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On orbifold Gromov–Witten classes

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Abstract. We discuss some questions about Gromov–Witten classes of target stacks.

0. Introduction

Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack over $\mathbb{C}$. The stack

$$\mathcal{K}_{g,n}(\mathcal{X}, d),$$

which parametrizes degree d stable maps from genus g orbifold curves (orbifold curves are also called *twisted curves*; orbifold nodes are always assumed to be *balanced*.) with n possibly orbifold markings (the marked gerbes are *not* trivialized), is constructed in [4] (see also [19]). It is Deligne–Mumford and proper over $\mathbb{C}$.

There are several natural maps defined for $\mathcal{K}_{g,n}(\mathcal{X}, d)$:

(1) Restricting stable maps to marked points yields the *evaluation maps*

$$\mathrm{ev} : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \bar{I}\mathcal{X},$$

where $\bar{I}\mathcal{X}$ is the *rigidified* inertia stack of $\mathcal{X}$. See [3, Section 3] for a detailed discussion on inertia stacks and [3, Section 4.4] for the construction of evaluation maps.

(2) Forgetting stable maps to $\mathcal{X}$ but only retaining the domain curves yields the forgetful map

$$\pi : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \mathfrak{M}^{\mathrm{tw}}_{g,n},$$

where $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ is the stack of n-pointed genus g orbifold curves, see [19, Theorem 1.9]. Assuming $2g - 2 + n > 0$, then passing to coarse curves and stabilizing the domains yield another forgetful map

$$p : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \overline{\mathcal{M}}_{g,n},$$

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where $\overline{\mathcal{M}}_{g,n}$ is the stack of n-pointed genus g stable curves. There is an obvious commutative diagram

$$\begin{array}{ccc} \mathcal{K}_{g,n}(\mathcal{X},d) & & \\ {\scriptstyle\pi}\downarrow & \searrow^{p} & \\ \mathfrak{M}^{\mathrm{tw}}_{g,n} & \longrightarrow & \overline{\mathcal{M}}_{g,n}. \end{array}$$

A perfect obstruction theory for $\mathcal{K}_{g,n}(\mathcal{X},d)$ relative to π is introduced in [3], yielding a virtual fundamental class in Chow groups[1],

$$[\mathcal{K}_{g,n}(\mathcal{X},d)]^{\mathrm{vir}} \in \mathrm{CH}_*(\mathcal{K}_{g,n}(\mathcal{X},d)),$$

which may also be viewed as a homology class via the cycle map

$$\mathrm{CH}_*(\mathcal{K}_{g,n}(\mathcal{X},d)) \to H_*(\mathcal{K}_{g,n}(\mathcal{X},d)).$$

There are natural classes defined on $\mathcal{K}_{g,n}(\mathcal{X},d)$:

(1) Pulling back via evaluation maps yields

$$\mathrm{ev}_i^*(\gamma)$$

where γ is a Chow/cohomology class of $\bar{I}\mathcal{X}$.

(2) The *descendant classes*

$$\psi_i := c_1(L_i)$$

are the first Chern classes (taken in Chow or cohomology groups) of line bundles $L_i \to \mathcal{K}_{g,n}(\mathcal{X},d)$ formed by cotangent lines at the i-th marked points of the *coarse domain curves*.

Gromov–Witten theory of the stack $\mathcal{X}$ is the study of classes of the form

$$\prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \cap [\mathcal{K}_{g,n}(\mathcal{X},d)]^{\mathrm{vir}}, \tag{0.1}$$

where $\gamma_1,\ldots,\gamma_n$ are Chow/cohomology classes of $\bar{I}\mathcal{X}$ and $k_1,\ldots,k_n \in \mathbb{Z}_{\geq 0}$. Pushing (0.1) to a point yields Gromov–Witten *invariants* of $\mathcal{X}$, which have been studied extensively in the past 20 years. The purpose of this note is to discuss some questions arising from pushing forward (0.1) to other natural settings.

[1] Chow and (co)homology groups are taken with $\mathbb{Q}$-coefficients.

1. Tautological cohomology classes

In (0.1), take $\gamma_1, \dots, \gamma_n \in H^*(\bar{I}\mathcal{X})$ to be cohomology classes. Pushing forward (0.1) via p yields what is usually called *Gromov–Witten classes*

$$p_*\Big(\prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \cap [\mathcal{K}_{g,n}(\mathcal{X}, d)]^{\mathrm{vir}}\Big) \in H^*(\overline{\mathcal{M}}_{g,n}). \tag{1.1}$$

Without descendants, the classes (1.1) yield a system of multi-linear maps

$$\begin{aligned} &H^*(\bar{I}\mathcal{X})^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}), \\ &\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto p_*\Big(\prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \cap [\mathcal{K}_{g,n}(\mathcal{X}, d)]^{\mathrm{vir}}\Big). \end{aligned} \tag{1.2}$$

Properties of virtual fundamental classes imply that (1.2) is a *cohomological field theory*, a notion introduced in [16]. Discussions on further developments of cohomological field theories can be found in [20].

An important aspect of the study of $H^*(\overline{\mathcal{M}}_{g,n})$ is the (cohomological) *tautological ring*

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n}),$$

which can be defined as the smallest system of unital subrings of $H^*(\overline{\mathcal{M}}_{g,n})$ which is stable under push-forward and pull-back by the following maps:

(1) $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting one of the markings;

(2) $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ gluing two curves at a point;

(3) $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ gluing together two points on a curve.

More details can be found in e.g. [11].

Elements of $RH^*(\overline{\mathcal{M}}_{g,n})$ are called *tautological classes*. The following question, raised for smooth projective varieties [11], should obviously be asked for stacks:

Question 1. *Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack over $\mathbb{C}$. For $\gamma_1, \dots, \gamma_n \in H^*(\bar{I}\mathcal{X})$ and $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$, are the Gromov–Witten classes* (1.1) *tautological?*

Remark 1.1. While tautological rings inside the Chow ring $\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ can be defined, the Chow version of Question 1 is not expected to be true even for varieties. Hence we do not discuss the Chow version here.

Question 1 is known to be true for a number of classes of varieties. A summary of known results can be found in [6, Section 0.7]. Here, we provide two classes of Deligne–Mumford stacks for which Question 1 is true.

Theorem 1.1. *Question 1 is true for smooth semi-projective* toric *Deligne–Mumford stacks $\mathcal{X}$.*

Just like the case of toric varieties, Theorem 1.1 follows from virtual localization [12]. Virtual localization formula for Gromov–Witten theory of toric Deligne–Mumford stacks is written very explicitly in [18].

The virtual localization formula reduces Theorem 1.1 to studying the *Hurwitz–Hodge classes*. To address this, we first review the construction of Hurwitz–Hodge classes arising in the present setting. Let G be a finite abelian group. Let V be a finite dimensional $\mathbb{C}$-vector space that admits a G-action. Let T be an algebraic torus with an action on V (so that the G and T actions on V commute). The vector space V defines a T-equivariant vector bundle $\mathcal{V} \to BG$. Let $\mathcal{K}_{g,n}(BG)$ be the moduli stack of stable maps to $BG = [\mathrm{pt}/G]$. Consider the universal stable map,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & BG \\ {\scriptstyle q}\downarrow & & \\ \mathcal{K}_{g,n}(BG). & & \end{array}$$

The K-theory class

$$Rq_* f^* \mathcal{V} \in K_*(\mathcal{K}_{g,n}(BG))$$

is well defined. Furthermore, Hurwitz–Hodge classes are T-equivariant inverse Euler classes $e_T^{-1}(Rq_* f^* \mathcal{V})$ of this kind of K-theory objects.

Now, $e_T^{-1}(Rq_* f^* \mathcal{V})$ can be expressed in terms of Chern characters of $Rq_* f^* \mathcal{V}$. The (more general) Riemann–Roch calculation[2] of [21] implies that these Chern characters can be expressed in terms of ψ classes and boundary classes of $\mathcal{K}_{g,n}(BG)$. Hence, after pushing forward to $\overline{\mathcal{M}}_{g,n}$, Hurwitz–Hodge classes $e_T^{-1}(Rq_* f^* \mathcal{V})$ are tautological. This proves Theorem 1.1.

Remark 1.2.
(1) The above argument is valid for Deligne–Mumford stacks $\mathcal{X}$ admitting torus actions with isolated fixed points and 1-dimensional orbits.
(2) The above argument is valid in Chow groups, thus answering the Chow version of Question 1 in the affirmative for toric $\mathcal{X}$.
(3) Virtual localization was applied to study Gromov–Witten theory of *toric bundles* $E \to B$ in [9]. It is clear from the localization analysis in [9] and from the Riemann–Roch calculations in [10] and [8] that Gromov–Witten classes

[2] Strictly speaking, the Riemann–Roch calculation in [21] is done for a different moduli stack $\overline{\mathcal{M}}_{g,n}(BG)$ parametrizing stable maps *with sections to marked gerbes*. The answer can be easily adjusted to the present setting.

of the toric bundle E are tautological if Gromov–Witten classes of the base B are tautological. This gives another evidence for Question 1.

The second class of examples we consider is orbifold curves. It is known that a smooth orbifold curve $\mathcal{C}$ is obtained from its underlying coarse curve C (which is itself a smooth curve) by applying a finite number of *root constructions*. We refer to [3, Theorem 4.2.1] for more details of this description.

Theorem 1.2. *Question 1 is true for smooth projective orbifold curves $\mathcal{C}$.*

Question 1 is proven to be true for nonsingular curves in [13]. Our proof of Theorem 1.2 builds on that result, as follows.

We begin with some notations. Let $p_1, \dots, p_m \in \mathcal{C}$ be the orbifold points of $\mathcal{C}$, and $\overline{p}_1, \dots, \overline{p}_m \in C$ their images in the coarse curve. Note that $\overline{p}_1, \dots, \overline{p}_m$ are smooth points on C. Let $r_1, \dots, r_m \in \mathbb{N}$ be orders of stabilizers of the orbifold points $p_1, \dots, p_m$, respectively. Deformation to the normal cone construction can be applied to $p_1, \dots, p_m \in \mathcal{C}$ to give a degeneration of $\mathcal{C}$ to the following nodal curve:

$$C \bigcup \cup_{i=1}^m \mathbb{P}^1_{1,r_i}, \tag{1.3}$$

where $\overline{p}_i \in C$ is identified with the smooth point $0 \in \mathbb{P}^1_{1,r_i}$. More precisely, this degeneration is obtained by degenerating the coarse curve C to $C \bigcup \cup_{i=1}^m \mathbb{P}^1$ (where $\overline{p}_i$ is identified with $0 \in \mathbb{P}^1$), then applying the r_i-th root construction to the divisor in the total space formed as $\overline{p}_i$ moves.

Associated to the pairs $(C, \overline{p}_1, \dots, \overline{p}_m)$ and $\{(\mathbb{P}^1_{1,r_i}, 0)\}_{i=1}^m$ are their *relative* Gromov–Witten classes. Relative Gromov–Witten classes of a pair $(\mathcal{X}, \mathcal{D})$ of a smooth proper Deligne–Mumford stack $\mathcal{X}$ and a smooth divisor $\mathcal{D} \subset \mathcal{X}$ are defined in a manner similar to (1.1) by working with moduli stacks of stable relative maps to $(\mathcal{X}, \mathcal{D})$. Details of these moduli stacks can be found in [2].

The degeneration formula, proven in [2], applies to this setting and expresses Gromov–Witten classes (1.1) of $\mathcal{C}$ in terms of relative Gromov–Witten classes of $(C, \overline{p}_1, \dots, \overline{p}_m)$ and $\{(\mathbb{P}^1_{1,r_i}, 0)\}_{i=1}^m$. By [13, Theorem 1], relative Gromov–Witten classes of $(C, \overline{p}_1, \dots, \overline{p}_m)$ are tautological. The pair $(\mathbb{P}^1_{1,r_i}, 0)$ is toric, and the relative virtual localization formula may be applied. The argument described in the proof of Theorem 1.1 applies here to show that some terms in the relative virtual localization formula are tautological. The only terms not covered by this argument are the *double ramification cycles*, which are tautological by [11] or [14]. Therefore, relative Gromov–Witten classes of $(\mathbb{P}^1_{1,r_i}, 0)$ are tautological. This proves Theorem 1.2.

Remark 1.3. The above argument can be extended a little bit to show the following: for a smooth projective variety X and a smooth divisor $D \subset X$, Gromov–Witten classes of the stack $X_{D,r}$ of r-th roots of X along D are tautological if relative

Gromov–Witten classes of (X, D) and absolute Gromov–Witten classes of D are tautological. This proof encounters double ramification cycles with target D, which are tautological by the formula in [15], provided that Gromov–Witten classes of D are tautological.

Remark 1.4. It would be interesting to consider Question 1 in other examples. For instance, it follows from the product formula [5] that Gromov–Witten classes of a product stack $\mathcal{X} \times \mathcal{Y}$ are tautological if Gromov–Witten classes of $\mathcal{X}$ and $\mathcal{Y}$ are tautological. With efforts, one can hope that the approach in [6] can be extended to complete intersections in weighted projective stacks.

2. Global finite group quotients

An important aspect of the Gromov–Witten theory of stacks $\mathcal{X}$ is the presence of orbifold structures in the domains of stable maps to $\mathcal{X}$. The morphism $p : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \overline{\mathcal{M}}_{g,n}$ forgets these orbifold structures. Therefore it is interesting to consider Gromov–Witten classes of $\mathcal{X}$ in suitable settings where these orbifold structures are not forgotten.

Here, we discuss an attempt to retain these orbifold structures for target stacks of the form

$$\mathcal{X} = [M/G],$$

where M is a smooth (quasi)projective variety over $\mathbb{C}$ and G is a *finite* group. The constant map $M \to \mathrm{pt}$ is clearly G-equivariant, and yields a representable morphism

$$\mathcal{X} = [M/G] \to BG = [\mathrm{pt}/G].$$

Composing stable maps to $\mathcal{X}$ with this morphism and stabilizing yield a morphism of moduli stacks

$$p_G : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \mathcal{K}_{g,n}(BG),$$

which is proper. Pushing forward (0.1) via p_G yields the following classes

$$(p_G)_*\left(\prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \cap [\mathcal{K}_{g,n}(\mathcal{X}, d)]^{\mathrm{vir}}\right) \in H^*(\mathcal{K}_{g,n}(BG)). \tag{2.1}$$

Further pushing forward (2.1) via the natural map $\mathcal{K}_{g,n}(BG) \to \overline{\mathcal{M}}_{g,n}$ recovers (1.1).

An interesting subring of $H^*(\mathcal{K}_{g,n}(BG))$ is the (cohomological) $\mathcal{H}$*-tautological ring*[3], see [17],

$$R_{\mathcal{H}}(\mathcal{K}_{g,n}(BG)) \subset H^*(\mathcal{K}_{g,n}(BG)).$$

[3]It is originally defined in the Chow theory.

Question 2. *Are (2.1) contained in the $\mathcal{H}$-tautological ring of $\mathcal{K}_{g,n}(BG)$?*

When M is toric, and the G-action commutes with the torus action on M, the stack $\mathcal{X} = [M/G]$ admits a torus action and the above approach to Theorem 1.1 applies to show that Question 2 is true in this case. However, in this case (2.1) are contained in some smaller subset of $R_{\mathcal{H}}(\mathcal{K}_{g,n}(BG))$. Indeed, the virtual localization formula and Riemann–Roch calculations show that (2.1) are obtained from push-forwards of combinations of ψ classes via the following natural morphisms:

(1) the morphism that forgets a non-stacky marking, as discussed in [3, Proposition 8.1.1];

(2) the boundary gluing morphisms, as discussed in [3, Proposition 5.2.1].

It should be possible to define a subring of $H^*(\mathcal{K}_{g,n}(BG))$ using the definition of $RH^*(\overline{\mathcal{M}}_{g,n})$, recalled in Section 1, with these maps. If defined, this subring is smaller than the $\mathcal{H}$-tautological ring. Still, (2.1) lie in such a subring.

Whether the formulation of Question 2 really requires the $\mathcal{H}$-tautological rings remains unclear.

For more general $\mathcal{X}$, it is not clear how to construct variants of Gromov–Witten classes of $\mathcal{X}$ that retain orbifold structures on the domains. The natural place for keeping the domain orbifold curves is the stack $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ of orbifold curves. However, the morphism $\pi : \mathcal{K}_{g,n}(\mathcal{X}, d) \to \mathfrak{M}^{\mathrm{tw}}_{g,n}$ is not necessarily proper and cannot be used to produce interesting classes on $\mathfrak{M}^{\mathrm{tw}}_{g,n}$, although the tautological Chow ring $R^*(\mathfrak{M}^{\mathrm{tw}}_{g,n})$ can be defined[4].

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