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Homology of weighted path complexes and directed hypergraphs

Yuri Muranov, Anna Szczepkowska, and Vladimir Vershinin

Abstract. We introduce the weighted path homology on the category of weighted directed hypergraphs and describe conditions of homotopy invariance of weighted path homology groups. We give several examples that explain the nontriviality of the introduced notions.

1. Introduction

One of the tools for the investigation of global structures in graph theory are (co)homology theories which were constructed recently by many authors for various categories of (di)graphs, multigraphs, quivers, and hypergraphs [4, 6, 7, 10, 12, 14–16].

The notion of directed hypergraphs arises in discrete mathematics as a natural generalization of digraphs and hypergraphs [3, 8]. At present time, there are many mathematical models of various complex systems in which the system is presented by a network whose interacting pairs of nodes are connected by links. Such models are naturally presented by hypergraphs with additional structures [2]. The notions of path, cycle and weight structure on hypergraphs arise naturally for the directed hypergraphs. These concepts allow to investigate global structures on hypergraphs by methods of algebraic topology and, in particular, homology theory. The excellent survey of application of directed hypergraphs to various aspects of computer sciences is given in paper [1]. The application of homology theory of hypergraphs in the pharmaceutical industries is described, for example, in [5, 13].

In the present paper, we introduce several categories of weighted directed hypergraphs and the notion of homotopy in these categories. Then we define functorial homology theories on such categories using the homology theory of path complexes constructed in [9, 10, 12]. We describe also the conditions of homotopy invariance of

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introduced homology groups and we give several examples. In the paper, we consider only finite digraphs, path complexes and directed hypergraphs.

2. Path homology of weighted path complexes

A *path complex* [9, Sec. 3.1] $\Pi = (V, P)$ consists of a set V of vertices and a set P of *elementary paths* $(i_0 \cdots i_n)$ of vertices such that one-element sequence (i) is in P and if $(i_0 \cdots i_n) \in P$ then $(i_0 \cdots i_{n-1}) \in P$, $(i_1 \cdots i_n) \in P$. The *length* of a path $(i_0 \cdots i_n)$ is equal to n. We denote by $\Pi_V = (V, P_V)$ the path complex for which P_V consists of all paths of finite length on V.

A *morphism of path complexes* $f \colon (V, P) \to (W, Q)$ is given by the pair of maps (f_V, f_P) where $f_V \colon V \to W$ and $f_P(i_0 \cdots i_n) := (f_V(i_0) \cdots f_V(i_n))$ lays in P. Thus we obtain a category $\mathcal{P}$ whose objects are path complexes and whose morphisms are morphisms of path complexes.

Let $J = \{0, 1\}$ be the two-element set. For any set $V = \{0, \ldots, n\}$, let $V \times J$ be as usual the Cartesian product. Let V' be a copy of the set V with the elements $\{0', \ldots, n'\}$ where $i' \in V'$ corresponds to $i \in V$. Then we can identify $V \times J$ with $V \coprod V'$ in such a way that $(i, 0)$ corresponds to i and $(i, 1)$ corresponds to i' for all $i \in V$. Thus, V is identified with $V \times \{0\} \subset V \times J$ and V' is identified with $V \times \{1\} \subset V \times J$. The natural bijection $V \cong V'$ defines the set of paths P' on the set V' by the condition: $(i'_0 \cdots i'_n) \in P'$ iff $(i_0 \cdots i_n) \in P$. We define a path complex $\Pi^{\uparrow} = (V \times J, P^{\uparrow})$ where

$$
\begin{aligned}
P^{\uparrow} &= P \cup P' \cup P^{\#}, \\
P^{\#} &= \left\{ q_k^{\#} = (i_0 \cdots i_k \, i'_k, i'_{k+1} \cdots i'_n) \mid q = (i_0 \cdots i_k \, i_{k+1} \cdots i_n) \in P \right\}.
\end{aligned}
$$

There are morphisms $i_{\bullet} \colon \Pi \to \Pi^{\uparrow}$ and $j_{\bullet} \colon \Pi \to \Pi^{\uparrow}$ which are induced by the natural inclusion of V in $V \times \{0\}$ and in $V \times \{1\}$, respectively.

Definition 2.1. (i) A path complex $\Pi = (V, P)$ is called *weighted* if it is equipped with a function $\delta_V \colon V \to R$ where R is a unitary commutative ring. This function is called the *weight function*. We denote such a path complex by $\Pi^{\delta} = (V, P, \delta_V)$.

(ii) Let (V, P, δ_V) and (W, S, δ_W) be weighted path complexes. A morphism $f \colon (V, P) \to (W, S)$ of path complexes is a *weighted morphism* if $\delta_W(f_V(v)) = \delta_V(v)$ for every $v \in V$. We denote such a morphism by f^{δ}.

Weighted path complexes with weighted morphisms form a category we denote by $\mathcal{P}^{\delta}$.

Let $\Pi^\delta = (V, P, \delta_V)$ be a weighted path complex. Then the path complex $\Pi^\uparrow$ has the natural structure of the weighted path complex $\Pi^{\uparrow^\delta}$ with the weighted function $\delta_{V\times J}(v,0) = \delta_{V\times J}(v,1) = \delta_V(v)$.

Definition 2.2. (i) We call weighted morphisms $f^\delta, g^\delta\colon \Pi^\delta \to \Sigma^\delta$ of weighted path complexes *weighted one-step homotopic* if morphisms f, g are one-step homotopic (see [10, Def. 3.1]) and the homotopy $F\colon \Pi^\uparrow \to \Sigma$ is a weighted morphism.

(ii) Two weighted morphisms $f^\delta, g^\delta\colon \Pi^\delta \to \Sigma^\delta$ of weighted path complexes are *homotopic* $f^\delta \simeq f^\delta$ if there exists a sequence of weighted morphisms

$$f^\delta = f_0^\delta, f_1^\delta, \dots, f_n^\delta = g^\delta\colon \Pi^\delta \to \Sigma^\delta$$

such that any two consequent morphisms are one-step homotopic.

Thus, we obtain homotopy category $h\mathcal{P}^\delta$ whose objects are weighted path complexes and morphisms are classes of weighted homotopic weighted morphisms.

Proposition 2.3. *Let $f^\delta, g^\delta\colon \Pi^\delta \to \Sigma^\delta$ be weighted morphisms of weighted path complexes which are one-step homotopic as morphisms of path complexes. Then morphisms f^δ and g^δ are weighted one-step homotopic.*

Proof. It follows from the definition of the weight function $\delta_{V\times J}$. ∎

Let $\Pi^\delta = (V, P, \delta_V)$. We define the weighted path homology groups $H_*^{\mathrm{w}}(\Pi^\delta)$ with coefficients in a unitary commutative ring R similarly to the regular path homology groups [10, Sec. 2]. For $n \geq 0$, let $\mathcal{R}_n^{\mathrm{reg}}(V)$ be a module generated by all paths $(i_0 \cdots i_n)$ on the set V which are *regular*, that is $i_k \neq i_{k+1}$ for $0 \leq k \leq n-1$ and $\mathcal{R}_{-1}^{\mathrm{reg}}(V) = 0$. Let $\mathcal{R}_n^{\mathrm{reg}}(P) \subset \mathcal{R}_n^{\mathrm{reg}}(V)$ be submodules generated by paths from P for $n \geq 0$ and $\mathcal{R}_{-1}^{\mathrm{reg}}(P) = 0$. We define a *weighted boundary homomorphism*

$$\partial_n^w\colon \mathcal{R}_n(V) \to \mathcal{R}_{n-1}(V)$$

by putting $\partial_0^w = 0$ and

$$\partial_n^w(e_{i_0\cdots i_n}) = \sum_{s=0}^{n} (-1)^s \delta_V(i_s) e_{i_0\cdots\widehat{i_s}\cdots i_n}$$

for $n \geq 1$, where $\widehat{i_s}$ means omission of the index i_s. Then, similarly to the case of unweighted boundary homomorphisms in [14], we obtain $(\partial^w)^2 = 0$. Let us define R-modules $\Omega_*^w = \Omega_*^w(\Pi^\delta)$ by setting $\Omega_{-1}^w = 0$ and

$$\Omega_n^w = \left\{ v \in \mathcal{R}_n^{\mathrm{reg}}(P) \mid \partial^w v \in \mathcal{R}_{n-1}^{\mathrm{reg}}(P) \right\} \quad \text{for } n \geq 0.$$

Modules Ω_n^w with the differential induced by ∂^w form a chain complex. Homology groups $H_*^{\mathrm{w}}(\Pi^\delta)$ of this complex are called *weighted path homology groups*.

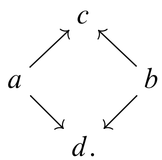
Recall that a digraph $G = (V_G, E_G)$ is called *weighted* if it is equipped with a weight function $\delta_{V_G}\colon V_G \to R$. Let $G^\delta = (V_G, E_G, \delta_{V_G})$ be a weighted digraph without loops. We define a weighted path complex $\mathfrak{D}^w(G) = (V_G, P_G, \delta_{V_G})$ in which a path $(i_0 \cdots i_n)$ on the set V_G lies in P_G iff for $k = 1, \dots, n$ every pair of consequent vertices i_{k-1}, i_k lies in the edge $(i_{k-1} \to i_k) \in E_G$. Thus, one gets a functor $\mathfrak{D}^w$ from the category of weighted digraphs $\mathcal{D}^\delta$ to the category of weighted path complexes $\mathcal{P}^\delta$. We define the *weighted path homology* of the weighted digraph by

$$H_*^{\mathrm{w}}(G^\delta) := H_*^{\mathrm{w}}\big(\mathfrak{D}^w(G^\delta)\big).$$

The homology group $H_*^{\mathrm{w}}(G^\delta)$ is a functor from the category $\mathcal{D}^\delta$ of weighted digraphs to the category of R-modules.

Example 2.4. We give two examples of weighted path complexes Π^δ for which the regular homology groups $H_*(\Pi)$ and $H_*^{\mathrm{w}}(\Pi^\delta)$ with coefficients in $\mathbb{Z}$ are non-isomorphic.

(i) Let $\Pi^\delta = (V, P, \delta_V)$ with $V = \{a, b, c, d\}$, $P = \{a, b, c, d, ac, ad, bc, bd\}$ and $\delta_V(a) = \delta_V(b) = 1$, $\delta_V(c) = \delta_V(d) = 0$. The path homology groups of $\Pi = (V, P)$ coincide with the regular path homology groups of the digraph



We have

$$H_i(\Pi) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 1, \\ 0 & \text{for } i \geq 2, \end{cases}$$

and

$$H_i^{\mathrm{w}}(\Pi^\delta) = \begin{cases} \mathbb{Z}^2 & \text{for } i = 0, 1, \\ 0 & \text{for } i \geq 2. \end{cases}$$

(ii) Let $\Pi^\delta = (V, P, \delta_V)$ with $V = \{a, b\}$, $P = \{a, b, ab\}$, and $\delta_V(a) = 2$, $\delta_V(b) = 4$. The path homology groups of $\Pi = (V, P)$ coincide with the regular path homology groups of the digraph $a \to b$. We have

$$H_i(\Pi) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{for } i \geq 1, \end{cases}$$

and

$$H_i^{\mathrm{w}}(\Pi^\delta) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$$

Let us fix a weighted path complex $\Pi^\delta = (V, P, \delta)$ with a weighted function $\delta: V \to R$ such that $\delta(i)$ is an invertible element for all $i \in V$. We define a function $\gamma: V \to R$ by putting $\gamma(i) = (\delta(i))^{-1}$. Recall that for the weighted path complex $\Pi^{\uparrow\delta} = (V \times J, P^\uparrow, \delta_{V\times J})$, we identify $V \times \{0\}$ with V, $V \times \{1\}$ with V' and, hence,

$$\delta_{V\times J}(v, 0) = \delta_{V\times J}(v, 1) = \delta_V(v) = \delta_V(v') \quad \text{for } v \in V.$$

Thus, for a path $e_{i_0\cdots i_n} \in \mathcal{R}_n^{\mathrm{reg}}(P)$ and $0 \le k \le n$, we have $\delta(i_k) = \delta(i_k')$, $\gamma(i_k) = \gamma(i_k')$. For $n \ge 0$, we define a homomorphism $\tau: \mathcal{R}_n^{\mathrm{reg}}(V) \to \mathcal{R}_{n+1}^{\mathrm{reg}}(V \times J)$ on basic regular n-paths $v = e_{i_0\cdots i_n}$ by

$$\tau(v) = \sum_{k=0}^{n} \gamma(i_k)(-1)^k e_{i_0\cdots i_k i_k'\cdots i_n'}. \tag{2.1}$$

We set $\mathcal{R}_{-1}^{\mathrm{reg}}(V) = 0$ and define $\tau = 0: \mathcal{R}_{-1}^{\mathrm{reg}}(V) \to \mathcal{R}_0^{\mathrm{reg}}(V \times J)$.

Lemma 2.5. *For $n \ge 0$ and any path $v \in \mathcal{R}_n^{\mathrm{reg}}(V)$, we have*

$$\partial^w \tau(v) + \tau(\partial^w v) = v' - v. \tag{2.2}$$

Proof. It is sufficient to prove the statement for basis elements $v = e_{i_0\cdots i_n}$. For $n = 0$ and $v = e_{i_0} \in \mathcal{R}_0^{\mathrm{reg}}(V)$, we have

$$\begin{aligned}
\partial^w \tau(e_{i_0}) &= \partial^w(\gamma(i_0)e_{i_0 i_0'}) = \gamma(i_0)\cdot\big(\delta(i_0)e_{i_0'} - \delta(i_0{}')e_{i_0}\big)\\
&= \gamma(i_0)\cdot\delta(i_0)[e_{i_0'} - e_{i_0}] = e_{i_0'} - e_{i_0},\\
\tau(\partial^w v) &= \tau(0) = 0,
\end{aligned}$$

and so (2.2) is true. Let $v = e_{i_0\cdots i_n} \in \mathcal{R}_n^{\mathrm{reg}}(V)$ with $n \ge 1$. Then

$$\begin{aligned}
\partial^w(\tau(v)) &= \partial^w\left(\sum_{k=0}^{n}(-1)^k\gamma(i_k)e_{i_0\cdots i_k i_k'\cdots i_{n-1}' i_n'}\right)\\
&= \sum_{k=0}^{n}(-1)^k\left(\sum_{m=0}^{k}(-1)^m\delta(i_m)\gamma(i_k)e_{i_0\cdots\widehat{i}_m\cdots i_k i_k'\cdots i_n'}\right)\\
&\quad + \sum_{k=0}^{n}(-1)^k\left(\sum_{m=k}^{n}(-1)^{m+1}\delta(i_m')\gamma(i_k)e_{i_0\cdots i_k i_k'\cdots\widehat{i}_m'\cdots i_n'}\right)\\
&= \sum_{0\le m\le k\le n}(-1)^{k+m}\delta(i_m)\gamma(i_k)e_{i_0\cdots\widehat{i}_m\cdots i_k i_k'\cdots i_n'}\\
&\quad + \sum_{0\le k\le m\le n}(-1)^{k+m+1}\delta(i_m')\gamma(i_k)e_{i_0\cdots i_k i_k'\cdots\widehat{i}_m'\cdots i_n'}
\end{aligned}$$

and

$$\begin{aligned}
\tau(\partial^w v) &= \tau\left(\sum_{m=0}^{n}(-1)^m\delta(i_m)e_{i_0\cdots\widehat{i}_m\cdots i_n}\right)\\
&= \sum_{m=0}^{n}(-1)^m\left(\sum_{k=0}^{m-1}(-1)^k\delta(i'_m)\gamma(i_k)e_{i_0\cdots i_k i'_k\cdots\widehat{i}'_m\cdots i'_n}\right)\\
&\quad+\sum_{m=0}^{n}(-1)^m\left(\sum_{k=m+1}^{n}(-1)^{k-1}\delta(i_m)\gamma(i_k)e_{i_0\cdots\widehat{i}_m\cdots i_k i'_k\cdots i'_n}\right)\\
&= \sum_{0\le k<m\le n}(-1)^{k+m}\delta(i'_m)\gamma(i_k)e_{i_0\cdots i_k i'_k\cdots\widehat{i}'_m\cdots i'_n}\\
&\quad+\sum_{0\le m<k\le n}(-1)^{k+m-1}\delta(i_m)\gamma(i_k)e_{i_0\cdots\widehat{i}_m\cdots i_k i'_k\cdots i'_n}.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial^w\tau(v)+\tau(\partial^w v) &= \sum_{0\le k\le n}(-1)^{k+k}\underbrace{\delta(i_k)\gamma(i_k)}_{=1}e_{i_0\cdots\widehat{i}_k i'_k\cdots i'_n}\\
&\quad+\sum_{0\le k\le n}(-1)^{k+k+1}\underbrace{\delta(i'_k)\gamma(i_k)}_{=1}e_{i_0\cdots i_k\widehat{i}'_k\cdots i'_n}\\
&= \sum_{0\le k\le n}e_{i_0\cdots i_{k-1}i'_k\cdots i'_n}-\sum_{0\le k\le n}e_{i_0\cdots i_k i'_{k+1}\cdots i'_n}\\
&= e_{i'_0\cdots i'_n}+\sum_{1\le k\le n}e_{i_0\cdots i_{k-1}i'_k\cdots i'_n}\\
&\quad-\sum_{0\le k\le n-1}e_{i_0\cdots i_k i'_{k+1}\cdots i'_n}-e_{i_0\cdots i_n}\\
&= e_{i'_0\cdots i'_n}-e_{i_0\cdots i_n}\\
&\quad+\left(\sum_{0\le k-1\le n-1}e_{i_0\cdots i_{k-1}i'_k\cdots i'_n}-\sum_{0\le k\le n-1}e_{i_0\cdots i_k i'_{k+1}\cdots i'_n}\right)\\
&= e_{i'_0\cdots i'_n}-e_{i_0\cdots i_n}=v'-v.
\end{aligned}$$

■

Theorem 2.6. *Let $f^\delta\simeq g^\delta\colon\Pi^\delta=(V,P,\delta_V)\to(W,S,\delta_W)=\Sigma^\delta$ be weighted homotopic morphisms of weighted path complexes such that the elements $\delta_V(i)\in R$ are invertible for all $i\in V$. Then morphisms f^δ and g^δ induce the chain homotopic morphisms of weighted chain complexes*

$$f^\delta_*\simeq g^\delta_*\colon\Omega^w_*(P)\to\Omega^w_*(S)$$

and hence, the equal homomorphisms of weighted path homology groups.

Proof. It is sufficient to consider the case of the one-step weighted homotopy $F\colon \Pi^{\uparrow^{\delta}} \to \Sigma^{\delta}$ between f^{δ} and g^{δ}. By definition $F_* \circ [i_\bullet]_* = f_*$ and $F_* \circ [j_\bullet]_* = g_*$. We define a chain homotopy $L_n\colon \Omega_n^w(P) \to \Omega_{n+1}^w(S)$ such that

$$\partial^w L_n + L_{n-1}\partial^w = g_* - f_*.$$

Let the element $v \in \mathcal{R}_n^{\mathrm{reg}}(P)$ belong to $\Omega_n^w(P)$. Then, by definition, $\tau(v) \in \mathcal{R}_{n+1}^{\mathrm{reg}}(P^{\uparrow})$. To prove that $\tau(v) \in \Omega_{n+1}^w(P^{\uparrow})$, it is sufficient to check that $\partial^w \tau(v) \in \mathcal{R}_n^{\mathrm{reg}}(P^{\uparrow})$. By definition, $\partial^w v \in \mathcal{R}_{n-1}^{\mathrm{reg}}(P) \subset \Omega_{n-1}^w(V)$. By Lemma 2.5,

$$\partial^w \tau(v) = -\tau(\partial^w v) + v' - v$$

where the right summands belong to $\mathcal{R}_n^{\mathrm{reg}}(P^{\uparrow})$. Hence, $\tau(v) \in \Omega_{n+1}^w(P^{\uparrow})$. For $n \geq 0$, we define homomorphisms

$$L_n(v)\colon \Omega_n^w(P) \to \Omega_{n+1}^w(S)$$

by $L_n(v) := F_*(\tau(v))$. Now, we check that L_n is a chain homotopy:

$$\begin{aligned}(\partial^w L_n + L_{n-1}\partial^w)(v) &= \partial^w(F_*(\tau(v))) + F_*(\tau(\partial^w v))\\ &= F_*(\partial^w \tau(v)) + F_*(\tau(\partial^w v))\\ &= F_*(\partial^w \tau(v) + \tau(\partial^w v))\\ &= F_*(v' - v) = g_*^{\delta}(v) - f_*^{\delta}(v).\end{aligned}$$

■

3. Path homology of weighted directed hypergraphs

Definition 3.1. (i) A *directed hypergraph* $G = (V, E)$ consists of a finite set of *vertices* V and a set of *arrows* $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_i \in E$ is an ordered pair (A_i, B_i) of disjoint non-empty subsets of V such that $V = \bigcup_{\mathbf{e}_i \in E}(A_i \cup B_i)$. The set $A = \mathrm{orig}(A \to B)$ is called the *origin* of the arrow and the set $B = \mathrm{end}(A \to B)$ is called the *end* of the arrow. The elements of A are called the *initial* vertices of $A \to B$ and the elements of B are called its *terminal* vertices.

(ii) A directed hypergraph $G = (V, E)$ is called *weighted* if it is equipped with a function $\delta_V\colon V \to R$ where R is a unitary commutative ring. This function is called the *weight function*. We denote such a directed hypergraph by $G^{\delta} = (V, E, \delta_V)$.

For a set X let $\mathbf{P}(X)$ denote as usual its power set. We define a set $\mathbb{P}(X) := \{\mathbf{P}(X) \setminus \emptyset\} \times \{\mathbf{P}(X) \setminus \emptyset\}$. Every map $f\colon V \to W$ induces a map $\mathbb{P}(f)\colon \mathbb{P}(V) \to \mathbb{P}(W)$. For a directed hypergraph $G = (V, E)$, by Definition 3.1, we have the natural map $\varphi_G\colon E \to \mathbb{P}(V)$ defined by $\varphi_G(A \to B) := (A, B)$. Let $X \subset V$ be a subset of the set of vertices of a weighted directed hypergraph G^δ. Define the *weight* $|X|$ of a set X by setting

$$|X| := \sum_{x\in X} \delta_V(x).$$

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two directed hypergraphs. By [14, Def. 2.1], the morphism $f\colon G \to H$ is given by a pair of maps $f_V\colon V_G \to V_H$ and $f_E\colon E_G \to E_H$ such that the diagram

$$\begin{array}{ccc} E_G & \xrightarrow{\varphi_G} & \mathbb{P}(V_G) \\ \Big\downarrow f_E & & \Big\downarrow \mathbb{P}(\mathfrak{f}_{\mathbb{V}}) \\ E_H & \xrightarrow{\varphi_H} & \mathbb{P}(V_H) \end{array} \tag{3.1}$$

is commutative. Let $G^\delta = (V_G, E_G, \delta_{V_G})$ and $H^\delta = (V_H, E_H, \delta_{V_H})$ be two weighted directed hypergraphs and let $f\colon G \to H$ be a morphism.

Definition 3.2. (i) The morphism f is called *vertex-weighted* if $\delta_{V_H}(f_V(v)) = \delta_{V_G}(v)$ for every $v \in V_G$. We denote such morphism by f^v.

(ii) The morphism f is called *edge-weighted* if for every $(A \to B) \in E_G$ and $f_E(A \to B) = A' \to B'$ the conditions $|A| = |A'|$, $|B| = |B'|$ are satisfied. We denote such a morphism by f^e.

(iii) The morphism f is called *strong-weighted* if it is *vertex-weighted* and *edge-weighted* simultaneously. We denote such a morphism by f^s.

Example 3.3. (i) Let $G^\delta = (V_G, E_G, \delta_{V_G})$ be the weighted directed hypergraph with $V_G = \{0, 1, 2\}$, $E_G = \{\{0\} \to \{1, 2\}\}$ and $\delta_{V_G}(i) = 1$ for $i \in V_G$. Let $H^\delta = (V_H, E_H, \delta_{V_H})$ be the weighted directed hypergraph with $V_H = \{0, 1\}$, $E_G = \{\{0\} \to \{1\}\}$ and $\delta_{V_H}(i) = 1$ for $i \in V_H$. Consider the morphism of directed hypergraphs $f\colon G \to H$ where $f_V(0) = 0$, $f_V(1) = f_V(2) = 1$ and $f_E(\{0\} \to \{1, 2\}) = (\{0\} \to \{1\})$. Then f is a vertex-weighted morphism but it is not an edge-weighted morphism.

(ii) Let G and H be the same directed hypergraphs as in (i). Define the weight functions δ_{V_G} and δ_{V_H} setting $\delta_{V_G}(0) = 1$, $\delta_{V_G}(1) = \delta_{V_G}(2) = 1/2$ and $\delta_{V_H}(i) = 1$ for $i \in V_H$. Then the morphism $f\colon G \to H$ defined in (i) is an edge-weighted morphism but it is not a vertex-weighted morphism.

(iii) For any weighted directed hypergraph G^δ the identity morphism Id: $G^\delta \to G^\delta$ provides an example of a strong-weighted morphism.

(iv) Let G^δ be the directed hypergraph as in (i) with the weight function $\delta_{V_G}(0) = \delta_{V_G}(1) = 1$, $\delta_{V_G}(2) = 0$. Let $F^\delta = (V_F, E_F, \delta_{V_F})$ be the weighted directed hypergraph with $V_F = \{0, 1, 2, 3\}$, $E_F = \{\{0\} \to \{1, 2, 3\}\}$ and $\delta_{V_F}(0) = \delta_{V_F}(1) = 1$, $\delta_{V_F}(2) = \delta_{V_F}(3) = 0$. Consider the morphism of directed hypergraphs $g\colon F \to G$ where $f_V(0) = 0$, $f_V(1) = 1$, $f_V(2) = f_V(3) = 2$ and $f_E(\{0\} \to \{1, 2, 3\}) = (\{0\} \to \{1, 2\})$. Then g is the strong-weighted morphism.

Thus, we obtain categories $\mathcal{DH}^v$, $\mathcal{DH}^e$, and $\mathcal{DH}^s$ of vertex-weighted, edge-weighted, and strong-weighted directed hypergraphs, respectively.

Let $G = (V_G, E_G)$ be a directed hypergraph. We define subsets $\mathbf{P}_0(G)$, $\mathbf{P}_1(G)$, and $\mathbf{P}_{01} = \mathbf{P}_0(G) \cup \mathbf{P}_1(G)$ of $\mathbf{P}(V_G) \setminus \emptyset$ by setting

$$\begin{aligned}\mathbf{P}_0(G) &= \{A \in \mathbf{P}(V_G) \setminus \emptyset \mid \exists B \in \mathbf{P}(V_G) \setminus \emptyset : A \to B \in E_G\},\\ \mathbf{P}_1(G) &= \{B \in \mathbf{P}(V_G) \setminus \emptyset \mid \exists A \in \mathbf{P}(V_G) \setminus \emptyset : A \to B \in E_G\}.\end{aligned}$$

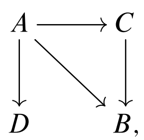
We define a digraph $\mathfrak{N}(G) = (V^n_G, E^n_G)$ where $V^n_G = \{C \in \mathbf{P}(V) \setminus \emptyset \mid C \in \mathbf{P}_{01}(G)\}$ and $E^n_G = \{A \to B \mid (A \to B) \in E\}$. Note that $\mathfrak{N}$ is a functor from the category $\mathcal{DH}$ of directed hypergraphs [14] to the category $\mathcal{D}$ of digraphs. We equip the digraph $\mathfrak{N}(G) = (V^n_G, E^n_G)$ with a structure of a *weighted digraph* by defining a weight function in the following way: $\delta_{V^n_G}(A) \overset{\text{def}}{:=} |A|$ where $|A|$ is the weight of the set A. We denote such a weighted digraph by $\mathfrak{N}^w(G)$.

Proposition 3.4. *Every edge-weighted morphism of weighted directed hypergraphs* $f^e\colon G^\delta \to H^\delta$ *defines a morphism of weighted digraphs*

$$[\mathfrak{N}^w(f^e)] = (f^n_V, f^n_E)\colon (V^n_G, E^n_G, \delta_{V^n_G}) \to (V^n_H, E^n_H, \delta_{V^n_H})$$

by $f^n_V(C) := [\mathbf{P}(f^e_V)](C)$ *and* $f^n_E(A \to B) = (f^e_V(A) \to f^e_V(B)) \in E^n_H$. *Thus, we obtain a functor* $\mathfrak{N}^w$ *from the category* $\mathcal{DH}^e$ *of edge-weighted directed hypergraphs to the category* $\mathcal{D}^\delta$ *of weighted digraphs.*

Example 3.5. Let $G^\delta = (V_G, E_G, \delta_{V_G})$ be the weighted directed hypergraph with $V_G = \{0, 1, 2, 3\}$, $E_G = \{\{0\} \to \{1, 2\}, \{1, 2\} \to \{3\}, \{0\} \to \{2, 3\}, \{0\} \to \{3\}\}$ and $\delta_{V_G}(i) = 1$ for $i \in V_G$. Then $\mathfrak{N}^w(G^\delta)$ is the following weighted digraph:



in which $A = \{0\}$, $B = \{3\}$, $C = \{1, 2\}$, $D = \{2, 3\}$ and $\delta_{V^n_G}(A) = \delta_{V^n_G}(B) = 1$, $\delta_{V^n_G}(C) = \delta_{V^n_G}(D) = 2$.

The composition $\mathfrak{D}^w \circ \mathfrak{N}^w$ is a functor $\mathcal{DH}^e \to \mathcal{P}^\delta$, and we define the *edge-weighted path homology groups* of the weighted directed hypergraph G^δ by

$$H^{\mathbf{e}}_*(G) := H^{\mathbf{w}}_*(\mathfrak{D}^w \circ \mathfrak{N}^w(G)).$$

Now we use the notion of a *line digraph* $I_n = (V_{I_n}, E_{I_n})$ $(n \geq 0)$ from [11, Sec. 3.1]. Note that we have two digraphs I_1, namely $0 \to 1$ and $1 \to 0$. For a digraph G the *box product* $G\Box I_n$ is a digraph with $V_{G\Box I_n} = V_G \times V_{I_n}$ and with the set of arrows $E_{G\Box I_n}$ such that there is an arrow $((x,i) \to (y,j)) \in E_{G\Box H}$ if and only if either $x = y$ and $i \to j$ or $x \to y$ and $i = j$, see [11]. If G^δ is a weighted digraph, we equip the digraph $G\Box I_n$ with the structure of a weighted digraph by setting $\delta_{V_{G\Box I_n}}(v,i) = \delta_{V_G}(v)$ for $v \in V_G, i \in V_{I_n}$. We denote this weighted digraph by $G^\delta \Box I_n$.

For a directed hypergraph G, the *box product* $G\Box I_n = (V_{G\Box I_n}, E_{G\Box I_n})$ is a directed hypergraph with $V_{G\Box I_n} = V_G \times V_{I_n}$ and the set of arrows $E_{G\Box I_n}$ being a union of $\{A\times i \to B \times i\}$, $(A \to B) \in E_G, i \in V_{I_n}$ and $\{A\times i \to A\times j\}$, $(i \to j)$ $\in E_{I_n}$, $A \in \mathbf{P}_{01}(G)$. If G^δ is a weighted directed hypergraph, we equip the directed hypergraph $G\Box I_n$ with the structure of a weighted directed hypergraph by setting $\delta_{V_{G\Box I_n}}(v\times i) = \delta_{V_G}(v)$ for $v \in V_G$, $i \in V_{I_n}$. We denote this weighted directed hypergraph by $G^\delta \Box I_n$. Recall, see e.g. [14], that two morphisms $f_0, f_1 \colon G \to H$ of directed hypergraphs are called *one-step homotopic* $f_0 \simeq_1 f_1$, if there exists a morphism $F\colon G\Box I_1 \to H$, such that

$$F|_{G\Box\{0\}} = f_0 \colon G\Box\{0\} \to H, \quad F|_{G\Box\{1\}} = f_1 \colon G\Box\{1\} \to H.$$

Two morphisms $f, g \colon G \to H$ of directed hypergraphs are called *homotopic* $f \simeq g$, if there exists a sequence of morphisms $f_i \colon G \to H$ for $i = 0, \dots, n$ such that $f = f_0 \simeq_1 f_1 \simeq_1 \cdots \simeq_1 f_n = g$.

Proposition 3.6. *Let $f_0^\chi, f_1^\chi \colon G^\delta \to H^\delta$ be weighted morphisms of weighted directed hypergraphs where χ is v, e, or s. If f_0 and f_1 are one-step homotopic as morphisms of directed hypergraphs by means of a homotopy F, then F can be considered as a weighted morphism $F^\chi \colon G^\delta \times I_1 \to H^\delta$.*

Proof. It follows from the definition of $G^\delta \Box I_1$. ∎

If χ means one of the values v, e, or s we obtain a *homotopy category of χ-weighted directed hypergraphs $h\mathcal{DH}^\chi$* whose objects are weighted directed hypergraphs and morphisms are classes of homotopic χ-weighted morphisms.

Lemma 3.7. *Let $I_1 = (0 \to 1)$ and G^δ be a weighted digraph. Then there is an equality $[\mathfrak{D}^w(G^\delta)]^{\uparrow^\delta} = \mathfrak{D}^w(G^\delta \Box I_1)$ of weighted path complexes.*

Proof. It follows from the definitions of the functor $\mathfrak{D}^w$, the box product, and the path complex $\Pi^\uparrow$. ∎

Lemma 3.8. *Let G^δ be a weighted directed hypergraph and $I_1 = (0 \to 1)$. Then there is an equality $[\mathfrak{D}^w\mathfrak{N}^w(G)]^{\uparrow^\delta} = \mathfrak{D}^w\mathfrak{N}^w(G\Box I_1)$ of the weighted path complexes.*

Proof. By Lemma 3.7, $[\mathfrak{D}^w(\mathfrak{N}^w(G))]^{\uparrow^\delta} = \mathfrak{D}^w[\mathfrak{N}^w(G)\Box I_1]$. The weighted digraphs $\mathfrak{N}^w(G)\Box I_1$ and $\mathfrak{N}^w(G\Box I_1)$ are equal by definition of the box product and the functor $\mathfrak{N}^w$. Hence, $\mathfrak{D}^w[\mathfrak{N}^w(G)\Box I_1] = \mathfrak{D}^w\mathfrak{N}^w(G\Box I_1)$ and the result follows. ■

Theorem 3.9. *Let $f^e, g^e\colon G^\delta \to H^\delta$ be edge-weighted homotopic morphisms of weighted directed hypergraphs and assume that all elements of the set $\mathbb{K} = \{k \in R \mid k = |A|, A \in \mathbf{P}_{01}(G)\}$ are invertible in R. Then the induced homomorphisms*

$$f_*, g_*\colon H_*^{\mathbf{e}}(G, R) \to H_*^{\mathbf{e}}(H, R)$$

coincide.

Proof. It follows from Lemma 3.8 and Theorem 2.6. ■

Let us consider a subcategory $h\mathcal{DH}^1$ of the category $h\mathcal{DH}^e$ in which $\delta_V \equiv 1$ for every weighted directed hypergraph.

Theorem 3.10. *If the ring R is an algebra over rationals, then the weighted homology groups $H_*^{\mathbf{e}}(-, R)$ are homotopy invariant on the category $h\mathcal{DH}^1$.*

Proof. Similar to the proof of Theorem 3.9. ■

Now, we describe several weighted path homology theories on the category $\mathcal{DH}^v$ which are similar to the path homology theories on $\mathcal{DH}$ constructed in [14, Sec. 3]. We present here only functorial and homotopy invariant theories. The other path homology theories from [14] can be transferred to the case of weighted hypergraphs similarly. Let us consider the functors $\mathfrak{C}$, $\mathfrak{B}$, and the functor given by the composition $\mathfrak{H}^q \circ \mathfrak{E}$ from the category $\mathcal{DH}$ to the category $\mathcal{P}$ of path complexes [14]. We recall the definition of these functors for objects, then it will be clear for morphisms.

Let $G=(V,E)$ be a directed hypergraph. Define a path complex $\mathfrak{C}(G)=(V^c, P_G^c)$ by $V^c = V$ and a path $(i_0\cdots i_n) \in P_V$ lies in P_G^c iff for any pair of consequent vertices (i_k, i_{k+1}) either $i_k = i_{k+1}$ or there is an edge $\mathbf{e} = (A \to B) \in E$ such that i_k is the initial vertex and i_{k+1} is the terminal vertex of $\mathbf{e}$.

Recall the definition of the *concatenation* $p \vee q$ of two paths $p = (i_0, \dots, i_n)$ and $q = (j_0, \dots, j_m)$ on a set V. The concatenation is well defined only for $i_n = j_0$ and in such a case, it is a path on V given by $p \vee q = (i_0 \cdots i_n j_1 \cdots j_m)$. Define a path complex $\mathfrak{B}(G) = (V_G^b, P_G^b)$ where $V_G^b = V$ and a path $q = (i_0 \cdots i_n) \in P_V$ lies in P_G^b iff there is a sequence of edges $(A_0 \to B_0), \dots, (A_r \to B_r)$ in E such that $B_i \cap A_{i+1} \neq \emptyset$ for $0 \leq i \leq r-1$ and the path q can be presented in the form

$$\big(p_0 \vee v_0w_0 \vee p_1 \vee v_1w_1 \vee p_2 \vee \cdots \vee p_r \vee v_rw_r \vee p_{r+1}\big)$$

where $p_0 \in P_{A_0}$, $p_{r+1} \in P_{B_r}$, $v_i \in A_i$, $w_i \in B_i$, $p_i \in P_{B_{i-1}} \cap P_{A_i}$ for $1 \leq i \leq r$ and all concatenations are well defined.

Recall that a *hypergraph* $G = (V, E)$ consists of a non-empty set V of vertices and a set of edges $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ which are distinct subsets of V such that $\bigcup_{i=1}^{n} \mathbf{e}_i = V$ and every $\mathbf{e}_i$ contains strictly more than one element. For a directed hypergraph G, we define a hypergraph $\mathfrak{E}(G) = (V^e, E^e)$ where $V^e = V$ and

$$E^e = \{C \in \mathbf{P}(V) \setminus \emptyset \mid C = A \cup B, (A \to B) \in E\}.$$

For a hypergraph $G = (V, E)$, define a path complex $\mathfrak{H}^2(G) = (V^2, P_G^2)$ of *density two* where $V^2 = V$ and a path $(i_0 \cdots i_n) \in P_V$ lies in P_G^2 iff every two consequent vertices of this path lie in an edge $\mathbf{e} \in E$, see [10]. Thus for every directed hypergraph G the composition $\mathfrak{H}^2 \circ \mathfrak{E}$ defines a path complex $\mathfrak{H}^2 \circ \mathfrak{E}(G)$.

For a weighted directed hypergraph $G^\delta = (V, E, \delta_V)$, the path complexes $\mathfrak{C}(G)$, $\mathfrak{B}(G)$, and $\mathfrak{H}^2 \circ \mathfrak{E}(G)$ have the natural structure of weighted path complexes given by the weight function $\delta_V : V \to R$. We denote weighted path complexes by $\mathfrak{C}^v(G^\delta)$, $\mathfrak{B}^v(G^\delta)$, and $[\mathfrak{H}^2\mathfrak{E}(G^\delta)]^v$, respectively.

Proposition 3.11. *If $f^v : G^\delta \to H^\delta$ is a vertex-weighted morphism of weighted directed hypergraphs, then the morphisms*

$$\mathfrak{C}(f) : \mathfrak{C}(G) \to \mathfrak{C}(H),$$
$$\mathfrak{B}(f) : \mathfrak{B}(G) \to \mathfrak{B}(H),$$
$$\mathfrak{H}^2\mathfrak{E}(f) : \mathfrak{H}^2\mathfrak{E}(G) \to \mathfrak{H}^2\mathfrak{E}(H)$$

are vertex weighted for the weighted function δ_V. Thus, we have the functors $\mathfrak{C}^v$, $\mathfrak{B}^v$, and $[\mathfrak{H}^2\mathfrak{E}]^v$ from $\mathcal{DH}^v$ to $\mathcal{P}^\delta$.

Proof. It follows from the definition of morphisms in the category $\mathcal{DH}^v$ and from the definition of the weight function on the objects of corresponding categories. ■

For any weighted directed hypergraph $G^\delta = (V, E, \delta_V)$, we call

$$\begin{aligned} H_*^{\mathbf{c}/\mathbf{v}}(G^\delta) &:= H_*^{\mathbf{w}}(\mathfrak{C}^v(G^\delta)), \\ H_*^{\mathbf{b}/\mathbf{v}}(G^\delta) &:= H_*^{\mathbf{w}}(\mathfrak{B}^v(G^\delta)), \\ H_*^{2/v}(G^\delta) &:= H_*^{\mathbf{w}}([\mathfrak{H}^2\mathfrak{E}]^v(G^\delta)), \end{aligned} \tag{3.2}$$

as *weighted path homology groups*, *weighted bold path homology groups*, and *weighted non-directed path homology groups of density two*, respectively.

Now we reformulate [14, Lemma 3.5, Lemma 3.10, and Lemma 3.16] for the case of functors $\mathfrak{C}^v$, $\mathfrak{B}^v$, and $[\mathfrak{H}^2\mathfrak{E}]^v$ from $\mathcal{DH}^v$ to $\mathcal{P}^\delta$. Let $G^\delta = (V, E, \delta_V)$ be a weighted directed hypergraph and $I_1 = (0 \to 1)$.

Lemma 3.12. (i) *There is a natural isomorphism* $\mathfrak{C}^v(G^\delta \Box I_1) \cong [[\mathfrak{C}^v(G^\delta)]^\uparrow]^\delta$ *of weighted path complexes.*

(ii) *There exists an inclusion* $\lambda^v\colon [[\mathfrak{B}^v(G^\delta)]^\uparrow]^\delta \to \mathfrak{B}^{\mathbf{v}}(G^\delta \Box I_1)$ *of weighted path complexes. The restrictions of* λ^v *to the images of the morphisms* $i_\bullet$ *and* $j_\bullet$*, defined in Section 2, are the natural identifications.*

(iii) *There is the inclusion*

$$\mu^v\colon \Big[\big[[\mathfrak{H}^2 \circ \mathfrak{E}]^v(G^\delta)\big]^\uparrow\Big]^\delta \to \big[\mathfrak{H}^2 \circ \mathfrak{E}\big]^v(G^\delta \Box I_1)$$

of weighted path complexes.

Proof. By definition, the weight functions on the vertex set $V \times J$ are given by the function $\delta_{V\times J}(v,0) = \delta_{V\times J}(v,1) = \delta_V(v)$ and hence, coincide for all path complexes above. Now the result follows from [14]. ∎

Now, let $H_*^{\chi/\mathbf{v}}(G^\delta)$ denote one of the homology groups $H_*^{\mathbf{c}/\mathbf{v}}(G^\delta)$, $H_*^{\mathbf{b}/\mathbf{v}}(G^\delta)$, or $H_*^{2/v}(G^\delta)$ defined in (3.2).

Theorem 3.13. *Let* $f^v, g^v\colon G_1^\delta = (V_1, E_1, \delta_{V_1}) \to G_2^\delta = (V_2, E_2, \delta_{V_2})$ *be vertex-weighted homotopic morphisms of weighted directed hypergraphs and assume that all elements of the set* $\mathbb{K} = \{k \in R \mid k = \delta_{V_1}(v), v \in V_1\}$ *are invertible in* R*. Then the induced homomorphisms*

$$f_*^v, g_*^v\colon H_*^{\chi/\mathbf{v}}(G_1^\delta, R) \to G_*^{\chi/\mathbf{v}}(G_2^\delta, R)$$

coincide.

Proof. It follows from Lemma 3.12 and Theorem 2.6. ∎

All data generated or analysed during this study are included in this published article.

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Yuri Muranov
Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710 Olsztyn, Poland; muranov@matman.uwm.edu.pl

Anna Szczepkowska
Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710 Olsztyn, Poland; anna.szczepkowska@matman.uwm.edu.pl

Vladimir Vershinin
Département des Sciences Mathématiques, Université de Montpellier, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France; vladimir.verchinine@umontpellier.fr; and Sobolev Institute of Mathematics, Novosibirsk, Russia