|_{p,k} \lesssim |\partial^m \hbar^{\mathcal{N}}|_{p,k} + |\partial_t^m h_{00}^{\mathcal{N}}|_{p,k} \lesssim |\partial^m \psi^{\mathcal{N}}|_{p,k} + |\partial^m h^\star|_{p,k} + |\partial_t^m h_{00}^{\mathcal{N}}|_{p,k}.$$

In view of Proposition 5.9 and (3.1a), we arrive at (5.47) and (5.48).

Proof of (5.50). We only outline the argument. For the bounds on $|\partial u|$ and $|\partial\partial u|$, we only need to observe that $h_{\alpha\beta} = h_{\alpha\beta}^\star + u_{\alpha\beta}$ together with (3.1a). For the bounds on $|\partial H|$ and $|\partial\partial H|$, namely $|\partial h^{\alpha\beta}|$ and $|\partial\partial h^{\alpha\beta}|$, we recall that $-h^{\alpha\beta} = h_{\alpha\beta} + \mathbb{A}^{\alpha\beta}[h]$ and apply (5.54).

Proof of (5.49). We rely on Proposition 5.7, and the proof is quite similar to that of the source in [30, Proposition 16.1], while the only difference is that here we have better pointwise bounds (5.47), (5.48) and (5.50). More precisely, by (4.19), (4.20), (4.21) and (3.1b), the terms $\mathbb{I}^\star[u]$, $u^{\mu\nu}\partial_\mu \partial_\nu h^\star$, $\mathbb{T}[\phi]$ and $^{(w)}R_{\alpha\beta}^\star$ are bounded (provided $\kappa \geq 5\delta$ and $\mu \geq 3/4 + (7/4)\delta$) by

$$\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta}\mathbf{X}^{-1+\delta}.$$

We only need to improve the estimate of $|\Box u_{\alpha\beta}|_{N-5}$ in $\mathcal{M}_{\ell,[s_0,s_1]}^{\text{near}}$. More precisely, we have

$$|\mathbb{P}^\star[u]|_k + |\mathbb{Q}^\star[u]|_k$$
$$\lesssim \begin{cases} (\ell^{-\delta} + \delta^{-4})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2}, & k = 0, \\ (\ell^{-\delta} + \delta^{-4})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2+C(\varepsilon_\star+C_1\varepsilon)}, & 1 \leq k \leq N-4. \end{cases}$$
$$(5.55)$$

Similarly, considering the most challenging component $h^{\mathcal{N}00}\partial_t \partial_t u$ we improve the estimate of quasi-linear terms as

$$|h^{\mu\nu}\partial_\mu \partial_\nu u|_k \lesssim \begin{cases} (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2}, & k = 0, \\ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2+C(\varepsilon_\star+C_1\varepsilon)}, & 1 \leq k \leq N-5, \end{cases}$$
$$(5.56)$$

provided $\kappa \geq 1/2 + (5/4)\delta$. We thus conclude that, in $\mathcal{M}_{[s_0,s_1]}^{\text{near}}$,

$$|\Box u_{\alpha\beta}|_k \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2-3\delta}\mathbf{X}^{-1+\delta}$$
$$+ \begin{cases} (\ell^{-\delta} + \delta^{-4})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2}, & k = 0, \\ (\ell^{-\delta} + \delta^{-4})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta}r^{-2+C(\varepsilon_\star+C_1\varepsilon)}, & 1 \leq k \leq N-4. \end{cases}$$
$$(5.57)$$

In $\mathcal{M}^{\text{far}}_{\ell,[s_0,s_1]}$ and for all $k \leq N - 5$, we use $\mathbf{X}^{-1} \lesssim \ell^{-1}\langle r\rangle^{-1}$ together with (4.6), (4.9) and (5.12), and arrive at

$$
\begin{aligned}
|\partial u\, \partial u|_{N-5} &\lesssim \ell^{-4\delta}(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta}\mathbf{X}^{-1+\delta}, \\
|h^{\alpha\beta}\partial_\alpha\partial_\beta u|_{N-5} &\lesssim \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta} r^{-1-\kappa}\mathbf{X}^{-1+\kappa}.
\end{aligned}
\tag{5.58}
$$

In conclusion, we have a control of the wave operator

$$
\begin{aligned}
|\Box u_{\alpha\beta}|_k \lesssim\ & (\ell^{-4\delta} + \delta^{-1})(\varepsilon_\star + C_1\varepsilon)^2 r^{-2-3\delta}\mathbf{X}^{-1+\delta} \\
&+ \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 t^{-1+(3/2)\delta} r^{-1-\kappa}\mathbf{X}^{-1+\kappa} \\
&+ (\ell^{-\delta} + \delta^{-4})(\varepsilon_\star + C_1\varepsilon)^2 \mathbf{X}^{-1-\delta} r^{-2+C(\varepsilon_\star + C_1\varepsilon)^{1/2}}.
\end{aligned}
\tag{5.59}
$$

Here, for simplicity we have replaced the decay factors r^{-1} and $r^{-2+C(\varepsilon_\star + C_1\varepsilon)}$ by $r^{-2+C(\varepsilon_\star + C_1\varepsilon)^{1/2}}$. Applying Proposition 5.7 for the case $k \geq 1$, we obtain

$$
|u_{\text{sour}}|_k \lesssim \underbrace{(\ell^{-4\delta} + \delta^{-1})\delta^{-2}(\varepsilon_\star + C_1\varepsilon)^2 r^{-1}}_{\text{Case 2 with } \mu=\delta, \nu=3\delta} + \underbrace{\ell^{-1}\delta^{-3}(\varepsilon_\star + C_1\varepsilon)^2 r^{-1}}_{\substack{\text{Case 1 with } \upsilon=(3/2)\delta \\ \mu=1-\min(\lambda,\kappa), \nu=\min(\lambda,\kappa), \\ \nu-\mu-\upsilon\geq\delta}}
$$

$$
+ \underbrace{(\ell^{-\delta} + \delta^{-4})\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^{3/2} r^{-1+C(\varepsilon_\star + C_1\varepsilon)^{1/2}}}_{\text{Case 4 with } \mu=\delta, \nu=C(\varepsilon_\star + C_1\varepsilon)^{1/2}, C(\varepsilon_\star + C_1\varepsilon)^{1/2}\leq\delta/2}.
$$

Provided $(\ell^{-\delta} + \delta^{-4})\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^{1/2} \lesssim 1$, we arrive at (5.49).

Proof of (5.51). We rely on the decomposition

$$
h_{\alpha\beta} = h^\star_{\alpha\beta} + u_{\alpha\beta} = h^\star_{\alpha\beta} + u_{\text{init},\alpha\beta} + u_{\text{sour},\alpha\beta}
$$

and, by recalling (3.1a) and (1.20) together with (5.49), we can control $|h_{\alpha\beta}|$. Further, for $|h^{\alpha\beta}|$, we only need to observe that $\max_{\alpha,\beta} |h^{\alpha\beta}| \lesssim \max_{\alpha,\beta} |h_{\alpha\beta}|$, provided that $\max_{\alpha,\beta} |h_{\alpha\beta}|$ is sufficiently small.

6. Closing the bootstrap estimates

6.1. Improved energy estimate for general metric components

Objective. This section is devoted to the following result. The proof begins with the energy estimate (2.12) applied to (4.12). Sufficient integrable L^2 decay must be checked for the terms arising on the right-hand side of the energy estimate.

Proposition 6.1 (Improved energy estimates for the matter field). *Under the bootstrap assumptions, the matter field satisfies*

$$
\mathcal{F}^{\mathcal{E}\mathcal{M},N}_\kappa(s,u) \leq \frac{C_1}{2}\varepsilon s^\delta, \qquad s \in [s_0, s_1].
\tag{6.1}
$$

Nonlinearities.

Proposition 6.2 (Sharp energy estimates for nonlinearities). *Under the bootstrap assumptions and as a consequence of the pointwise metric estimates* (5.50)*, for all* $s \in [s_0, s_1]$ *one has*

$$
\begin{aligned}
\left\| \mathbf{X}^{\kappa} J \zeta^{-1} |\widetilde{\Box}_g u|_{p,k} \right\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} &\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1 \varepsilon)\big(s^{-1} \mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M}, p, k}(s, u) \\
&\quad + s^{-1 + C(\varepsilon_{\star} + C_1 \varepsilon)} \mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M}, p, k-1}(s, u) \big) \\
&\quad + (\varepsilon_{\star} + C_1 \varepsilon)^2 s^{-1-\delta} + R_{\text{err}}^{\star}(s),
\end{aligned}
\tag{6.2}
$$

where $R_{\text{err}}^{\star}(s)$ *denotes the Ricci upper bound in* (3.3).

Proof. Null semi-linear terms near the light cone. Recalling the structure of the null terms, we find

$$
\begin{aligned}
|\mathbb{Q}^{\star}[u]|_{p,k} &:= \max_{\alpha, \beta} |\mathbb{Q}_{\alpha\beta}^{\star}[u]|_{p,k} \\
&\lesssim \sum_{\substack{p_1 + p_2 = p \\ k_1 + k_2 = k}} |\partial u|_{p_1, k_1} |\partial^{\mathcal{N}} u|_{p_2, k_2} + |h^{\star}|_p \sum_{\substack{p_1 + p_2 = p \\ k_1 + k_2 = k}} |\partial u|_{p_1, k_1} |\partial u|_{p_2, k_2}
\end{aligned}
\tag{6.3}
$$

and, in combination with (5.50),

$$
\begin{aligned}
|\mathbb{Q}^{\star}[u]|_{p,k} &\lesssim \sum_{\substack{p_1 + p_2 = p \\ k_1 + k_2 = k}} |\partial u|_{p_1, k_1} |\partial^{\mathcal{N}} u|_{p_2, k_2} + \varepsilon_{\star} r^{-1} |\partial u|_{p,k} |\partial u|_{p_1, k_1} \\
&\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1 \varepsilon) \\
&\quad \cdot \big(\langle r \rangle^{-1} |\partial^{\mathcal{N}} u|_{p,k} + \langle r \rangle^{-1 + C(\varepsilon_{\star} + C_1 \varepsilon)} |\partial^{\mathcal{N}} u|_{p-1, k-1} \big) \\
&\quad + (\varepsilon_{\star} + C_1 \varepsilon) s^{\delta} r^{-1-\kappa} |\partial u|_p =: G_1 + G_2.
\end{aligned}
\tag{6.4}
$$

We integrate the above inequality so that, thanks to (2.10),

$$
\begin{aligned}
\left\| \mathbf{X}^{\kappa} J \zeta^{-1} G_2 \right\|_{L^2(\mathcal{M}_{\ell,s}^{\text{near}})} &\lesssim (\varepsilon_{\star} + C_1 \varepsilon) s^{-1-2\kappa+\delta} \left\| \mathbf{X}^{\kappa} \zeta |\partial u|_p \right\|_{L^2(\mathcal{M}_{\ell,s}^{\text{near}})} \\
&\lesssim (\varepsilon_{\star} + C_1 \varepsilon)^2 s^{-1-\delta}.
\end{aligned}
\tag{6.5}
$$

On the other hand, we have

$$
\begin{aligned}
&\left\| \mathbf{X}^{\kappa} J \zeta^{-1} G_1 \right\|_{L^2(\mathcal{M}_{\ell,s}^{\text{near}})} \\
&\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1 \varepsilon) s^{-1} \left\| \mathbf{X}^{\kappa} \zeta |\partial^{\mathcal{N}} u|_{p,k} \right\|_{L^2(\mathcal{M}_{\ell,s}^{\text{near}})} \\
&\quad + (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1 \varepsilon) s^{-1 + C(\varepsilon_{\star} + C_1 \varepsilon)^{1/2}} \left\| \mathbf{X}^{\kappa} \zeta |\partial^{\mathcal{N}} u|_{p-1, k-1} \right\|_{L^2(\mathcal{M}_{\ell,s}^{\text{near}})} \\
&\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1 \varepsilon) \\
&\quad \cdot \big(s^{-1} \mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M}, p, k}(s, u) + s^{-1 + C(\varepsilon_{\star} + C_1 \varepsilon)} \mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M}, p-1, k-1}(s, u) \big).
\end{aligned}
$$

Quasi-null terms near the light cone. Recalling [30, Lemma 11.2], we have

$$
\begin{aligned}
|\mathbb{P}^{\star}[u]|_{p,k} &\lesssim |\mathbb{P}_{00}^{\star}[u]|_{p,k} + |\mathbb{P}^{\star}[u]|_{p,k} \\
&\lesssim \underbrace{\sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} |\partial \psi^{\mathcal{N}}|_{p_1,k_1} |\partial \psi^{\mathcal{N}}|_{p_2,k_2}}_{G_3} \\
&\quad + \underbrace{\sum_{p_1+p_2=p} \left(|\partial^{\mathcal{N}} u|_{p_1} |\partial u|_{p_2} + |\mathbb{S}_{p_1}^{\mathcal{E}\mathcal{M}}[u]| |\partial u|_{p_2} \right)}_{G_{4a}} \\
&\quad + \underbrace{\sum_{p_1+p_2+p_3=p} |h^{\star}|_{p_3} |\partial u|_{p_1} |\partial u|_{p_2}}_{G_{4b}}
\end{aligned}
$$

The terms in G_{4a} and G_{4b} are null terms and high-order terms, bounded similarly to G_2, so we can focus our attention on G_3. By applying (5.46a), we find

$$
\begin{aligned}
\|s\mathbf{X}^{\kappa}\zeta G_3\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} &\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_{\star} + C_1\varepsilon) \\
&\quad \cdot \left(s^{-1}\mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M},p,k}(s,u) + s^{-1+C(\varepsilon_{\star}+C_1\varepsilon)}\mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M},p-1,k-1}(s,u) \right).
\end{aligned}
$$

Quadratic semi-linear terms away from the light cone. It remains to derive the desired bound in $\mathcal{M}_{\ell,s}^{\mathrm{far}}$. We observe that $\mathbf{X}^{-1} \lesssim \ell^{-1}s^{-2}$ so, thanks to (6.3) and our bound on $|\mathbb{P}^{\star}[u]|_{p,k}$,

$$
\begin{aligned}
|\mathbb{Q}^{\star}[u]|_{p,k} + |\mathbb{P}^{\star}[u]|_{p,k} &\lesssim (1 + |h^{\star}|_N)|\partial u \partial u|_N \lesssim \sum_{\substack{p_1+p_2=p \\ k_1+k_2=k}} |\partial u|_{p_1,k_1} |\partial u|_{p_2,k_2} \\
&\lesssim |\partial u|_{p,k} |\partial u|_{[N/2]} \lesssim (\varepsilon_{\star} + C_1\varepsilon)r^{-1}\mathbf{X}^{-\kappa}s^{\delta}|\partial u|_{p,k} \\
&\lesssim \ell^{-\delta/2}(\varepsilon_{\star} + C_1\varepsilon)r^{-1}\mathbf{X}^{-\kappa+\delta/2}|\partial u|_{p,k}.
\end{aligned}
$$

We thus find

$$
\begin{aligned}
\|\mathbf{X}^{\kappa}J\zeta^{-1}G_3\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})} &\lesssim \ell^{-\delta/2}(\varepsilon_{\star} + C_1\varepsilon)\|r^{-1}s\mathbf{X}^{\kappa}\zeta|\partial u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})} \\
&\lesssim \ell^{-\delta/2}(\varepsilon_{\star} + C_1\varepsilon)s^{-1}\mathcal{F}_{\kappa}^{\mathcal{E}\mathcal{M},p,k}(s,u). \qquad \blacksquare
\end{aligned}
$$

Commutators.

Proposition 6.3 (Sharp energy estimates for commutators). *As a consequence of Lemma* 5.13, *for all* $\operatorname{ord}(Z) = p$ *and* $\operatorname{rank}(Z) = k$ *one has*

$$
\begin{aligned}
\|\mathbf{X}^\kappa & J\zeta^{-1}[Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]u\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\lesssim \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p,k}(s,u) \\
&\quad + \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star + C_1\varepsilon)}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p-1,k-1}(s,u) \\
&\quad + \|\mathbf{X}^\kappa J\zeta^{-1}|\widetilde{\Box}_g u|_{p-1,k-1}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}.
\end{aligned}
$$

Proof. Estimates away from the light cone. For simplicity in the notation, we define

$$
\mathbf{A}_{p,k}(s) := \sum_{\substack{\operatorname{ord}(Z) \le p \\ \operatorname{rank}(z) \le k}} \|\mathbf{X}^\kappa J\zeta^{-1}[Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]u\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})},
$$

$$
\mathbf{B}_{p,k}(s) := \|\mathbf{X}^\kappa J\zeta^{-1}|\partial\partial u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})}.
$$

We recall Proposition 5.2, which gives us

$$
\begin{aligned}
\mathbf{B}_{p,k}(s) &\lesssim \mathbf{A}_{p,k}(s) + \ell^{-1}\|\mathbf{X}^\kappa J\zeta^{-1}|\widetilde{\Box}_g u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})} \\
&\quad + \ell^{-1}s^{-1}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p+1,k+1}(s,u).
\end{aligned}
\tag{6.6}
$$

We then rely on (2.16) and control each term in this inequality as follows.

- For $|Y_{\mathrm{rot}}h|_{p_1-1}|\partial\partial u|_{p_2,k_2}$ we find

$$
|Y_{\mathrm{rot}}h|_p \lesssim |Y_{\mathrm{rot}}h^\star|_p + |Y_{\mathrm{rot}}u|_p \lesssim \varepsilon_\star\langle r\rangle^{-1} + |Y_{\mathrm{rot}}u|_p,
$$

  which leads us to

$$
\begin{aligned}
\|\mathbf{X}^\kappa & J\zeta^{-1}|Y_{\mathrm{rot}}h|_{p_1-1}|\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})} \\
&\lesssim \begin{cases} \delta^{-1}(\varepsilon_\star + C_1\varepsilon)\mathbf{B}_{p-1,k-1}(s), & p_1 \le N-3, \\ \varepsilon_\star\mathbf{B}_{p-1,k-1}(s) + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}, & p_1 \ge N-2, \end{cases}
\end{aligned}
\tag{6.7}
$$

  where for the first case we have applied the third bound of (5.9), while for the second bound we have applied (4.5) and (5.12), provided $\kappa \ge 1/2 + (3/2)\delta$.

- The term $|H||\partial\partial u|_{p-1,k-1}$ is also bounded by the first case of (6.7).

- The term $|\partial h|_{p_1-1,k_1}|\partial\partial u|_{p_2,k_2}$ is easier since a partial derivative is acting on h. We observe that

$$
|\partial h|_p \lesssim |\partial h^\star|_p + |\partial u|_p \lesssim \varepsilon_\star\langle r\rangle^{-2} + |\partial u|_p
\tag{6.8}
$$

  and, using the inequality $\mathbf{X}^{-1} \lesssim \ell^{-1}\langle r\rangle^{-1}$ valid in $\mathcal{M}_{\ell,[s_0,s_1]}^{\mathrm{far}}$,

$$
\|\mathbf{X}^\kappa J\zeta^{-1}|\partial h|_{p_1-1,k_1}|\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})} \lesssim \ell^{-\delta/2}(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p,k}(s,u).
\tag{6.9}
$$

We then deduce that

$$\mathbf{A}_{p,k}(s) \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)\mathbf{B}_{p-1,k-1}(s) + \ell^{-\delta/2}(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{EM},p,k}(s,u)$$
$$+ \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}. \tag{6.10}$$

Combining (6.10) and (6.6) and observing that $\delta^{-1}(\varepsilon_\star + C_1\varepsilon) \ll 1$, we obtain

$$\sum_{\substack{\mathrm{ord}(Z)\leq p \\ \mathrm{rank}(z)\leq k}} \|\mathbf{X}^\kappa J\zeta^{-1}[Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]u\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})}$$

$$\lesssim \|\mathbf{X}^\kappa J\zeta^{-1}|\widetilde{\square}_g u|_{p-1,k-1}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{far}})}$$
$$+ \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{EM},p,k}(s,u)$$
$$+ \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}, \tag{6.11}$$

which completes the proof of Proposition 6.3 in $\mathcal{M}_{\ell,[s_0,s_1]}^{\mathrm{far}}$.

Estimates near the light cone. We also introduce the notation

$$\mathbf{A}_{p,k}(s) := \sum_{\substack{\mathrm{ord}(Z)\leq p \\ \mathrm{rank}(z)\leq k}} \|\mathbf{X}^\kappa J\zeta^{-1}[Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]u\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})},$$

$$\mathbf{B}_{p,k}(s) := \|t^{-1}\mathbf{X}^{1+\kappa}J\zeta^{-1}|\partial\partial u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})}.$$

Then, in view of (5.5) we get

$$\mathbf{B}_{p,k}(s) \lesssim \mathbf{A}_{p,k}(s) + \|\mathbf{X}^\kappa J\zeta^{-1}|\widetilde{\square}_g u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} + s^{-1}\mathcal{F}_\kappa^{\mathcal{EM},p+,k+1}(s,u). \tag{6.12}$$

On the other hand, we recall (2.15) and we need to control each term in its right-hand side.

• The term $(|H^{\mathcal{N}00}| + t^{-1}|r - t||H|)|\partial\partial u|_{p-1,k-1}$, thanks to (5.14) and (5.9) (the third bound), is directly controlled by

$$\left\|\mathbf{X}^\kappa J\zeta^{-1}(|H^{\mathcal{N}00}| + t^{-1}|r - t||H|)|\partial\partial u|_{p-1,k-1})\right\| \lesssim (\varepsilon_\star + C_1\varepsilon)\mathbf{B}_{p-1,k-1}(s). \tag{6.13}$$

• Thanks to (5.14) and (5.9) (i.e. the third bound therein), the term $T_{p,k}^{\mathrm{hier}}[H, u]$ is controlled as follows. First of all, we have

$$|LH^{\mathcal{N}00}|_{N-5} + t^{-1}|r - t||H|_{N-5} \lesssim (\varepsilon_\star + C_1\varepsilon)\mathbf{X}r^{-1}$$

and

$$|LH^{\mathcal{N}00}|_{k-1} + t^{-1}|r - t||H|_{k-1} \lesssim |h^\star|_k + |Lu|_{k-1} + t^{-1}\mathbf{X}|u|_k$$
$$\lesssim \varepsilon_\star\langle r\rangle^{-1} + |Y_{\mathrm{rot}}u|_{k-1} + t^{-1}\mathbf{X}|u|_k.$$

We thus deduce that

$$
\begin{aligned}
\|\mathbf{X}^\kappa J \zeta^{-1} & T_{p,k}^{\mathrm{hier}}[H,u]\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \lesssim (\varepsilon_\star + C_1\varepsilon)\|\mathbf{X}^\kappa J\zeta^{-1}\mathbf{X}r^{-1}|\partial\partial u|_{p-1,k-1}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \quad + \sum_{k_1=N-4}^{k} \||\mathbf{X}^\kappa J\zeta^{-1}|\partial\partial u|_{p-k_1,k-k_1}|Y_{\mathrm{rot}}u|_{k_1-1}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \quad + (\varepsilon_\star + C_1\varepsilon)s^{2\delta}\|\mathbf{X}^\kappa J\zeta^{-1}r^{-2}\mathbf{X}^{-\kappa}|u|_N\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})},
\end{aligned}
$$

where (4.23) and (5.11) were used. For the latter term, we apply (4.5) which is seen to be bounded by $\delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^{-2}$ (provided $\kappa \geq 1/2 + (3/2)\delta$). For the second term, we recall (2.20) together with (5.50) and obtain

$$
\begin{aligned}
\||\mathbf{X}^\kappa J\zeta^{-1}&|\partial\partial u|_{p-k_1,k-k_1}|Y_{\mathrm{rot}}u|_{k_1-1}|\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \lesssim \begin{cases} (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\|\zeta\mathbf{X}^{-1+\kappa}|Y_{\mathrm{rot}}u|_{k-1}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{when } k_1 = k), \\
(\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star + C_1\varepsilon)}\|\zeta\mathbf{X}^{-1+\kappa}|Y_{\mathrm{rot}}u|_{k-2}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{when } k_1 \leq k - 1), \end{cases} \\
& \lesssim (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p,k}(s,u) \\
& \quad + (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star + C_1\varepsilon)}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p-1,k-1}(s,u).
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\|\mathbf{X}^\kappa J\zeta^{-1} & T_{p,k}^{\mathrm{hier}}[H,u]\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \lesssim (\varepsilon_\star + C_1\varepsilon)\mathbf{B}_{p-1,k-1}(s) + (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p,k}(s,u) \\
& \quad + (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star + C_1\varepsilon)}\mathcal{F}_\kappa^{\mathcal{E}\mathcal{M},p-1,k-1}(s,u) \\
& \quad + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^{-2}. \qquad\qquad\qquad\qquad\qquad\qquad (6.14)
\end{aligned}
$$

- We consider next the terms T^{easy} and T^{super}, and we observe that T^{super} contains a favorable factor t^{-1} while the second term of T^{easy} contains the factor $t^{-1}|r-t|$, supplying sufficient decay in $\langle r\rangle^{-1}$. Thus, these terms are relatively easier to handle and we only discuss now the most challenging term, namely $|h^{\mathcal{N}00}|_{p_1-1,k_1}|\partial\partial u|_{p_2,k_2}$. When $p_1 - 1 \leq N - 3$, we apply (4.23) and obtain

$$
\begin{aligned}
\|\mathbf{X}^\kappa J\zeta^{-1}&|\partial h^{\mathcal{N}00}|_{p_1-1,k_1}|\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)s^{-1-2\kappa+\delta}\|\mathbf{X}^\kappa\zeta|\partial\partial u|_{p,k}\|_{L^2(\mathcal{M}_{\ell,s}^{\mathrm{near}})} \\
& \lesssim \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}.
\end{aligned}
$$

When $p_1 - 1 \geq N - 2$, we have $p_2 \leq 1 \leq N - 5$ and thus

$$
\|\mathbf{X}^\kappa J \zeta^{-1} |\partial h^{\mathcal{N}00}|_{p_1-1,k_1} |\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}^{\text{near}}_{\ell,s})}
$$
$$
\lesssim s \|\mathbf{X}^\kappa \zeta |\partial h^\star|_{p_1-1,k_1} |\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}^{\text{near}}_{\ell,s})}
$$
$$
+ s \|\mathbf{X}^\kappa \zeta |\partial u|_{p_1-1,k_1} |\partial\partial u|_{p_2,k_2}\|_{L^2(\mathcal{M}^{\text{near}}_{\ell,s})}
$$
$$
\lesssim (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1} \mathcal{F}^{\mathcal{EM},p,k}_\kappa(s,u)
$$
$$
+ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}^{\mathcal{EM},p-1,k-1}_\kappa(s,u).
$$

The weighted L^2 norm of the remaining terms is bounded by $\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}$.

Now we conclude with (6.13), (6.14), and the bounds established on T^{easy} and T^{super} that

$$
\mathbf{A}_{p,k} \lesssim (\varepsilon_\star + C_1\varepsilon)\mathbf{B}_{p-1,k-1}(s) + (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1} \mathcal{F}^{\mathcal{EM},p,k}_\kappa(s,u)
$$
$$
+ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}^{\mathcal{EM},p-1,k-1}_\kappa(s,u)
$$
$$
+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}.
$$

Substituting (6.12) in the above estimate and recalling $(\varepsilon_\star + C_1\varepsilon) \ll 1$, we obtain

$$
\sum_{\substack{\text{ord}(Z)\leq p \\ \text{rank}(z)\leq k}} \|\mathbf{X}^\kappa J \zeta^{-1} [Z, h^{\alpha\beta} \partial_\alpha \partial_\beta] u\|_{L^2(\mathcal{M}^{\text{near}}_{\ell,s})}
$$
$$
\lesssim \|\mathbf{X}^\kappa J \zeta^{-1} |\widetilde{\Box}_g u|_{p,k}\|_{L^2(\mathcal{M}^{\text{near}}_{\ell,s})} + (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1} \mathcal{F}^{\mathcal{EM},p,k}_\kappa(s,u)
$$
$$
+ (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}^{\mathcal{EM},p-1,k-1}_\kappa(s,u)
$$
$$
+ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-2}. \tag{6.15}
$$

In conclusion, by combining (6.11) and (6.15) we arrive at the desired result. ∎

Conclusion. We summarize our result so far as follows.

Proposition 6.4. *Under the conclusions of Propositions* 6.2 *and* 6.3*, the following estimate holds for all* $\text{ord}(Z) \leq p$ *and* $\text{rank}(Z) \leq k$:

$$
\|\mathbf{X}^\kappa J \zeta^{-1} \widetilde{\Box}_g Z u_{\alpha\beta}\|_{L^2(\mathcal{M}^{\mathcal{EM}}_s)}
$$
$$
\lesssim \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)
$$
$$
\cdot \left(s^{-1} \mathcal{F}^{\mathcal{EM},p,k}_\kappa(s,u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}^{\mathcal{EM},p-1,k-1}_\kappa(s,u) \right)
$$
$$
+ \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2 s^{-1-\delta} + R^\star_{\text{err}}(s). \tag{6.16}
$$

At this juncture it is useful to state the following elementary property.

Lemma 6.5 (Comparison of energy functionals). *Given* $(\eta, c^*, w) = (\kappa, 0, u)$ *or* $(\eta, c^*, w) = (\mu, c, \phi)$ *one has*

$$\mathscr{E}_{g,\eta,c^*}^{\mathscr{E}\mathscr{M}}(s, u) \geq (1/4)\mathscr{E}_{\eta,c^*}^{\mathscr{E}\mathscr{M}}(s, u).$$

Proof. The assumptions required in [30, Section 12.1] are valid in the present context, especially the inequality $h^{\mathscr{N}00} \leq 0$ in $\mathcal{M}_{\ell,[s_0,s_1]}^{\text{near}}$ and the decay

$$|h| \lesssim (\varepsilon_\star + C_1\varepsilon)r^{-\kappa}s^\delta \leq (\varepsilon_\star + C_1\varepsilon)r^{-1/2},$$

and we can thus follow the arguments in [30]. ∎

In order to apply (2.12), we still need the following two bounds for (ord$(Z) \leq p$ and rank$(Z) \leq k$) and $(\eta, w) = (\kappa, u)$ or (μ, ϕ):

$$\int_{\mathcal{M}_s^{\mathscr{E}\mathscr{M}}} |JG_{g,\eta}[Zw]| \, dx \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathscr{E}_\eta^{\mathscr{E}\mathscr{M}}(s, Zw), \qquad (6.17)$$

$$\int_{\mathcal{M}_s^{\mathscr{E}\mathscr{M}}} \eta \mathbf{X}^{2\eta-1}\aleph'(r-t)h^{\mathscr{N}00}|\partial_t Zw|^2 \, Jdx \lesssim (\varepsilon_\star + C_1\varepsilon)s^{-1}\mathscr{E}_\eta^{\mathscr{E}\mathscr{M}}(s, Zw). \quad (6.18)$$

The first bound (6.17) is directly derived by applying (5.50) (with $k = 0$) and (6.18) is nothing but (5.14). Finally, we are ready to apply (2.12) with $(\eta, w, c^*) = (\kappa, u, 0)$. Taking (6.16), (6.17), (6.18) and (3.3) into account, we obtain

$$\begin{aligned}
\frac{d}{ds}&\mathscr{E}_{g,\kappa}^{\mathscr{E}\mathscr{M}}(s, Zu) + \frac{d}{ds}\mathscr{E}_{g,c}^{\mathscr{L}}(s, Zu; s_0) \\
&+ 2\kappa \int_{\mathcal{M}_s^{\mathscr{E}\mathscr{M}}} \left(g^{\mathscr{N}ab}\not{\partial}_a^{\mathscr{N}} Zu \not{\partial}_b^{\mathscr{N}} Zu + c^2|Zu|^2\right)\mathbf{X}^{2\kappa-1}\aleph'(r-t) \, Jdx \\
&\lesssim (\ell^{-\delta} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathscr{E}_\kappa^{\mathscr{E}\mathscr{M}}(s, Zu)ds \\
&\quad + \|\zeta\mathbf{X}^\kappa \partial_t Zu\|_{L^2(\mathcal{M}_s^{\mathscr{E}\mathscr{M}})}\|\mathbf{X}^\kappa J\zeta^{-1}\widetilde{\Box}_g Zu\|_{L^2(\mathcal{M}_s^{\mathscr{E}\mathscr{M}})} \\
&\lesssim \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
&\quad \cdot \left(s^{-1}\mathscr{E}_\kappa^{\mathscr{E}\mathscr{M},p,k}(s, u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)}\mathscr{F}_\kappa^{\mathscr{E}\mathscr{M},p-1,k-1}(s, u)\mathscr{F}_\kappa^{\mathscr{E}\mathscr{M},p,k}(s, u)\right) \\
&\quad + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2(1 + C_{R\star})s^{-1-\delta}\mathscr{F}_\kappa^{\mathscr{E}\mathscr{M},p,k}(s, u). \qquad (6.19)
\end{aligned}$$

Now, let us focus our attention on the left-hand side. In the third term we observe that $\aleph'(r-t) \geq 0$ in $\mathcal{M}_s^{\mathscr{E}\mathscr{M}}$. Furthermore, since $|h| \ll 1$ we have $\sum_a |\not{\partial}_a^{\mathscr{N}} u|^2 \lesssim g^{\mathscr{N}ab}\not{\partial}_a^{\mathscr{N}} u \not{\partial}_b^{\mathscr{N}} u$ and this leads us to

$$\sum_{\substack{\text{ord}(Z)\leq p \\ \text{rank}(Z)\leq k}} \int_{\mathcal{M}_s^{\mathscr{E}\mathscr{M}}} \mathbf{X}^{2\kappa-1}\aleph'(r-t)g^{\mathscr{N}ab}\not{\partial}_a^{\mathscr{N}} Zu \not{\partial}_b^{\mathscr{N}} Zu \, Jdx \geq 0. \qquad (6.20)$$

For the second term of the left-hand side, we recall (2.13) and (3.11) and obtain

$$\frac{d}{ds}\mathcal{E}_{g,c}^{\mathcal{L}}(s, Zu; s_0) \geq 0. \tag{6.21}$$

Summing up, in view of (6.19), for all $\mathrm{ord}(Z) \leq p, \mathrm{rank}(Z) \leq k$ we find

$$
\begin{aligned}
\frac{d}{ds}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) &\lesssim \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
&\quad \cdot \left(s^{-1}\mathcal{F}_\kappa^{\mathcal{EM},p,k}(s, u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)}\mathcal{F}_\kappa^{\mathcal{EM},p-1,k-1}(s, u)\right) \\
&\quad + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2(1 + C_{R^\star})s^{-1-\delta} \\
&\lesssim \delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
&\quad \cdot \left(s^{-1}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) + \underline{s^{-1+C(\varepsilon_\star+C_1\varepsilon)}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p-1,k-1}(s, u)}\right) \\
&\quad + \ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2(1 + C_{R^\star})s^{-1-\delta}.
\end{aligned}
$$

Observe that when $k = 0$ the *underlined* term in the right-hand side *does not exist*. We rewrite the above inequality in the form

$$
\begin{aligned}
\frac{d}{ds}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) &\leq C\delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
&\quad \cdot \left(s^{-1}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)}\mathcal{F}_{g,\kappa}^{\mathcal{EM},p-1,k-1}(s, u)\right) \\
&\quad + C\ell^{-1}\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2(1 + C_{R^\star})s^{-1-\delta}
\end{aligned}
$$

with C a constant determined by N. Then, by Gronwall's inequality and by an induction argument on k, we deduce that

$$\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) - \mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s_0, u)s^{C(\varepsilon_\star+C_1\varepsilon)^{1/3}} \leq C(\varepsilon_\star + C_1\varepsilon)^{4/3}s^{C(\varepsilon_\star+C_1\varepsilon)^{1/3}},$$

provided $C_0 < C_1$ and

$$
\begin{aligned}
\ell^{-1}\delta^{-2}(1 + C_{R^\star})(\varepsilon_\star + C_1\varepsilon)^{1/3} &\leq 1, \\
\delta^{-1}(\ell^{-1} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)^{1/3} &\leq 1.
\end{aligned}
$$

Now we take $C(\varepsilon_\star + C_1\varepsilon)^{1/3} \leq \delta$ and $C_1 > 4C_0$ together with

$$\varepsilon_\star + C_1\varepsilon \leq (C_1 - 4C_0)^3/(4CC_1)^3$$

and we arrive at

$$\mathcal{F}_\kappa^{\mathcal{EM},N}(s, u) \leq 2\mathcal{F}_{g,\kappa}^{\mathcal{EM},p,k}(s, u) \leq \frac{1}{2}(\varepsilon_\star + C_1\varepsilon)s^\delta.$$

6.2. Improved energy estimate for the matter field

In comparison with the issues arising with metrics with lower decay, the control of the matter field is significantly simpler in the harmonic regime for the metric, and we do not need to distinguish between various components of the metric. We are going to establish the following result on commutators valid in $\mathcal{M}_{[s_0,s_1]}^{\mathcal{E}\mathcal{M}}$:

$$
\begin{aligned}
&\|\mathbf{X}^\mu J\zeta^{-1}[Z, H^{\alpha\beta}\partial_\alpha\partial_\beta]\phi\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\big(\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,\phi) + s^{C(\varepsilon_\star+C_1\varepsilon)}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi)\big) \\
&\quad + \begin{cases} 0, & k \leq N-5, \\ \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2, & k \geq N-4. \end{cases}
\end{aligned}
\tag{6.22}
$$

To this end, we rely on the general commutator inequality (2.16) (with $u = \phi$ therein). Recalling (5.50) (case $k = 0$) we find

$$
\begin{aligned}
&\|\mathbf{X}^\mu J\zeta^{-1}|H||\partial\partial\phi|_{p-1,k-1}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s\|\langle r\rangle^{-1}\mathbf{X}^\mu\zeta|\partial\partial\phi|_{p-1,k-1}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi).
\end{aligned}
\tag{6.23}
$$

For the second term of the right-hand side of (2.16), we observe that, when $p_1 \leq N-5$ and thanks to (5.51),

$$
\begin{aligned}
&\|\mathbf{X}^\mu J\zeta^{-1}|Y_{\text{rot}}H|_{p_1-1}|\partial\partial\phi|_{p_2,k_2}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim (\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)^{1/2}}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi),
\end{aligned}
\tag{6.24}
$$

where we used $p_1 \geq 1$ and $p_1 + p_2 = p, k_1 + k_2 = k$. When $p_1 \geq N-4$, we have

$$
|Y_{\text{rot}}H|_{p-1} \lesssim |h^\star|_{p_1} + |u|_{p_1} \lesssim \varepsilon_\star\langle r\rangle^{-1} + |u|_{p_1}.
$$

Then, we have

$$
\begin{aligned}
&\|\mathbf{X}^\mu J\zeta^{-1}|Y_{\text{rot}}H|_{p_1-1}|\partial\partial\phi|_{p_2,k_2}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim \varepsilon_\star\|\mathbf{X}^\mu J\zeta^{-1}\langle r\rangle^{-1}|\partial\partial\phi|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} + \|\mathbf{X}^\mu J\zeta^{-1}|u|_{p_1}|\partial\partial\phi|_{p_2,k_2}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim \varepsilon_\star s^{-1}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi) + (\varepsilon_\star + C_1\varepsilon)s^{2\delta}\|\mathbf{X}^\mu J\zeta^{-1}r^{-2}\mathbf{X}^{1-\mu}|u|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim \varepsilon_\star s^{-1}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi) + (\varepsilon_\star + C_1\varepsilon)s^{1+2\delta}\|\mathbf{X}^{2-\kappa}\zeta r^{-2}\mathbf{X}^{-1+\kappa}|u|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
&\quad \lesssim \varepsilon_\star s^{-1}\mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi) + (\varepsilon_\star + C_1\varepsilon)s^{1-2\kappa+2\delta}\|\mathbf{X}^{-1+\kappa}|u|_{p,k}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})},
\end{aligned}
$$

where we applied Lemma 4.1. Then in view of (4.5) and (6.24), we obtain

$$
\left\| \mathbf{X}^\mu J \zeta^{-1} |Y_{\mathrm{rot}} H|_{p_1-1} |\partial\partial\phi|_{p_2,k_2} \right\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})}
$$
$$
\lesssim
\begin{cases}
\delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2, & N-4 \le p \le N, \\
(\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \\
\qquad \cdot \mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi), & p \le N-5.
\end{cases}
\tag{6.25}
$$

Finally, for the terms $|\partial H|_{p_1-1,k_1} |\partial\partial\phi|_{p_2,k_2}$, when $p_1 \le N-3$, we apply (5.50) and obtain

$$
\begin{aligned}
\big\| \mathbf{X}^\mu & J\zeta^{-1} |\partial H|_{p_1-1,k_1} |\partial\partial\phi|_{p_2,k_2} \big\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
& \lesssim \big\| \mathbf{X}^\mu J\zeta^{-1} |\partial H|_{N-4,0} |\partial\partial\phi|_{p,k} \big\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
& \quad + \big\| \mathbf{X}^\mu J\zeta^{-1} |\partial H|_{p_1-1,k_1} |\partial\phi|_{p,k-1} \big\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})} \\
& \lesssim (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1} \mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,\phi) \\
& \quad + (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}_{\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,\phi),
\end{aligned}
$$

which leads us to (6.22).

Recalling (6.17), (6.18) and (6.21), which are also valid for ϕ, we apply (2.12) and obtain

$$
\begin{aligned}
\frac{d}{ds} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,u) \lesssim\ & (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
& \cdot \big(s^{-1} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,u)\big)
\end{aligned}
$$

for $p \le N-5$ and

$$
\begin{aligned}
\frac{d}{ds} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,u) \lesssim\ & (\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon) \\
& \cdot \big(s^{-1} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k}(s,u) + s^{-1+C(\varepsilon_\star+C_1\varepsilon)} \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},p,k-1}(s,u)\big) \\
& + \delta^{-1}(\varepsilon_\star + C_1\varepsilon)^2
\end{aligned}
$$

for $p \ge N-4$. Here, C is a constant determined by N. Then we take $C_0 < C_1$ and $(\ell^{-\delta/2} + \delta^{-2})(\varepsilon_\star + C_1\varepsilon)^{1/3} \le 1$ and, by Gronwall's inequality and an induction on k, we have

$$
\begin{aligned}
\mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},N-5}(s,\phi) \le\ & \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},N-5}(s_0,\phi)s^{C(\varepsilon_\star+C_1\varepsilon)^{1/3}} \\
& + C(\varepsilon_\star + C_1\varepsilon)^{4/3} s^{C(\varepsilon_\star+C_1\varepsilon)^{1/3}}, \\
\mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},N}(s,\phi) \le\ & \mathcal{F}_{g,\mu,c}^{\mathcal{E}\mathcal{M},N}(s_0,\phi)s^{1+C(\varepsilon_\star+C_1\varepsilon)^{1/3}} \\
& + C(\varepsilon_\star + C_1\varepsilon)^{4/3} s^{1+C(\varepsilon_\star+C_1\varepsilon)^{1/3}},
\end{aligned}
\tag{6.26}
$$

with (another) constant $C > 0$ determined by N. We require that

$$C(\varepsilon_\star + C_1\varepsilon)^{1/3} \leq \delta, \qquad C_1 > 4C_0, \qquad \varepsilon_\star + C_1\varepsilon < \Big(\frac{C_1 - 4C_0}{4CC_1}\Big)^3.$$

We have established the refined energy bound for the matter field, and the proof of Theorem 3.1 is completed.

References

[1] L. Bieri, An extension of the stability theorem of the Minkowski space in general relativity. *J. Differential Geom.* **86** (2010), no. 1, 17–70   Zbl 1213.83025   MR 2772545

[2] L. Bieri and N. Zipser, *Extensions of the stability theorem of the Minkowski space in general relativity*. AMS/IP Stud. Adv. Math. 45, American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2009   Zbl 1172.83001   MR 2531716

[3] L. Bigorgne, *Propriétés asymptotiques des solutions à données petites du système de Vlasov-Maxwell.* Ph.D. thesis, Université Paris-Saclay, 2019

[4] L. Bigorgne, D. Fajman, J. Joudioux, J. Smulevici, and M. Thaller, Asymptotic stability of Minkowski space-time with non-compactly supported massless Vlasov matter. *Arch. Ration. Mech. Anal.* **242** (2021), no. 1, 1–147   Zbl 1471.83003   MR 4302757

[5] J.-P. Bourguignon, Stabilité par déformation non-linéaire de la métrique de Minkowski (d'après D. Christodoulou et S. Klainerman). In *Séminaire Bourbaki, Vol. 1990/91*. Exp. No. 730-744, Astérisque No. 201-203 (1991), 321–358   Zbl 0754.53060   MR 1157847

[6] T. Candy and H. Lindblad, Long range scattering for the cubic Dirac equation on $\mathbb{R}^{1+1}$. *Differ. Integral Equ.* **31** (2018), no. 7-8, 507–518   Zbl 1463.35422   MR 3801822

[7] Y. Choquet-Bruhat, *General relativity and the Einstein equations*. Oxford Math. Monogr., Oxford University Press, Oxford, 2009   Zbl 1157.83002   MR 2473363

[8] Y. Choquet-Bruhat and R. Geroch, Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.* **14** (1969), 329–335   Zbl 0182.59901   MR 250640

[9] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*. Princeton Math. Ser. 41, Princeton University Press, Princeton, NJ, 1993   Zbl 0827.53055   MR 1316662

[10] S. Dong, P. G. LeFloch, and Z. Wyatt, Global evolution of the U(1) Higgs Boson: nonlinear stability and uniform energy bounds. *Ann. Henri Poincaré* **22** (2021), no. 3, 677–713   Zbl 1469.35182   MR 4226448

[11] D. Fajman, J. Joudioux, and J. Smulevici, A vector field method for relativistic transport equations with applications. *Anal. PDE* **10** (2017), no. 7, 1539–1612   Zbl 1373.35046   MR 3683922

[12] D. Fajman, J. Joudioux, and J. Smulevici, Sharp asymptotics for small data solutions of the Vlasov–Nordström system in three dimensions. 2017, arXiv:1704.05353

[13] D. Fajman, J. Joudioux, and J. Smulevici, The stability of the Minkowski space for the Einstein–Vlasov system. *Anal. PDE* **14** (2021), no. 2, 425–531   Zbl 1476.83007   MR 4241806

[14] Y. Fourès-Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.* **88** (1952), 141–225   Zbl 0049.19201   MR 53338

[15] P. Hintz and A. Vasy, The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.* **220** (2018), no. 1, 1–206   Zbl 1391.83061   MR 3816427

[16] P. Hintz and A. Vasy, Stability of Minkowski space and polyhomogeneity of the metric. *Ann. PDE* **6** (2020), no. 1, Paper No. 2   Zbl 1462.35389   MR 4105742

[17] C. Huneau and A. Stingo, Global well-posedness for a system of quasilinear wave equations on a product space. 2021, arXiv:2110.13982

[18] M. Ifrim and A. Stingo, Almost global well-posedness for quasilinear strongly coupled wave-Klein–Gordon systems in two space dimensions. 2019, arXiv:1910.12673

[19] A. D. Ionescu and B. Pausader, Global solutions of quasilinear systems of Klein-Gordon equations in 3D. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 11, 2355–2431   Zbl 1316.35180   MR 3283401

[20] A. D. Ionescu and B. Pausader, On the global regularity for a wave-Klein-Gordon coupled system. *Acta Math. Sin. (Engl. Ser.)* **35** (2019), no. 6, 933–986   Zbl 1420.35057   MR 3952698

[21] A. D. Ionescu and B. Pausader, *The Einstein-Klein-Gordon coupled system: global stability of the Minkowski solution*. Ann. Math. Stud. 213, Princeton University Press, Princeton, NJ, 2022   Zbl 07438194   MR 4422074

[22] C. Kauffman and H. Lindblad, Global stability of minkowski space for the Einstein-Maxwell-Klein–Gordon system in generalized wave coordinates. 2021, arXiv:2109.03270

[23] S. Klainerman, Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Comm. Pure Appl. Math.* **38** (1985), no. 5, 631–641   Zbl 0597.35100   MR 803252

[24] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space $\mathbf{R}^{n+1}$. *Comm. Pure Appl. Math.* **40** (1987), no. 1, 111–117   Zbl 0686.46019   MR 865359

[25] P. G. LeFloch, An introduction to self-gravitating matter, Graduate course given at Institute Henri Poincaré, Paris, Fall 2015, available at http://www.youtube.com/user/PoincareInstitute

[26] P. G. LeFloch and Y. Ma, *The hyperboloidal foliation method*. Series in Applied and Computational Mathematics 2, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014   Zbl 1320.53001   MR 3362362

[27] P. G. LeFloch and Y. Ma, The global nonlinear stability of Minkowski space for self-gravitating massive fields. *Comm. Math. Phys.* **346** (2016), no. 2, 603–665   Zbl 1359.83003   MR 3535896

[28] P. G. LeFloch and Y. Ma, The global nonlinear stability of Minkowski space for the Einstein equations in the presence of a massive field. *C. R. Math. Acad. Sci. Paris* **354** (2016), no. 9, 948–953   Zbl 1347.83008   MR 3535352

[29] P. G. LeFloch and Y. Ma, *The global nonlinear stability of Minkowski space for self-gravitating massive fields*. Ser. Appl. Comput. Math. 3, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018   Zbl 1434.83005   MR 3729443

[30] P. G. LeFloch and Y. Ma, Nonlinear stability of self-gravitating massive fields. 2017, arXiv:1712.10045

[31] P. G. LeFloch and Y. Ma, in preparation

[32] P. G. LeFloch and T.-C. Nguyen, The seed-to-solution method for the Einstein equations and the asymptotic localization problem. 2019, arXiv:1903.00243

[33] P. G. LeFloch and C. Wei, Boundedness of the total energy of relativistic membranes evolving in a curved spacetime. *J. Differ. Equations* **265** (2018), no. 1, 312–331 Zbl 1390.35357   MR 3782545

[34] H. Lindblad, J. Lührmann, and A. Soffer, Decay and asymptotics for the one-dimensional Klein-Gordon equation with variable coefficient cubic nonlinearities. *SIAM J. Math. Anal.* **52** (2020), no. 6, 6379–6411   Zbl 1455.35021   MR 4189725

[35] H. Lindblad and I. Rodnianski, Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.* **256** (2005), no. 1, 43–110   Zbl 1081.83003 MR 2134337

[36] H. Lindblad and I. Rodnianski, The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)* **171** (2010), no. 3, 1401–1477   Zbl 1192.53066   MR 2680391

[37] H. Lindblad and M. Taylor, Global stability of Minkowski space for the Einstein-Vlasov system in the harmonic gauge. *Arch. Ration. Mech. Anal.* **235** (2020), no. 1, 517–633 Zbl 1434.35234   MR 4062482

[38] Y. Ma, Global solutions of nonlinear wave-Klein-Gordon system in one space dimension. *Nonlinear Anal.* **191** (2020), 111641   Zbl 1439.35319   MR 4017462

[39] Y. Ma, Global solutions of nonlinear wave-Klein-Gordon system in two spatial dimensions: a prototype of strong coupling case. *J. Differ. Equations* **287** (2021), 236–294 Zbl 1462.35191   MR 4241615

[40] V. Moncrief and O. Rinne, Regularity of the Einstein equations at future null infinity. *Classical Quantum Gravity* **26** (2009), no. 12, 125010   Zbl 1170.83344   MR 2515671

[41] O. Rinne and V. Moncrief, Hyperboloidal Einstein-matter evolution and tails for scalar and Yang-Mills fields. *Classical Quantum Gravity* **30** (2013), no. 9, 095009 Zbl 1269.83015   MR 3046474

[42] J. Smulevici, Small data solutions of the Vlasov-Poisson system and the vector field method. *Ann. PDE* **2** (2016), no. 2, Art. 11   Zbl 1397.35033   MR 3595457

[43] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation. *Trans. Amer. Math. Soc.* **353** (2001), no. 2, 795–807 Zbl 0956.35088   MR 1804518

[44] Q. Wang, An intrinsic hyperboloid approach for Einstein Klein-Gordon equations. *J. Differ. Geom.* **115** (2020), no. 1, 27–109   Zbl 1444.53043   MR 4081931

[45] W. W. Y. Wong, A commuting-vector-field approach to some dispersive estimates. *Arch. Math.* **110** (2018), no. 3, 273–289 Zbl 1384.35108 MR 3761138

Philippe G. LeFloch
Laboratoire Jacques-Louis Lions and Centre National de la Recherche Scientifique, Sorbonne Université, 4 Place Jussieu, 75252 Paris, France; contact@philippelefloch.org

Yue Ma
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049 Shaanxi, P. R. China; yuemath@mail.xjtu.edu.cn