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Hardwiring truth in functional interpretations

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Abstract. We present four different approaches to prove the soundness theorem for variants with t-truth of functional interpretations. To showcase our different methods we focus on the intuitionistic nonstandard bounded functional interpretation of the nonstandard extensional Heyting arithmetic in all finite types because a version with t-truth for this interpretation has not been given before. Also, because it is a more involved interpretation than others since it includes both nonstandard principles and majorisability. This leads us to believe that if the approaches work for this more complicated functional interpretation, then they should also work for simpler functional interpretations (and realisabilities).

1. Introduction

Functional interpretations are maps of formulas from the language of one theory into the language of another theory, in such a way that provability is preserved. These interpretations typically replace logical relations by functional relations, for example they may transform a formula $\forall x\, \exists y\, \varphi(x, y)$, where y logically depends on x, into the formula $\exists f\, \forall x\, \varphi(x, f(x))$, where $y = f(x)$ is a function(al) of x.

Functional interpretations have many uses, such as

(1) relative consistency results ("a first theory is consistent if a second theory is also consistent");

(2) conservation results ("if a first theory proves a formula of a certain form, then a second theory also proves that formula");

(3) extraction of computational content from proofs ("if a formula of a certain form, e.g. $\forall x\, \exists y\, \varphi(x, y)$, is provable in a certain theory, then we can extract from the proof a computable function(al) giving y as a function of x").

The first functional interpretation was introduced by Gödel in his seminal *Dialectica* article [11, page 285]; it translates formulas of the language of Heyting arithmetic

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with functionals HA^ω (the intuitionistic counterpart of Peano arithmetic with functionals (or finite types) PA^ω) into itself; it has the feature of extracting witnesses (a witness for $\exists x\, \varphi(x)$ would be an x such that $\varphi(x)$ holds). A classical variant was introduced by Shoenfield in his classical textbook on mathematical logic [20, page 219]; it translates formulas of the language of PA^ω into formulas of the language of HA^ω.

A variant of the Dialectica interpretation which deals with the contraction axiom $\varphi \vee \varphi \to \varphi$ using a sequence of potential witnesses with the guarantee that one is indeed a witness, was introduced by Diller and Nahm [2, pages 54–55]; as the Dialectica, it translates formulas of the language of HA^ω into itself.

Later, the monotone functional interpretation was introduced by Kohlenbach [15, page 231]; again, it translates formulas of the language of HA^ω into itself and works by changing the Dialectica interpretation so that in the very last step one changes from extracting witnesses to upper bounds on witnesses (an upper bound for x in $\exists x\, \varphi(x)$ would be a y such that $\exists x \leq^* y\, \varphi(y)$ holds); this has the advantage of interpreting more axioms, sometimes uniformising terms (that is, eliminating their dependence on certain variables) and producing simpler terms [9, Paragraph 8.1].

Afterwards, the bounded functional interpretation was introduced by Ferreira and Oliva [7, Definition 4]; it translates formulas of the language of Heyting arithmetic $\mathsf{HA}^\omega_\trianglelefteq$ extended with some majorisability $\trianglelefteq$ into itself; it also has the feature of not extracting witnesses but upper bounds on witnesses, but in all steps instead of only the very last step. A classical variant was soon after introduced by Ferreira in [5, page 123]; it translates formulas of the language of $\mathsf{PA}^\omega_\trianglelefteq$ into formulas of the language of $\mathsf{HA}^\omega_\trianglelefteq$.

Subsequently, the nonstandard Herbrandised functional interpretation was introduced by Van den Berg, Briseid and Safarik [22, Definition 5.1]; it translates formulas of the language of Heyting arithmetic $\mathsf{E\text{-}HA}^{\omega*}_{\mathrm{st}}$ extended with standardness predicates st and finite sequences of types into itself; it extracts potential witnesses for existential quantifications ranging over standard elements while ignoring quantifications ranging over all (including nonstandard) elements. A classical variant was introduced in the same article [22, Definition 7.1]; it translates formulas of the language of $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ into itself.

Finally, the intuitionistic nonstandard bounded functional interpretation was introduced by Dinis and Gaspar [3, Definition 19]; it translates formulas of the language of Heyting arithmetic $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ without sequences of types into itself. It has the feature of giving upper bounds for existential quantifications ranging over standard elements while ignoring quantifications ranging over all elements. A classical variant had been previously introduced by Ferreira and Gaspar [5, Definition 1]; it translates formulas of the language of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ into formulas of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$.

A functional interpretation I gives information about an interpreted formula Φ^{I}. But sometimes we want information about the uninterpreted Φ itself. Without changing I, there are (at least) two ways of doing this:

(1) if Φ belongs to a special class of formulas Γ such that $\Phi^{\mathrm{I}} \to \Phi$, then the information can be transferred from Φ^{I} to Φ;

(2) if we strengthen the theory in which we are working to the so-called characterisation theory $\mathsf{T}^{\sharp}$, then we have $\Phi^{\mathrm{I}} \leftrightarrow \Phi$.

These two situations already work in many applications of I, for example in proof mining. However, in some investigations, it happens that Φ is not in Γ and we want to keep the theory weaker than $\mathsf{T}^{\sharp}$. In these cases the solution is different: we change I to a related interpretation It, in a process called "hardwiring truth", so that $\Phi^{\mathrm{It}} \to \Phi$ holds for all Φ and in a weaker theory. Historically, this idea first appeared, for realisability, by Kleene [14] (with a realisability rq such that $\Phi^{\mathrm{rq}} \to \Phi$ holds for more formulas than the ones in Γ, but not for all) and then completed by Grayson [12] (with a realisability rt such that $\Phi^{\mathrm{rt}} \to \Phi$ holds for all formulas). More details on functional interpretations and their applications can be found in [16] and the more recent [1].

In this article, we explore proof methods of the soundness theorem of functional interpretations with t-truth: the usual proof method of an induction on the length of proofs and three alternative proof methods to choose from. They are presented as a case-study together with a conviction of generalisation: we illustrate the methods applied to the intuitionistic nonstandard bounded functional interpretation; but it is our conviction that they are easily adapted to other interpretations.

Finally, we would like to point out that, empirically, we see no obstacle to apply the methods presented here to realisability, for example the modified realisability [17, Paragraph 3.5] and the bounded modified realisability [6, Definition 4]. A more rational argument could be that realisabilities can be presented as relations between potential witnesses and challenges, and so recast as functional interpretations [18, Section 3.1], [19, Section 2].

2. Framework

We adopt the same framework that we used before [3, Section 2]. Nevertheless, we give here a quick overview for the sake of completeness.

Let $\mathsf{E\text{-}PA}^{\omega}$ and $\mathsf{E\text{-}HA}^{\omega}$ be (respectively) Peano and Heyting arithmetics in all finite types with full extensionality and with primitive equality only at type 0.

Definition 1. The *Howard–Bezem strong majorisability* $\leq^*_\sigma$ is defined recursively by

(1) $s \leq^*_0 t :\equiv s \leq_0 t$;

(2) $s \leq^*_{\rho\to\sigma} t :\equiv \forall v\, \forall u \leq^*_\rho v\, (su \leq^*_\sigma tv \wedge tu \leq^*_\sigma tv)$.

We say that x^σ is *monotone* if and only if $x \leq^*_\sigma x$.

We recall some properties of the Howard–Bezem strong majorisability.

Proposition 2. *We have*

(1) $\mathsf{E\text{-}HA}^\omega \vdash x \leq^*_\sigma y \to y \leq^*_\sigma y$;

(2) $\mathsf{E\text{-}HA}^\omega \vdash x \leq^*_\sigma y \wedge y \leq^*_\sigma z \to x \leq^*_\sigma z$.

Theorem 3 (Howard's majorisability theorem [13]). *Let t^σ be a closed term of $\mathsf{E\text{-}HA}^\omega$. Then, there is a closed term $\tilde{t}^\sigma$ of $\mathsf{E\text{-}HA}^\omega$ such that $\mathsf{E\text{-}HA}^\omega \vdash t \leq^*_\sigma \tilde{t}$.*

We recall the variant $\mathsf{E\text{-}HA}^\omega_{\mathrm{st}}$ of $\mathsf{E\text{-}HA}^\omega$.

Definition 4. The *nonstandard Heyting arithmetic in all finite types with full extensionality* $\mathsf{E\text{-}HA}^\omega_{\mathrm{st}}$ is obtained from $\mathsf{E\text{-}HA}^\omega$ by enriching the language and the axioms of $\mathsf{E\text{-}HA}^\omega$ in the following way:

(1) adding *standard predicates* $\mathrm{st}^\sigma(t^\sigma)$ for each type σ;

(2) adding the axioms:

  (a) $x =_\sigma y \wedge \mathrm{st}^\sigma(x) \to \mathrm{st}^\sigma(y)$;

  (b) $\mathrm{st}^\sigma(y) \wedge x \leq^*_\sigma y \to \mathrm{st}^\sigma(x)$;

  (c) $\mathrm{st}^\sigma(t)$ for each closed term t;

  (d) $\mathrm{st}^{\sigma\to\tau}(x) \wedge \mathrm{st}^\sigma(y) \to \mathrm{st}^\tau(xy)$;

  for all types σ and τ;

(3) adding the *external induction rule*

$$\frac{\Phi(0) \quad \forall x^0\, (\mathrm{st}^0(x) \to (\Phi(x) \to \Phi(x+1)))}{\forall x^0\, (\mathrm{st}^0(x) \to \Phi(x))}.$$

The logical axioms (but not the arithmetical axioms) are extended to the formulas in the language of $\mathsf{E\text{-}HA}^\omega_{\mathrm{st}}$.

3. Hardwiring t-truth

We hardwire t-truth [12], [21, Exercise 9.7.11 in chapter 9] in the intuitionistic nonstandard bounded functional interpretation B [3, Definition 19] obtaining the intuitionistic nonstandard bounded functional interpretation with t-truth Bt. We follow

Oliva and Gaspar's [10, page 592], [9, Paragraph 3.1] method to hardwire t-truth: to add copies of the formulas under interpretations to the clauses of implication and universal quantification of the interpretation $\Phi^{\mathrm{I}} \equiv \dots \Phi_{\mathrm{I}}$; to be more clear, if Φ_{I} is defined as in the left column below, then Φ_{It} should be defined as in the right column in the following:

$$(\Phi \to \Psi)_{\mathrm{I}} :\equiv \cdots, \qquad (\Phi \to \Psi)_{\mathrm{It}} :\equiv \cdots \wedge (\Phi \to \Psi),$$
$$(\forall x\, \Phi)_{\mathrm{I}} :\equiv \cdots, \qquad (\forall x\, \Phi)_{\mathrm{It}} :\equiv \underbrace{\cdots}_{(\dagger)} \wedge \underbrace{\forall x\, \Phi}_{(\ddagger)}.$$

It turns out that in some cases the copy (‡) is redundant because it is implied by (†) and so can be removed: it is the case of modified realisability [10, page 592] and of our Bt.

Definition 5. The *intuitionistic nonstandard bounded functional interpretation with* t-*truth* Bt assigns to each formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the formula

$$\Phi^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{a}\, \tilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$$

of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ according to the following clauses (where $\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$ is the part inside square brackets, and r^0, s^0 and t^{ρ} are terms). For atomic formulas, we define

$$(r =_0 s)^{\mathrm{Bt}} :\equiv [r =_0 s],$$
$$\mathrm{st}(t)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a].$$

For the remaining formulas, if $\Phi^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\underline{a}\, \tilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$ and $\Psi^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\underline{c}\, \tilde{\forall}^{\mathrm{st}}\underline{d}\, \Psi_{\mathrm{Bt}}(\underline{c};\underline{d})$, then we define

$$(\Phi \wedge \Psi)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{a},\underline{c}\, \tilde{\forall}^{\mathrm{st}}\underline{b},\underline{d}\, [\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \wedge \Psi_{\mathrm{Bt}}(\underline{c};\underline{d})],$$
$$(\Phi \vee \Psi)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{a},\underline{c}\, \tilde{\forall}^{\mathrm{st}}\underline{e},\underline{f}\, [\tilde{\forall}\underline{b} \leq^* \underline{e}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \vee \tilde{\forall}\underline{d} \leq^* \underline{f}\, \Psi_{\mathrm{Bt}}(\underline{c};\underline{d})],$$
$$(\Phi \to \Psi)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{C},\underline{e}\, \tilde{\forall}^{\mathrm{st}}\underline{a},\underline{d}\, [(\tilde{\forall}\underline{b} \leq^* \underline{e}\underline{a}\underline{d}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{C}\,\underline{a};\underline{d})) \wedge (\Phi \to \Psi)],$$
$$(\forall x\, \Phi)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{a}\, \tilde{\forall}^{\mathrm{st}}\underline{b}\, [\forall x\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})],$$
$$(\exists x\, \Phi)^{\mathrm{Bt}} :\equiv \tilde{\exists}^{\mathrm{st}}\underline{a}\, \tilde{\forall}^{\mathrm{st}}\underline{c}\, [\exists x\, \tilde{\forall}\underline{b} \leq^* \underline{c}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})].$$

In the following proposition we prove that Bt really has truth, that is (recalling the explanation of truth in the introduction) $\Phi^{\mathrm{It}} \to \Phi$.

Proposition 6 (t-truth property). *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, we have $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Phi$.*

Proof. The proof is by an easy induction on the length of Φ. Let us see only the case of the universal quantifier. We assume the induction hypothesis $\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Phi$. Then

$$\begin{aligned} (\forall x\, \Phi)_{\mathrm{Bt}}(\underline{a};\underline{b}) &\equiv && \text{(Definition 5)}\\ \forall x\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) &\to && \text{(IH)}\\ \forall x\, \Phi. & && \end{aligned}$$

■

The following is a technical lemma often used in relation with bounded functional interpretations [7, inspired on Lemma 6]. Informally, the lemma says that if we have $\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$, and we think of $\underline{a}$ as being bounds on witnesses, then obviously we can increase the bounds to some $\underline{\tilde{a}}$ (with $\underline{a} \leq^* \underline{\tilde{a}}$), that is we have $\Phi_{\mathrm{Bt}}(\underline{\tilde{a}};\underline{b})$.

Lemma 7 (Monotonicity of Bt). *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \underline{a} \leq^* \underline{\tilde{a}} \wedge \underline{b} \leq^* \underline{b} \wedge \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Phi_{\mathrm{Bt}}(\underline{\tilde{a}};\underline{b}).$$

Proof. The proof is by induction on the length of Φ. Let us see only the case of implication. We assume

$$\underline{C} \leq^* \underline{\widetilde{C}}, \qquad \underline{e} \leq^* \underline{\tilde{e}}, \qquad \underline{a} \leq^* \underline{a}, \qquad \underline{d} \leq^* \underline{d},$$

so $\underline{C}\,\underline{a} \leq^* \underline{\widetilde{C}}\,\underline{a}$ and $\underline{e}\underline{a}\underline{d} \leq^* \underline{\tilde{e}}\underline{a}\underline{d}$. Then

$$\begin{aligned} (\Phi \to \Psi)_{\mathrm{Bt}}(\underline{C},\underline{e};\underline{a},\underline{d}) &\equiv && \text{(Definition 5)}\\ (\widetilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\; \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{C}\,\underline{a};\underline{d})) \wedge (\Phi \to \Psi) &\to && (\text{IH}, \underline{C}\,\underline{a} \leq^* \underline{\widetilde{C}}\,\underline{a})\\ (\widetilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\; \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{\widetilde{C}}\,\underline{a};\underline{d})) \wedge (\Phi \to \Psi) &\to && (\underline{e}\underline{a}\underline{d} \leq^* \underline{\tilde{e}}\underline{a}\underline{d})\\ (\widetilde{\forall} \underline{b} \leq^* \underline{\tilde{e}}\underline{a}\underline{d}\; \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{\widetilde{C}}\,\underline{a};\underline{d})) \wedge (\Phi \to \Psi) &\equiv && \text{(Definition 5)}\\ (\Phi \to \Psi)_{\mathrm{Bt}}(\underline{\widetilde{C}},\underline{\tilde{e}};\underline{a},\underline{d}). & && \end{aligned}$$

■

The interpretation Bt is sound in the usual sense, that is it maps theorems of the interpreted theory $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ to theorems of the interpreting theory $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$.

Theorem 8 (Soundness theorem of Bt). *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, if $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi$, then there are closed (and therefore standard) and monotone terms $\underline{t}$ such that $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}} \underline{b}\; \Phi_{\mathrm{Bt}}(\underline{t};\underline{b})$.*

4. Proof 1: The "standard" proof

First, let us make some remarks and establish some conventions.

(1) The proof is by induction on the length of derivations, so below we give the Bt-interpretations of (some of the) axioms and rules of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ and the terms witnessing the existential quantifications of the Bt-interpretations.

(2) If it is easy to check that the terms are closed, monotone and witness the existential quantifications, then we just present the terms without further explanation.

(3) Given an axiom Φ with Bt-interpretation $\widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$, we denote by $\underline{t_{\underline{a}}}$ the terms witnessing $\widetilde{\exists}^{\mathrm{st}}\underline{a}$.

(4) Given a rule $\frac{\Phi\quad\Psi}{\Omega}$ with Bt-interpretation

$$\frac{\widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})\quad\widetilde{\exists}^{\mathrm{st}}\underline{c}\,\widetilde{\forall}^{\mathrm{st}}\underline{d}\,\Psi_{\mathrm{Bt}}(\underline{c};\underline{d})}{\widetilde{\exists}^{\mathrm{st}}\underline{e}\,\widetilde{\forall}^{\mathrm{st}}\underline{f}\,\Omega_{\mathrm{Bt}}(\underline{e};\underline{f})},$$

we denote by $\underline{r_{\underline{a}}}$ and $\underline{s_{\underline{c}}}$ the terms that by induction hypothesis exist witnessing respectively $\widetilde{\exists}^{\mathrm{st}}\underline{a}$ and $\widetilde{\exists}^{\mathrm{st}}\underline{c}$, and by $\underline{t_{\underline{e}}}$ the terms that we have to present witnessing $\widetilde{\exists}^{\mathrm{st}}\underline{e}$.

(5) We write, for example, $\underline{t_{\underline{a}}}\underline{b}\underline{c} = \underline{b}$ instead of $\underline{t_{\underline{a}}} \equiv \lambda\underline{b},\underline{c}\,.\,\underline{b}$.

Proof. $\underline{\Phi \to \Phi \wedge \Phi}$

$$(\Phi \to \Phi \wedge \Phi)^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{C},\underline{E},\underline{g}\,\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{d},\underline{f}\,\big((\widetilde{\forall}\underline{b} \leq^* \underline{g}\underline{a}\underline{d}\underline{f}\,\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Phi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d}) \wedge \Phi_{\mathrm{Bt}}(\underline{E}\underline{a};\underline{f})) \wedge (\Phi \to \Phi \wedge \Phi)\big),$$

$$\underline{t_{\underline{C}}}\underline{a} := \underline{a}, \quad \underline{t_{\underline{E}}}\underline{a} := \underline{a}, \quad \underline{t_{\underline{g}}}\underline{a}\underline{d}\underline{f} := \max(\underline{d},\underline{f}).$$

The term $\max_\rho$ is a closed and monotone term such that

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall} a^\rho, b^\rho\,(a \leq^*_\rho \max_\rho(a,b) \wedge b \leq^*_\rho \max_\rho(a,b)),$$

which exists [8, Definition 133 and Lemma 149].

$\underline{\Phi \to \Phi \vee \Psi}$

$$(\Phi \to \Phi \vee \Psi)^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{C},\underline{E},\underline{i}\,\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{g},\underline{h}\,\big((\widetilde{\forall}\underline{b} \leq^* \underline{i}\underline{a}\underline{g}\underline{h}\,\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \widetilde{\forall}\underline{d} \leq^* \underline{g}\,\Phi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d}) \vee \widetilde{\forall}\underline{f} \leq^* \underline{h}\,\Psi_{\mathrm{Bt}}(\underline{E}\underline{a};\underline{f})) \wedge (\Phi \to \Phi \vee \Psi)\big),$$

$$\underline{t_{\underline{C}}}\underline{a} := \underline{a}, \quad \underline{t_{\underline{E}}}\underline{a} := \underline{\mathcal{O}}, \quad \underline{t_{\underline{i}}}\underline{a}\underline{g}\underline{h} := \underline{g}.$$

The term $\mathcal{O}^{0\underline{\rho}}$ is the term $\lambda\underline{x}^{\underline{\rho}}\,.\,0^0$. Analogously for $\perp \to \Phi$.

$\underline{\Phi \vee \Psi \to \Psi \vee \Phi}$

$$
\begin{gathered}
(\Phi \vee \Psi \to \Psi \vee \Phi)^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} \underline{E}, \underline{G}, \underline{m}, \underline{n}\, \widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{c}, \underline{k}, \underline{l} \\
\big((\widetilde{\forall} \underline{i}, \underline{j} \le^{*} \underline{m}\underline{a}\underline{c}\underline{k}\underline{l}, \underline{n}\underline{a}\underline{c}\underline{k}\underline{l}\ (\widetilde{\forall} \underline{b} \le^{*} \underline{i}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \vee \widetilde{\forall} \underline{d} \le^{*} \underline{j}\ \Psi_{\mathrm{Bt}}(\underline{c};\underline{d})) \to \\
\widetilde{\forall} \underline{f} \le^{*} \underline{k}\ \Psi_{\mathrm{Bt}}(\underline{E}\underline{a}\underline{c};\underline{f}) \vee \widetilde{\forall} \underline{h} \le^{*} \underline{l}\ \Phi_{\mathrm{Bt}}(\underline{G}\underline{a}\underline{c};\underline{h})\big) \wedge (\Phi \vee \Psi \to \Psi \vee \Phi)\big), \\
\underline{t}_{\underline{E}}\underline{a}\underline{c} := \underline{c}, \quad \underline{t}_{\underline{G}}\underline{a}\underline{c} := \underline{a}, \quad \underline{t}_{\underline{m}}\underline{a}\underline{c}\underline{k}\underline{l} := \underline{l}, \quad \underline{t}_{\underline{n}}\underline{a}\underline{c}\underline{k}\underline{l} := \underline{k}.
\end{gathered}
$$

Analogously for $\Phi \vee \Phi \to \Phi$, $\Phi \wedge \Psi \to \Phi$ and $\Phi \wedge \Psi \to \Psi \wedge \Phi$.

$\underline{\Phi[s/x] \to \exists x\, \Phi}$

$$
\begin{gathered}
(\Phi[s/x] \to \exists x\, \Phi)^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} \underline{C}, \underline{f}\, \widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{e}\, \big((\widetilde{\forall} \underline{b} \le^{*} \underline{f}\underline{a}\underline{e}\ \Phi[s/x]_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
\exists x\, \widetilde{\forall} \underline{d} \le^{*} \underline{e}\ \Phi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d})) \wedge (\Phi[s/x] \to \exists x\, \Phi)\big), \\
\underline{t}_{\underline{C}}\underline{a} := \underline{a}, \quad \underline{t}_{\underline{f}}\underline{a}\underline{e} := \underline{e}.
\end{gathered}
$$

Here we use $\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})[x/w] \equiv \Phi[x/w]_{\mathrm{Bt}}(\underline{a};\underline{b})$, where the variables $\underline{a}, \underline{b}, x, w$ are all distinct, which is easily proved by induction on the length of Φ. Analogously for $\forall x\, \Phi \to \Phi[s/x]$.

$\underline{\Phi \to \Psi,\ \Psi \to \Omega\ /\ \Phi \to \Omega}$

$$
\begin{aligned}
(\Phi \to \Psi)^{\mathrm{Bt}} &\equiv \widetilde{\exists}^{\mathrm{st}} \underline{C}, \underline{g}\, \widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{d}\, \big((\widetilde{\forall} \underline{b} \le^{*} \underline{g}\underline{a}\underline{d}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
&\qquad\qquad \Psi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d})) \wedge (\Phi \to \Psi)\big), \\
(\Psi \to \Omega)^{\mathrm{Bt}} &\equiv \widetilde{\exists}^{\mathrm{st}} \underline{E}, \underline{g}\, \widetilde{\forall}^{\mathrm{st}} \underline{c}, \underline{f}\, \big((\widetilde{\forall} \underline{d} \le^{*} \underline{g}\underline{c}\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{c};\underline{d}) \to \\
&\qquad\qquad \Omega_{\mathrm{Bt}}(\underline{E}\underline{c};\underline{f})) \wedge (\Psi \to \Omega)\big), \\
(\Phi \to \Omega)^{\mathrm{Bt}} &\equiv \widetilde{\exists}^{\mathrm{st}} \underline{E}, \underline{g}\, \widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{f}\, \big((\widetilde{\forall} \underline{b} \le^{*} \underline{g}\underline{a}\underline{f}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
&\qquad\qquad \Omega_{\mathrm{Bt}}(\underline{E}\underline{a};\underline{f})) \wedge (\Phi \to \Omega)\big), \\
\underline{t}_{\underline{E}}\underline{a} &:= \underline{s}_{\underline{E}}(\underline{r}_{\underline{C}}\underline{a}), \qquad \underline{t}_{\underline{g}}\underline{a}\underline{f} := \underline{r}_{\underline{g}}\underline{a}(\underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}).
\end{aligned}
$$

Dropping $\Phi \to \Psi$, $\Psi \to \Omega$ and $\Phi \to \Omega$ to simplify, by induction hypothesis we have (4.1) and (4.2) and we want to prove (4.3),

$$\widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{d}\, (\widetilde{\forall} \underline{b} \le^{*} \underline{r}_{\underline{g}}\underline{a}\underline{d}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{r}_{\underline{C}}\underline{a};\underline{d})), \tag{4.1}$$

$$\widetilde{\forall}^{\mathrm{st}} \underline{c}, \underline{f}\, (\widetilde{\forall} \underline{d} \le^{*} \underline{s}_{\underline{g}}\underline{c}\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{c};\underline{d}) \to \Omega_{\mathrm{Bt}}(\underline{s}_{\underline{E}}\underline{c};\underline{f})), \tag{4.2}$$

$$\widetilde{\forall}^{\mathrm{st}} \underline{a}, \underline{f}\, \big(\widetilde{\forall} \underline{b} \le^{*} \underline{r}_{\underline{g}}\underline{a}(\underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f})\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Omega_{\mathrm{Bt}}(\underline{s}_{\underline{E}}(\underline{r}_{\underline{C}}\underline{a});\underline{f})\big). \tag{4.3}$$

From (4.1) we get (4.4), from (4.2) we get (4.5), and from the last line of (4.4) and (4.5) we get (4.3),

$$\tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{f}\ \underbrace{\tilde{\forall}\underline{d} \leq^* \underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}\ (\tilde{\forall}\underline{b} \leq^* \underline{r}_{\underline{g}}\underline{a}\underline{d}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{r}_{\underline{C}}\underline{a};\underline{d})),}_{\underbrace{\tilde{\forall}\underline{d} \leq^* \underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}\ \tilde{\forall}\underline{b} \leq^* \underline{r}_{\underline{g}}\underline{a}\underline{d}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \tilde{\forall}\underline{d} \leq^* \underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{r}_{\underline{C}}\underline{a};\underline{d})}_{\tilde{\forall}\underline{b} \leq^* \underline{r}_{\underline{g}}\underline{a}(\underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f})\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \tilde{\forall}\underline{d} \leq^* \underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{r}_{\underline{C}}\underline{a};\underline{d})}} \tag{4.4}$$

$$\tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{f}\ \big(\tilde{\forall}\underline{d} \leq^* \underline{s}_{\underline{g}}(\underline{r}_{\underline{C}}\underline{a})\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{r}_{\underline{C}}\underline{a};\underline{d}) \to \Omega_{\mathrm{Bt}}(\underline{s}_{\underline{E}}(\underline{r}_{\underline{C}}\underline{a});\underline{f})\big). \tag{4.5}$$

Recovering $\Phi \to \Psi$, $\Psi \to \Omega$ and $\Phi \to \Omega$, since $\Phi \to \Psi$ and $\Psi \to \Omega$ are provable by induction hypothesis, so it is $\Phi \to \Omega$ by the rule under interpretation. Analogously for Φ, $\Phi \to \Psi$ / Ψ and $\Phi \to \Psi$ / $\Phi \vee \Omega \to \Psi \vee \Omega$.

$\underline{\Phi \wedge \Psi \to \Omega \ /\ \Phi \to (\Psi \to \Omega)}$

$$\begin{gathered}
(\Phi \wedge \Psi \to \Omega)^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\underline{E}, \underline{g}, \underline{h}\ \tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{c}, \underline{f} \\
\big((\tilde{\forall}\underline{b}, \underline{d} \leq^* \underline{g}\underline{a}\underline{c}\underline{f}, \underline{h}\underline{a}\underline{c}\underline{f}\ (\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \wedge \Psi_{\mathrm{Bt}}(\underline{c};\underline{d})) \to \\
\Omega_{\mathrm{Bt}}(\underline{E}\underline{a}\underline{c};\underline{f})) \wedge (\Phi \wedge \Psi \to \Omega)\big), \\
(\Phi \to (\Psi \to \Omega))^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\mathrm{E}, \underline{H}, \underline{G}\ \tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{c}, \underline{f}\ \big(\tilde{\forall}\underline{b} \leq^* \underline{G}\underline{a}\underline{c}\underline{f}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
\big((\tilde{\forall}\underline{d} \leq^* \underline{H}\underline{a}\underline{c}\underline{f}\ \Psi_{\mathrm{Bt}}(\underline{c};\underline{d}) \to \Omega_{\mathrm{Bt}}(\underline{E}\underline{a}\underline{c};\underline{f})) \wedge (\Psi \to \Omega)\big) \wedge (\Phi \to (\Psi \to \Omega))\big), \\
\underline{t}_{\underline{\mathrm{E}}}\underline{a}\underline{c} := \underline{s}_{\underline{E}}\underline{a}\underline{c}, \quad \underline{t}_{\underline{H}}\underline{a}\underline{c}\underline{f} := \underline{s}_{\underline{h}}\underline{a}\underline{c}\underline{f}, \quad \underline{t}_{\underline{G}}\underline{a}\underline{g}\underline{c}\underline{f} := \underline{s}_{\underline{g}}\underline{a}\underline{c}\underline{f}.
\end{gathered}$$

To prove $\Psi \to \Omega$ in $(\Phi \to (\Psi \to \Omega))^{\mathrm{Bt}}$, we use Proposition 6 saying $\Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Phi$, and $\Phi \wedge \Psi \to \Omega$ in $(\Phi \wedge \Psi \to \Omega)^{\mathrm{Bt}}$. Analogously for $\Phi \to (\Psi \to \Omega)$ / $\Phi \wedge \Psi \to \Omega$.

$\underline{\Phi \to \Psi \ /\ \exists x\, \Phi \to \Psi}$

$$\begin{gathered}
(\Phi \to \Psi)^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\underline{C}, \underline{g}\ \tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{d}\big((\tilde{\forall}\underline{b} \leq^* \underline{g}\underline{a}\underline{d}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
\Psi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d})) \wedge (\Phi \to \Psi)\big), \\
(\exists x\, \Phi \to \Psi)^{\mathrm{Bt}} \equiv \tilde{\exists}^{\mathrm{st}}\underline{C}, \underline{f}\ \tilde{\forall}^{\mathrm{st}}\underline{a}, \underline{d}\big((\tilde{\forall}\underline{e} \leq^* \underline{f}\underline{a}\underline{d}\ \exists x\ \tilde{\forall}\underline{b} \leq^* \underline{e}\ \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \\
\Psi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d}) \wedge (\exists x\, \Phi \to \Psi))\big), \\
\underline{t}_{\underline{C}}\underline{a} := \underline{s}_{\underline{g}}\underline{a}, \quad \underline{t}_{\underline{f}}\underline{a}\underline{d} := \underline{s}_{\underline{g}}\underline{a}\underline{d}.
\end{gathered}$$

To prove $\exists x\, \Phi \to \Psi$ in $(\exists x\, \Phi \to \Psi)^{\mathrm{Bt}}$, we use that by induction hypothesis we proved $\Phi \to \Psi$. Analogously for $\Phi \to \Psi$ / $\Phi \to \forall x\, \Psi$.

$\underline{\forall \underline{x}, \underline{y}, z\, (\underline{x} =_{\rho} \underline{y} \to z\underline{x} =_0 z\underline{y})}$ This axiom is an internal formula, so it is equivalent (provably in $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$) to its Bt-interpretation, thus is provable, and does not require witnessing terms. Analogously for the axioms for the combinators, recursors, equality, successor and the rule of (internal) induction.

<u>$\mathrm{st}(t^\rho)$</u>

$$\mathrm{st}(t^\rho)^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} a\, (t \leq^* a),$$
$$t_a :\equiv t^{\mathrm{m}}.$$

Since t^ρ is a closed term, then there exists a closed and monotone term $(t^{\mathrm{m}})^\rho$ such that $\mathsf{E\text{-}HA}^\omega_{\mathrm{st}} \vdash t \leq^*_\rho t^{\mathrm{m}}$ by Howard's theorem [13, Theorem 3.1].

<u>$x =_\rho y \wedge \mathrm{st}(x) \to \mathrm{st}(y)$</u>

$$(x =_\rho y \wedge \mathrm{st}(x) \to \mathrm{st}(y))^{\mathrm{Bt}} \leftrightarrow \widetilde{\exists}^{\mathrm{st}} B\, \widetilde{\forall}^{\mathrm{st}} a\big((x =_\rho y \wedge x \leq^* a \to y \leq^* Ba) \wedge$$
$$(x =_\rho y \wedge \mathrm{st}(x) \to \mathrm{st}(y))\big),$$
$$t_B a :\equiv a.$$

Analogously for $\mathrm{st}(y) \wedge x \leq^*_\rho y \to \mathrm{st}(x)$.

<u>$\Phi[0/w],\ \mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w]\ /\ \mathrm{st}(x) \to \Phi[x/w]$</u>

$$\Phi[0/w]^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} \underline{a}\, \widetilde{\forall}^{\mathrm{st}} \underline{b}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})[0/w],$$
$$(\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w])^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} \underline{D}, \underline{f}\, \widetilde{\forall}^{\mathrm{st}} c, \underline{a}, \underline{e}$$
$$\big((\widetilde{\forall} \underline{b} \leq^* \underline{f} c\underline{a}\underline{e}\, (x \leq^* c \wedge \Phi_{\mathrm{Bt}}(a;b)[x/w]) \to$$
$$\Phi_{\mathrm{Bt}}(\underline{D}c\underline{a};\underline{e})[\mathrm{S}x/w]\big) \wedge (\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w])\big),$$
$$(\mathrm{st}(x) \to \Phi[x/w])^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}} \underline{A}\, \widetilde{\forall}^{\mathrm{st}} c, \underline{b}\big((x \leq^* c \to \Phi_{\mathrm{Bt}}(\underline{A}c;\underline{b})[x/w]) \wedge$$
$$(\mathrm{st}(x) \to \Phi[x/w])\big),$$
$$\underline{t}_A 0 := \underline{r}_a, \quad \underline{t}_A(\mathrm{S}c) := \max(\underline{s}_D c(\underline{t}_A c), \underline{t}_A c).$$

Dropping $\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w]$ and $\mathrm{st}(x) \to \Phi[x/w]$ to simplify, by induction hypothesis we have (4.6) and (4.7) and we want to prove (4.8),

$$\widetilde{\forall}^{\mathrm{st}} \underline{b}\, \Phi_{\mathrm{Bt}}(\underline{r}_a;\underline{b})[0/w], \tag{4.6}$$
$$\widetilde{\forall}^{\mathrm{st}} c, \underline{a}, \underline{e}\, \big(\widetilde{\forall} \underline{b} \leq^* \underline{s}_f c\underline{a}\underline{e}\, (x \leq^* c \wedge \Phi_{\mathrm{Bt}}(a;b)[x/w]) \to \Phi_{\mathrm{Bt}}(\underline{s}_D c\underline{a};\underline{e})[\mathrm{S}x/w]\big), \tag{4.7}$$
$$\widetilde{\forall}^{\mathrm{st}} c, \underline{b}\, (x \leq^* c \to \Phi_{\mathrm{Bt}}(\underline{t}_A c;\underline{b})[x/w]). \tag{4.8}$$

(1) We prove (†) $\forall^{\mathrm{st}} c\, \forall x\, (x \leq^* c \to \underline{t}_A x \leq^* \underline{t}_A c)$ by induction on c.

(2) We prove (‡) $\mathrm{st}(x) \to \widetilde{\forall}^{\mathrm{st}} \underline{b}\, \Phi_{\mathrm{Bt}}(\underline{t}_A x;\underline{b})[x/w]$ by induction on x: the base case is equivalent to (4.6); for the induction step, we assume (‡) and $\mathrm{st}(x)$, we take $c = x$ and $\underline{a} = \underline{t}_A x$ in (4.7) getting essentially its premise

$$\widetilde{\forall}^{\mathrm{st}} \underline{e}\, \widetilde{\forall} \underline{b} \leq^* \underline{s}_f x(\underline{t}_A x)\underline{e}\, \Phi_{\mathrm{Bt}}(\underline{t}_A x;\underline{b})[x/w]$$

and therefore essentially its conclusion

$$\widetilde{\forall}^{\mathrm{st}} \underline{e}\ \Phi_{\mathrm{Bt}}(\underline{s}_{\underline{D}} x (\underline{t}_{\underline{A}} x); \underline{e})[\mathrm{S}x/w],$$

from which by Lemma 7 we get

$$\widetilde{\forall}^{\mathrm{st}} \underline{b}\ \Phi_{\mathrm{Bt}}(\underline{t}_{\underline{A}}(\mathrm{S}x); \underline{b})[\mathrm{S}x/w].$$

(3) We prove (4.8) using Lemma 7, (†) and (‡).

Recovering $\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w]$ and $\mathrm{st}(x) \to \Phi[x/w]$, since $\Phi[0/w]$ and $\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w]$ are provable by induction hypothesis, so it is $\mathrm{st}(x) \to \Phi[x/w]$ by the rule under interpretation. ■

5. Proof 2: A detour through q-truth

We hardwire q-truth [14] in the intuitionistic nonstandard bounded functional interpretation B obtaining the intuitionistic nonstandard bounded functional interpretation with q-truth Bq. We follow Stephen C. Kleene's method to hardwire q-truth: to add copies of the formulas under interpretations to the clauses of disjunction, implication and existential quantification of the interpretation $\Phi^{\mathrm{I}} \equiv \ldots \Phi_{\mathrm{I}}$; to be more clear, if Φ_{I} is defined as in the left column below, then Φ_{Iq} should be defined like in the right column below,

$$\begin{aligned} (\Phi \vee \Psi)_{\mathrm{I}} &:\equiv \cdots \vee \cdots, & (\Phi \vee \Psi)_{\mathrm{Iq}} &:\equiv (\cdots \wedge \Phi) \vee (\cdots \wedge \Psi), \\ (\Phi \to \Psi)_{\mathrm{I}} &:\equiv \cdots \to \cdots, & (\Phi \to \Psi)_{\mathrm{Iq}} &:\equiv \cdots \wedge \Phi \to \cdots, \\ (\exists x\, \Phi)_{\mathrm{I}} &:\equiv \exists x\ \cdots, & (\exists x\, \Phi)_{\mathrm{Iq}} &:\equiv \exists x\, (\cdots \wedge \Phi). \end{aligned}$$

Definition 9. The *intuitionistic nonstandard bounded functional interpretation with* q-*truth* Bq assigns to each formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the formula

$$\Phi^{\mathrm{Bq}} :\equiv \widetilde{\exists}^{\mathrm{st}} \underline{a}\ \widetilde{\forall}^{\mathrm{st}} \underline{b}\ \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b})$$

of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ according to the following clauses (where $\Phi_{\mathrm{Bq}}(\underline{a}; \underline{b})$ is the part inside square brackets, and r^0, s^0 and t^ρ are terms). For atomic formulas, we define

$$\begin{aligned} (r =_0 s)^{\mathrm{Bq}} &:\equiv [r =_0 s], \\ \mathrm{st}(t)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}} a\, [t \leq^* a]. \end{aligned}$$

For the remaining formulas, if

$$\Phi^{\mathrm{Bq}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \qquad \text{and} \qquad \Psi^{\mathrm{Bq}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{c}\,\widetilde{\forall}^{\mathrm{st}}\underline{d}\,\Psi_{\mathrm{Bq}}(\underline{c};\underline{d}),$$

then we define

$$\begin{aligned}
(\Phi \wedge \Psi)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}}\underline{a},\underline{c}\,\widetilde{\forall}^{\mathrm{st}}\underline{b},\underline{d}\,[\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Psi_{\mathrm{Bq}}(\underline{c};\underline{d})],\\
(\Phi \vee \Psi)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}}\underline{a},\underline{c}\,\widetilde{\forall}^{\mathrm{st}}\underline{e},\underline{f}\,[(\widetilde{\forall}\underline{b} \leq^* \underline{e}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Phi)\vee\\
&\qquad\qquad\qquad\qquad (\widetilde{\forall}\underline{d} \leq^* \underline{f}\,\Psi_{\mathrm{Bq}}(\underline{c};\underline{d}) \wedge \Psi)],\\
(\Phi \to \Psi)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}}\underline{C},\underline{e}\,\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{d}\,[\widetilde{\forall}\underline{b} \leq^* \underline{e}\underline{a}\underline{d}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Phi \to \Psi_{\mathrm{Bq}}(\underline{C}\,\underline{a};\underline{d})],\\
(\forall x\,\Phi)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,[\forall x\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b})],\\
(\exists x\,\Phi)^{\mathrm{Bq}} &:\equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{c}\,[\exists x\,(\widetilde{\forall}\underline{b} \leq^* \underline{c}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Psi)].
\end{aligned}$$

In the following proposition we prove that Bq really has truth, but only for disjunctions and existential quantifications.

Proposition 10 (q-truth property). *For all formulas Φ of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ *of the form $\Phi \equiv \Psi \vee \Omega$ or $\Phi \equiv \exists x\,\Psi$, we have* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \to \Phi$.

Proof. Let us see only the case of disjunction: we have

$$(\Psi \vee \Omega)_{\mathrm{Bq}}(\underline{a},\underline{c};\underline{e},\underline{f}) \equiv \underbrace{(\widetilde{\forall}\underline{b} \leq^* \underline{e}\,\Psi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Psi)}_{\to\Psi} \vee \underbrace{(\widetilde{\forall}\underline{d} \leq^* \underline{f}\,\Omega_{\mathrm{Bq}}(\underline{c};\underline{d}) \wedge \Omega)}_{\to\Omega},$$

so $(\Psi \vee \Omega)_{\mathrm{Bq}}(\underline{a},\underline{c};\underline{e},\underline{f}) \to \Psi \vee \Omega$. ∎

The following is an analogous of Lemma 7 for the Bq interpretation.

Lemma 11 (Monotonicity of Bq). *For all formulas Φ of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \underline{a} \leq^* \underline{\tilde{a}} \wedge \underline{b} \leq^* \underline{b} \wedge \Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \to \Phi_{\mathrm{Bq}}(\underline{\tilde{a}};\underline{b}).$$

The next result, inspired by a similar result in [10, Proposition 7.6], not only relates q-truth and t-truth but even "explains" why q-truth is weaker than t-truth: we need to strengthen Φ_{Bq} by "adding" Φ in order to get Φ_{Bt}.

Proposition 12 (Factorisation Bq $\wedge$ id $=$ Bt). *For all formulas Φ of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *being $\Phi^{\mathrm{Bq}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b})$ the* Bq*-interpretation of Φ, then the* Bt*-interpretation of Φ will be of the form $\Phi^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\,\widetilde{\forall}^{\mathrm{st}}\underline{b}\,\Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$ (for the same variables $\underline{a},\underline{b}$ as in Φ^{Bq}) and we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{b}\,(\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Phi \leftrightarrow \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})).$$

Proof. The proof is by induction on the length of Φ. We will implicitly assume that the variables $\underline{a}, \underline{b}$ are monotone and standard, so that (†) $\underline{t} \leq^* \underline{a} \to \mathrm{st}(\underline{t})$ and (‡) $\tilde{\forall} \underline{a} \leq^* \underline{b}\, \Phi \wedge \Psi \leftrightarrow \tilde{\forall} \underline{a} \leq^* \underline{b}\, (\Phi \wedge \Psi)$ with $\underline{a}$ not free in Ψ.

$\underline{s =_0 t}$

$$\begin{aligned}
(s =_0 t)_{\mathrm{Bq}}(;) \wedge s =_0 t &\equiv && \text{(Definition 9)}\\
s =_0 t \wedge s =_0 t &\leftrightarrow && \text{(logic)}\\
s =_0 t &\equiv && \text{(Definition 5)}\\
(s =_0 t)_{\mathrm{Bt}}(;)&.
\end{aligned}$$

$\underline{\mathrm{st}(t)}$

$$\begin{aligned}
\mathrm{st}(t)_{\mathrm{Bq}}(a;) \wedge \mathrm{st}(t) &\equiv && \text{(Definition 9)}\\
t \leq^* a \wedge \mathrm{st}(t) &\leftrightarrow && \text{(†)}\\
t \leq^* a &\equiv && \text{(Definition 5)}\\
\mathrm{st}(t)_{\mathrm{Bt}}(a;)&.
\end{aligned}$$

$\underline{\Phi \vee \Psi}$

$$\begin{aligned}
(\Phi \vee \Psi)_{\mathrm{Bq}}(\underline{a}, \underline{c}; \underline{e}, \underline{f}) \wedge (\Phi \vee \Psi) &\equiv && \text{(Definition 9)}\\
\big((\tilde{\forall} \underline{b} \leq^* \underline{e}\, \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi) \vee (\tilde{\forall} \underline{d} \leq^* \underline{f}\, \Psi_{\mathrm{Bq}}(\underline{c}; \underline{d}) \wedge \Psi)\big) \wedge &\\
(\Phi \vee \Psi) &\leftrightarrow && \text{(logic)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{e}\, \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi) \vee (\tilde{\forall} \underline{d} \leq^* \underline{f}\, \Psi_{\mathrm{Bq}}(\underline{c}; \underline{d}) \wedge \Psi) &\leftrightarrow && \text{(‡)}\\
\tilde{\forall} \underline{b} \leq^* \underline{e}\, (\Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi) \vee \tilde{\forall} \underline{d} \leq^* \underline{f}\, (\Psi_{\mathrm{Bq}}(\underline{c}; \underline{d}) \wedge \Psi) &\leftrightarrow && \text{(IH)}\\
\tilde{\forall} \underline{b} \leq^* \underline{e}\, \Phi_{\mathrm{Bt}}(\underline{a}; \underline{b}) \vee \tilde{\forall} \underline{d} \leq^* \underline{f}\, \Psi_{\mathrm{Bt}}(\underline{c}; \underline{d}) &\equiv && \text{(Definition 5)}\\
(\Phi \vee \Psi)_{\mathrm{Bt}}(\underline{a}, \underline{c}; \underline{e}, \underline{f})&.
\end{aligned}$$

Analogously for $\Psi \wedge \Phi$, $\forall x\, \Phi$ and $\exists x\, \Psi$.

$\underline{\Phi \to \Psi}$

$$\begin{aligned}
(\Phi \to \Psi)_{\mathrm{Bq}}(\underline{C}, \underline{e}; \underline{a}, \underline{d}) \wedge (\Phi \to \Psi) &\equiv && \text{(Definition 9)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\, \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi \to \Psi_{\mathrm{Bq}}(\underline{C}\underline{a}; \underline{d})) \wedge (\Phi \to \Psi) &\leftrightarrow && \text{(logic)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\, \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi \to \Psi_{\mathrm{Bq}}(\underline{C}\underline{a}; \underline{d}) \wedge \Psi) \wedge (\Phi \to \Psi) &\leftrightarrow && \text{(‡)}\\
\big(\tilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\, (\Phi_{\mathrm{Bq}}(\underline{a}; \underline{b}) \wedge \Phi) \to \Psi_{\mathrm{Bq}}(\underline{C}\underline{a}; \underline{d}) \wedge \Psi\big) \wedge (\Phi \to \Psi) &\leftrightarrow && \text{(IH)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{e}\underline{a}\underline{d}\, \Phi_{\mathrm{Bt}}(\underline{a}; \underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{C}\underline{a}; \underline{d})) \wedge (\Phi \to \Psi) &\equiv && \text{(Proposition 6)}\\
(\Phi \to \Psi)_{\mathrm{Bt}}(\underline{C}, \underline{e}; \underline{a}, \underline{d})&. \qquad \blacksquare
\end{aligned}$$

As a curious side remark, observe that from the factorisation $\mathrm{Bq} \wedge \mathrm{id} = \mathrm{Bt}$ the terms witnessing Φ^{Bq} and Φ^{Bt} are the same (and it can be proved that they are also the terms witnessing Φ^{B}).

The interpretation Bq is sound in the usual sense, that it maps theorems of the interpreted theory $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ into theorems of the interpreting theory $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$.

Theorem 13 (soundness theorem of Bq). *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, if $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi$, then there are closed (and therefore standard) and monotone terms $\underline{t}$ such that $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bq}}(\underline{t}; \underline{b})$.*

Finally, we give our second proof of the soundness theorem of Bt (Theorem 8). The proof can be summarised by the following diagram:

$$\begin{tikzcd}
\Phi \arrow[r, "\mathrm{Bq}"] \arrow[dr] & \Phi_{\mathrm{Bq}}(\underline{t}; \underline{b}) \arrow[r] & \Phi_{\mathrm{Bq}}(\underline{t}; \underline{b}) \wedge \Phi \arrow[r, "\mathrm{Bq}\wedge\mathrm{id}=\mathrm{Bt}"] & \Phi_{\mathrm{Bt}}(\underline{t}; \underline{b}). \\
& \Phi \arrow[ur] & &
\end{tikzcd}$$

Proof. The proof is similar to Jaime Gaspar and Paulo Oliva's proof [10, Theorem 7.7], using the soundness theorem of Bq (Theorem 13) and the factorisation $\mathrm{Bq} \wedge \mathrm{id} = \mathrm{Bt}$ (Proposition 12): for all formulas Φ in the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, being $\Phi^{\mathrm{Bq}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\, \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bq}}(\underline{a}; \underline{b})$ the Bq-interpretation of Φ and $\Phi^{\mathrm{Bt}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{a}\, \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bt}}(\underline{a}; \underline{b})$ the Bt-interpretation of Φ (by the factorisation $\mathrm{Bq} \wedge \mathrm{id} = \mathrm{Bt}$), if $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi$ then there exist closed and monotone terms $\underline{t}$ such that $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bq}}(\underline{t}; \underline{b})$ (by the soundness theorem of Bq), so $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bq}}(\underline{t}; \underline{b}) \wedge \Phi$, where $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\, (\Phi_{\mathrm{Bq}}(\underline{t}; \underline{b}) \wedge \Phi \leftrightarrow \Phi_{\mathrm{Bt}}(\underline{t}; \underline{b}))$ (by the factorisation $\mathrm{Bq} \wedge \mathrm{id} = \mathrm{Bt}$), thus the same terms $\underline{t}$ are such that $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\, \Phi_{\mathrm{Bt}}(\underline{t}; \underline{b})$. ∎

6. Proof 3: The copies-only method

To explain the "copies-only method", we consider the case of the axiom $\Phi \to \Phi \vee \Psi$. Let us see how we check that B interprets this axiom.

(1) First, we give the B-interpretation of the axiom,

$$(\Phi \to \Phi \vee \Psi)^{\mathrm{B}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{C}, \underline{E}, \underline{i}\; \widetilde{\forall}^{\mathrm{st}}\underline{a}, \underline{g}, \underline{h}\; (\widetilde{\forall}\underline{b} \leq^{*} \underline{i}\underline{a}\underline{g}\underline{h}\; \Phi_{\mathrm{B}}(\underline{a}; \underline{b}) \to \widetilde{\forall}\underline{d} \leq^{*} \underline{g}\; \Phi_{\mathrm{B}}(\underline{C}\underline{a}; \underline{d}) \vee \widetilde{\forall}\underline{f} \leq^{*} \underline{h}\; \Psi_{\mathrm{B}}(\underline{E}\underline{a}; \underline{f})),$$

and then the terms supposedly witnessing the interpreted formula,

$$\underline{t}_{\underline{C}}\underline{a} := \underline{a}, \quad \underline{t}_{\underline{E}}\underline{a} := \underline{\mathcal{O}}, \quad \underline{t}_{\underline{i}}\underline{a}\underline{g}\underline{h} := \underline{g}.$$

(2) Secondly, we replace the terms by their definitions in the interpreted formula, getting

$$\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{g},\underline{h}\,(\underline{\widetilde{\forall}\underline{b} \le^{*} \underline{g}\,\Phi_{\mathrm{B}}(\underline{a};\underline{b})} \to \underline{\widetilde{\forall}\underline{d} \le^{*} \underline{g}\,\Phi_{\mathrm{B}}(\underline{a};\underline{d})} \vee \widetilde{\forall}\underline{f} \le^{*} \underline{h}\,\Psi_{\mathrm{B}}(\underline{\mathcal{O}};\underline{f})),$$

and we verify that the result is provable, which essentially consists in noticing that the first underlined subformula implies the second underlined subformula because they are the same modulo an irrelevant renaming of the variables $\underline{b}$ to $\underline{d}$.

Now let us see how Bq interprets the axiom $\Phi \to \Phi \vee \Psi$.

(1) First, we give the Bq-interpretation of the axiom,

$$(\Phi \to \Phi \vee \Psi)^{\mathrm{Bq}} \equiv \widetilde{\exists}^{\mathrm{st}}\underline{C},\underline{E},\underline{i}\;\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{g},\underline{h}\,\big(\widetilde{\forall}\underline{b} \le^{*} \underline{i}\underline{a}\underline{g}\underline{h}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b}) \wedge \Phi \to$$
$$(\widetilde{\forall}\underline{d} \le^{*} \underline{g}\,\Phi_{\mathrm{Bq}}(\underline{C}\underline{a};\underline{d}) \wedge \Phi) \vee (\widetilde{\forall}\underline{f} \le^{*} \underline{h}\,\Psi_{\mathrm{Bq}}(\underline{E}\underline{a};\underline{f}) \wedge \Psi)\big),$$

and then the terms supposedly witnessing the interpreted formula (the same as for B),

$$\underline{t_C}\underline{a} := \underline{a}, \quad \underline{t_E}\underline{a} := \underline{\mathcal{O}}, \quad \underline{t_i}\underline{a}\underline{g}\underline{h} := \underline{g}.$$

(2) Secondly, we replace the terms by their definitions in the interpreted formula, getting

$$\widetilde{\forall}^{\mathrm{st}}\underline{a},\underline{g},\underline{h}\,\big(\underline{\widetilde{\forall}\underline{b} \le^{*} \underline{g}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b})} \wedge \underline{\underline{\Phi}} \to$$
$$(\underline{\widetilde{\forall}\underline{d} \le^{*} \underline{g}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{d})} \wedge \underline{\underline{\Phi}}) \vee (\widetilde{\forall}\underline{f} \le^{*} \underline{h}\,\Psi_{\mathrm{Bq}}(\underline{\mathcal{O}};\underline{f}) \wedge \Psi)\big),$$

and we verify that the result is provable, which essentially consists in noticing that

(†) the first underlined subformula implies the second underlined subformula because they are the same modulo an irrelevant renaming of the variables $\underline{b}$ to $\underline{d}$;

(‡) the first double-underlined Φ implies the second double-underlined Φ.

The previous points reveal that the verification that the terms interpret the axiom splits into two *disjoint* tasks (†) and (‡), which we reformulate now:

(†) essentially, to verify, ignoring the ellipsis, that the terms work, that is that they are such that

$$\cdots\big(\underline{\widetilde{\forall}\underline{b} \le^{*} \underline{g}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{b})} \wedge \cdots \to$$
$$(\underline{\widetilde{\forall}\underline{d} \le^{*} \underline{g}\,\Phi_{\mathrm{Bq}}(\underline{a};\underline{d})} \wedge \cdots) \vee (\widetilde{\forall}\underline{f} \le^{*} \underline{h}\,\Psi_{\mathrm{Bq}}(\underline{\mathcal{O}};\underline{f}) \wedge \cdots)\big);$$

(‡) to verify, ignoring the ellipsis, that the copies work, that is that they do not spoil provability

$$\cdots(\cdots \wedge \underline{\underline{\Phi}} \to (\cdots \wedge \underline{\underline{\Phi}}) \vee (\cdots \wedge \Psi)).$$

Analogous considerations work for t-truth. At this point we should conclude what we learned as a statement of the "copies-only method".

In proving the soundness theorem of a functional interpretation with truth Iq *or* It*, it is superfluous to do the task (†) since it is just a repetition of the same task for the functional interpretation without truth* I*, so it suffices to do only the task (‡).*

We should point out, however, that the "copies-only method" does not always work. One situation in where it does not work is when the proof of the soundness theorem of I depends on some crucial property of the formulas $\Phi_{\mathrm{I}}(\underline{a};\underline{b})$ that does not hold for the formulas $\Phi_{\mathrm{Iq}}(\underline{a};\underline{b})$ or $\Phi_{\mathrm{It}}(\underline{a};\underline{b})$. This problem does not occur with Bq (nor Bt) but occurs, for example, if we try to hardwire q-truth (or t-truth) in Gödel's functional interpretation D [11, page 285] obtaining Dq, which we explain now. To illustrate the problem, let us prove that D interprets the axiom $\Phi \to \Phi \wedge \Phi$. Its D-interpretation is

$$\exists \underline{C}, \underline{E}, \underline{B}\, \forall \underline{a}, \underline{d}, \underline{f}\, (\Phi_{\mathrm{D}}(\underline{a}; \underline{B}\,\underline{a}\,\underline{d}\,\underline{f}) \to \Phi_{\mathrm{D}}(\underline{C}\,\underline{a};\underline{d}) \wedge \Phi_{\mathrm{D}}(\underline{E}\,\underline{a};\underline{f}))$$

and the terms witnessing the interpretation are

$$\underline{t}_{\underline{C}}\,\underline{a} = \underline{a}, \quad \underline{t}_{\underline{E}}\,\underline{a} = \underline{a}, \quad \underline{t}_{\underline{B}}\,\underline{a}\,\underline{d}\,\underline{f} = \begin{cases} \underline{f} & \text{if } \ \Phi_{\mathrm{D}}(\underline{a};\underline{d}), \\ \underline{d} & \text{if } \neg\Phi_{\mathrm{D}}(\underline{a};\underline{d}). \end{cases}$$

The terms $\underline{t}_{\underline{B}}$ can be defined by cases because of the crucial property that the formula $\Phi_{\mathrm{D}}(\underline{a};\underline{d})$ is quantifier-free. But, and here is the problem, the formula $\Phi_{\mathrm{Dq}}(\underline{a};\underline{d})$ is no longer quantifier-free (because of the possible non-quantifier-free copies added to it), so we cannot apply the "copies-only method", as we wanted to illustrate. Other properties, which look more innocent but could also be a source of problems, include

(1) $\Phi[t/x]_{\mathrm{Dq}}(\underline{a};\underline{b})$ and $\Phi_{\mathrm{Dq}}(\underline{a};\underline{b})[t/x]$ should be syntactically equal (which plays a crucial role in interpreting the axioms $\forall x\, \Phi \to \Phi[t/x]$ and $\Phi[t/x] \to \exists x\, \Phi$);

(2) the free variables of $\Phi_{\mathrm{Dq}}(\underline{a};\underline{b})$ should be exactly the free variables of Φ and the variables $\underline{a}, \underline{b}$ (which plays a crucial role in interpreting the rules $\Phi \to \Psi$ / $\Phi \to \forall x\, \Psi$ and $\Phi \to \Psi$ / $\exists x\, \Phi \to \Psi$);

(3) for all quantifier-free formulas φ, we should have $\varphi_{\mathrm{Dq}}(\underline{a};\underline{b}) \leftrightarrow \varphi$ and the tuples $\underline{a}$ and $\underline{b}$ should be empty (which plays a crucial role in interpreting axioms and rules restricted to quantifier-free formulas).

After presenting the "copies-only method", we finally go to the actual proof of Theorem 8 using this method.

Proof. First, we notice that there is no need to check the copies for axioms and rules whose premises and conclusions are both unnested implications, as in $\Phi \to \Psi$ versus $\Phi \to (\Psi \to \Omega)$, because their Bt-interpretations, which are very roughly $(\Phi_{\mathrm{Bt}} \to \Psi_{\mathrm{Bt}}) \wedge (\Phi \to \Psi)$, are provable since $\Phi \to \Psi$ is an axiom or the proved premise or conclusion of a rule. In the present setting, there are only two axioms or rules with nested implications, namely

$$\frac{\Phi \wedge \Psi \to \Omega}{\Phi \to (\Psi \to \Omega)}, \qquad \frac{\Phi \to (\Psi \to \Omega)}{\Phi \wedge \Psi \to \Omega}.$$

So we only have to check the theorem for these two rules, which was already done in the "standard" proof in Section 4. ■

7. Proof 4: The translations $\mathfrak{t}$ and $\mathfrak{o}$

The strategy for this final proof is to

(1) define an extension $\mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}}$ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ where there are copies Φ_{c} of every formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$;

(2) a translation $\mathfrak{t}$ from $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ to $\mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}}$ that puts copies in strategical places, namely after implications and universal quantifications;

(3) a translation Bc from $\mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}}$ to itself which is essentially the same as B except that a clause is added to deal with the translation of copies by leaving them unchanged;

(4) a translation $\mathfrak{o}$ from $\mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}}$ to $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ that replaces the copies Φ_{c} with the original formulas Φ.

The idea is summarised in the following diagram (compare with Proposition 24):

$$\begin{tikzcd}
\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \arrow[r, "\mathfrak{t}"] \arrow[rrr, bend right, "\mathrm{Bt}"'] & \mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}} \arrow[r, "\mathrm{Bc}"] & \mathsf{E\text{-}HA}^{\omega}_{\substack{\mathrm{c}\\ \mathrm{st}}} \arrow[r, "\mathfrak{o}"] & \mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}.
\end{tikzcd}$$

Theorem 8 is then essentially an immediate consequence of the soundness of the translations $\mathfrak{t}$, Bc and $\mathfrak{o}$ and of the factorisation $\mathrm{Bt} = \mathfrak{o} \circ \mathrm{Bc} \circ \mathfrak{t}$ (Proposition 24). This recasts a previous idea [9, Section 13.4].

Definition 14. The theory $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$ is obtained by adding to $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the following.

(1) A fresh formula Φ_{c}, called the *copy* of Φ, for each formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, which we declare to be internal if and only if Φ is internal, and atomic if and only if Φ is atomic.

(2) The axioms and rules of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ extended to the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$. For example, for an internal formula φ_{c}, the internal rule of induction now holds.

(3) The axioms $(\Phi_1)_{\mathrm{c}} \to \cdots \to (\Phi_n)_{\mathrm{c}}$ for each theorem $\Phi_1 \to \cdots \to \Phi_n$ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$. Note that the implication associates to the right so, for example, $\Phi_1 \to \Phi_2 \to \Phi_3$ means $\Phi_1 \to (\Phi_2 \to \Phi_3)$. Note also that n can be equal to 1, so if Φ is a theorem of $\mathsf{E\text{-}HA}^{\omega}$, then Φ_{c} is an axiom of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$.

(4) The axiom $\varphi \to \varphi_{\mathrm{c}}$, for each internal formula φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$.

(5) The free variables $\mathrm{FV}(\cdot)$ are extended to the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$ by $\mathrm{FV}(\Phi_{\mathrm{c}}) := \mathrm{FV}(\Phi)$.

(6) We say that a term t is free for a variable x in Φ_{c} if and only if t is free for x in Φ.

(7) Substitution $[\cdot/\cdot]$ is extended to the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$ by $\Phi_{\mathrm{c}}[t/x] :\equiv \Phi[t/x]_{\mathrm{c}}$.

We have to define the translations $\mathfrak{t}$, $\mathfrak{o}$ and the functional interpretation Bc, and to prove their soundness theorems. Let us start with $\mathfrak{t}$.

Definition 15. The translation $\mathfrak{t}$ assigns to each formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the formula $\Phi^{\mathfrak{t}}$ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}^{\mathrm{c}}}$ according to the following clauses:

$$
\begin{aligned}
(s =_0 t)^{\mathfrak{t}} &:\equiv (s =_0 t), \\
\mathrm{st}(t)^{\mathfrak{t}} &:\equiv \mathrm{st}(t), \\
(\Phi \wedge \Psi)^{\mathfrak{t}} &:\equiv (\Phi^{\mathfrak{t}} \wedge \Psi^{\mathfrak{t}}), \\
(\Phi \vee \Psi)^{\mathfrak{t}} &:\equiv (\Phi^{\mathfrak{t}} \vee \Psi^{\mathfrak{t}}), \\
(\Phi \to \Psi)^{\mathfrak{t}} &:\equiv (\Phi^{\mathfrak{t}} \to \Psi^{\mathfrak{t}}) \wedge (\Phi \to \Psi)_{\mathrm{c}}, \\
(\exists x\, \Phi)^{\mathfrak{t}} &:\equiv \exists x\, \Phi^{\mathfrak{t}}, \\
(\forall x\, \Phi)^{\mathfrak{t}} &:\equiv \forall x\, \Phi^{\mathfrak{t}} \wedge (\forall x\, \Phi)_{\mathrm{c}}.
\end{aligned}
$$

It should be clear from the previous definition that, for all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, we have $\mathrm{FV}(\Phi^{\mathfrak{t}}) = \mathrm{FV}(\Phi)$. Moreover, for all terms t, variables x and formulas Φ, t is free for x in $\Phi^{\mathfrak{t}}$ if and only if t is free for x in Φ, and we have $\Phi[t/x]^{\mathfrak{t}} = \Phi^{\mathfrak{t}}[t/x]$.

The translation $\mathfrak{t}$ has a sort of "truth property" since the translated formulas $\Phi^{\mathfrak{t}}$ imply the copies Φ_{c} of the original formulas, instead of implying the original formulas Φ (as in a proper truth property).

Proposition 16 ("Truth property"). *For all formulas* Φ *of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}{}^{\mathrm{c}} \vdash \Phi^{\mathrm{t}} \to \Phi_{\mathrm{c}}$.

Proof. The proof is by induction on the length of Φ. We present only the case of conjunction. By induction hypothesis, we assume (IH_1) $\Phi^{\mathrm{t}} \to \Phi_{\mathrm{c}}$ and (IH_2) $\Psi^{\mathrm{t}} \to \Psi_{\mathrm{c}}$. The formula $\Phi \to (\Psi \to (\Phi \wedge \Psi))$ is a theorem of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, so $\Phi_{\mathrm{c}} \to (\Psi_{\mathrm{c}} \to (\Phi \wedge \Psi)_{\mathrm{c}})$ is an axiom of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}{}^{\mathrm{c}}$. Hence $\Phi_{\mathrm{c}} \wedge \Psi_{\mathrm{c}} \to (\Phi \wedge \Psi)_{\mathrm{c}}$. Using this together with (IH_1) and (IH_2), we derive

$$\begin{aligned}&(\Phi \wedge \Psi)^{\mathrm{t}} \equiv \\ &\Phi^{\mathrm{t}} \wedge \Psi^{\mathrm{t}} \to \\ &\Phi_{\mathrm{c}} \wedge \Psi_{\mathrm{c}} \to \\ &(\Phi \wedge \Psi)_{\mathrm{c}}.\end{aligned}$$

The remaining cases are either immediate or similar. ∎

Theorem 17 (Soundness theorem of t). *For all formulas* Φ *of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi$, *then* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}{}^{\mathrm{c}} \vdash \Phi^{\mathrm{t}}$.

Proof. The proof is by induction on the length of derivations.

$\underline{\Phi \wedge \Psi \to \Psi \wedge \Phi}$ We have

$$(\Phi \wedge \Psi \to \Psi \wedge \Phi)^{\mathrm{t}} \equiv (\Phi^{\mathrm{t}} \wedge \Psi^{\mathrm{t}} \to \Psi^{\mathrm{t}} \wedge \Phi^{\mathrm{t}}) \wedge (\Phi \wedge \Psi \to \Psi \wedge \Phi)_{\mathrm{c}}.$$

The formula $\Phi \wedge \Psi \to \Psi \wedge \Phi$ is a theorem, so $(\Phi \wedge \Psi \to \Psi \wedge \Phi)_{\mathrm{c}}$ is an axiom and the result follows. Analogously for $\Phi \vee \Phi \to \Phi$, $\Phi \to \Phi \vee \Psi$, $\Phi \to \Phi \wedge \Phi$, $\Phi \wedge \Psi \to \Phi$, $\Phi \vee \Psi \to \Psi \vee \Phi$ and $\bot \to \Phi$.

$\underline{\forall x\, \Phi \to \Phi[t/x]}$ We have

$$(\forall x\, \Phi \to \Phi[t/x])^{\mathrm{t}} \equiv (\forall x\, \Phi^{\mathrm{t}} \wedge (\forall x\, \Phi)_{\mathrm{c}} \to \Phi[t/x]^{\mathrm{t}}) \wedge (\forall x\, \Phi \to \Phi[t/x])_{\mathrm{c}}.$$

Here we use the fact that $\Phi^{\mathrm{t}}[t/x] \equiv \Phi[t/x]^{\mathrm{t}}$ and that t is free for x in Φ if and only if t is free for x in Φ^{t}. Analogously for $\Phi[t/x] \to \exists x\, \Phi$.

$\underline{\Phi \wedge \Psi \to \Omega \;/\; \Phi \to (\Psi \to \Omega)}$ We have

$$\begin{aligned}(\Phi \wedge \Psi \to \Omega)^{\mathrm{t}} &\equiv (\Phi^{\mathrm{t}} \wedge \Psi^{\mathrm{t}} \to \Omega^{\mathrm{t}}) \wedge (\Phi \wedge \Psi \to \Omega)_{\mathrm{c}}, \\ (\Phi \to (\Psi \to \Omega))^{\mathrm{t}} &\equiv (\Phi^{\mathrm{t}} \to (\Psi^{\mathrm{t}} \to \Omega^{\mathrm{t}}) \wedge (\Psi \to \Omega)_{\mathrm{c}}) \wedge (\Phi \to (\Psi \to \Omega))_{\mathrm{c}}.\end{aligned}$$

The formula $\Phi^{\mathrm{t}} \wedge \Psi^{\mathrm{t}} \to \Omega^{\mathrm{t}}$ is a theorem (because it follows from the t-translation of the premise of the rule), so (1) $\Phi^{\mathrm{t}} \to (\Psi^{\mathrm{t}} \to \Omega^{\mathrm{t}})$ is a theorem. By Proposition 16, we have (2) $\Phi^{\mathrm{t}} \to \Phi_{\mathrm{c}}$. The formula $\Phi \to (\Psi \to \Omega)$ is a theorem, so (3) $\Phi_{\mathrm{c}} \to (\Psi \to \Omega)_{\mathrm{c}}$

and (4) $(\Phi \to (\Psi \to \Omega))_{\mathrm{c}}$ are axioms. From (1), (2), (3) and (4), we get the t-translation of the conclusion of the rule. Analogously for $\frac{\Phi \quad \Phi\to\Psi}{\Psi}$, $\frac{\Phi\to\Psi \quad \Psi\to\Omega}{\Phi\to\Omega}$, $\frac{\Phi\to(\Psi\to\Omega)}{\Phi\wedge\Psi\to\Omega}$, $\frac{\Phi\to\Psi}{\Phi\vee\Omega\to\Psi\vee\Omega}$, $\frac{\Phi\to\Psi}{\Phi\to\forall x\,\Psi}$ and $\frac{\Phi\to\Psi}{\exists x\,\Phi\to\Psi}$.

<u>External induction</u> The translation of the rule is

$$\frac{\Phi^{\mathrm{t}}[0/w] \quad (\mathrm{st}(x) \wedge \Phi^{\mathrm{t}}[x/w] \to \Phi^{\mathrm{t}}[\mathrm{S}x/w]) \wedge (\mathrm{st}(x) \wedge \Phi[x/w] \to \Phi[\mathrm{S}x/w])_{\mathrm{c}}}{(\mathrm{st}(x) \to \Phi^{\mathrm{t}}[x/w]) \wedge (\mathrm{st}(x) \to \Phi[x/w])_{\mathrm{c}}}.$$

Although the formula Φ^{t} may contain copies, it is an instance of the rule in $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ because in $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ the rule was extended to the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{c}\,\mathrm{st}}$.

The remaining cases are trivial in the sense that either the axiom is an internal formula or the translation of the axiom is essentially an instance of the same axiom-scheme. ∎

Now we present the translation $\mathfrak{o}$ and prove its soundness theorem.

Definition 18. The translation $\mathfrak{o}$ assigns to each formula Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{c}\,\mathrm{st}}$ the formula $\Phi^{\mathfrak{o}}$ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ according to the following clauses:

$$\begin{aligned}
(\Phi_{\mathrm{c}})^{\mathfrak{o}} &:\equiv \Phi,\\
(s =_0 t)^{\mathfrak{o}} &:\equiv (s =_0 t),\\
\mathrm{st}(t)^{\mathfrak{o}} &:\equiv \mathrm{st}(t),\\
(\Phi \wedge \Psi)^{\mathfrak{o}} &:\equiv \Phi^{\mathfrak{o}} \wedge \Psi^{\mathfrak{o}},\\
(\Phi \vee \Psi)^{\mathfrak{o}} &:\equiv \Phi^{\mathfrak{o}} \vee \Psi^{\mathfrak{o}},\\
(\Phi \to \Psi)^{\mathfrak{o}} &:\equiv \Phi^{\mathfrak{o}} \to \Psi^{\mathfrak{o}},\\
(\forall x\, \Phi)^{\mathfrak{o}} &:\equiv \forall x\, \Phi^{\mathfrak{o}},\\
(\exists x\, \Phi)^{\mathfrak{o}} &:\equiv \exists x\, \Phi^{\mathfrak{o}}.
\end{aligned}$$

Theorem 19 (Soundness theorem of $\mathfrak{o}$). *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{c}\,\mathrm{st}}$, if $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{c}\,\mathrm{st}} \vdash \Phi$ then $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi^{\mathfrak{o}}$.*

Proof. The assertion is proved by a simple induction on the length of the derivation. Observe that for the nonstandardness axioms the result is trivial since for those axioms we have $\Phi \equiv \Phi^{\mathfrak{o}}$. ∎

Proposition 20. *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash (\Phi^{\mathrm{t}})^{\mathfrak{o}} \leftrightarrow \Phi.$$

Proof. The proof is by induction on the length of Φ.

$\underline{\mathrm{st}(t)}$ We have

$$\begin{aligned} (\mathrm{st}(t)^{\mathfrak{t}})^{\mathfrak{o}} &\equiv && \text{(Definition 15)} \\ \mathrm{st}(t)^{\mathfrak{o}} &\equiv && \text{(Definition 18)} \\ \mathrm{st}(t). \end{aligned}$$

$\underline{\Phi \to \Phi}$ We have

$$\begin{aligned} ((\Phi \to \Psi)^{\mathfrak{t}})^{\mathfrak{o}} &\equiv && \text{(Definition 15)} \\ ((\Phi^{\mathfrak{t}} \to \Psi^{\mathfrak{t}}) \wedge (\Phi \to \Psi)_{\mathrm{c}})^{\mathfrak{o}} &\equiv && \text{(Definition 18)} \\ ((\Phi^{\mathfrak{t}})^{\mathfrak{o}} \to (\Psi^{\mathfrak{t}})^{\mathfrak{o}}) \wedge (\Phi \to \Psi) &\leftrightarrow && \text{(IH)} \\ (\Phi \to \Psi) \wedge (\Phi \to \Psi) &\leftrightarrow && \text{(logic)} \\ \Phi \to \Psi. \end{aligned}$$

$\underline{\forall x\, \Phi}$ We have

$$\begin{aligned} ((\forall x\, \Phi)^{\mathfrak{t}})^{\mathfrak{o}} &\equiv && \text{(Definition 15)} \\ ((\forall x\, \Phi^{\mathfrak{t}}) \wedge (\forall x\, \Phi)_{\mathrm{c}})^{\mathfrak{o}} &\equiv && \text{(Definition 18)} \\ \forall x\, (\Phi^{\mathfrak{t}})^{\mathfrak{o}} \wedge \forall x\, \Phi &\leftrightarrow && \text{(IH)} \\ \forall x\, \Phi \wedge \forall x\, \Phi &\leftrightarrow && \text{(logic)} \\ \forall x\, \Phi. \end{aligned}$$

The other cases are analogous. ∎

Now we present the functional interpretation Bc and prove its soundness theorem.

Definition 21. The *intuitionistic nonstandard bounded functional interpretation* Bc extended to copies Φ_{c} interprets formulas in the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}_{\mathrm{c}}}$ into formulas of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}_{\mathrm{c}}}$ in the same way that the interpretation B interprets formulas of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ into formulas of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, except that for formulas of the form Φ_{c} we define

$$(\Phi_{\mathrm{c}})^{\mathrm{Bc}} \equiv (\Phi_{\mathrm{c}})_{\mathrm{Bc}} :\equiv \Phi_{\mathrm{c}}.$$

Proposition 22. *For all formulas Φ of the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}_{\mathrm{c}}}$, we have that if $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}_{\mathrm{c}}} \vdash \Phi$, then there are closed (and therefore standard) and monotone terms $\underline{t}$ such that $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}_{\mathrm{c}}} \vdash \widetilde{\forall}^{\mathrm{st}}\underline{b}\; \Phi_{\mathrm{Bc}}(\underline{t};\underline{b})$.*

Proof. The proof is the soundness theorem of B plus the interpretation of the new axioms $(\Phi_1)_{\mathrm{c}} \to \cdots \to (\Phi_n)_{\mathrm{c}}$ by Bc, which is the theorem $\Phi_1 \to \cdots \to \Phi_n$ of $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$. ∎

Lemma 23. *For all formulas Φ of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have* $\Phi^{\mathfrak{o}} \equiv \Phi$.

Proof. The proof is a simple induction on the length of the formula Φ in which each case follows immediately from Definition 18 and the induction hypothesis. ∎

Proposition 24. *For all formulas Φ of the language of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

(1) $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash ((\Phi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{a};\underline{b}))^{\mathfrak{o}} \equiv \Phi_{\mathrm{Bt}}(\underline{a};\underline{b})$;

(2) $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash ((\Phi^{\mathfrak{t}})^{\mathrm{Bc}})^{\mathfrak{o}} \equiv \Phi^{\mathrm{Bt}}$.

So, Bt *is sound because* $\mathfrak{t}$, Bc *and* $\mathfrak{o}$ *are sound.*

Proof. We focus on the proof of the first item because the second item is an immediate consequence. The proof is by induction on the length of the formula Φ. The cases are all similar so we just present the case of implication.

$$\begin{aligned}
\left(((\Phi \to \Psi)^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{C},\underline{B};\underline{a},\underline{d})\right)^{\mathfrak{o}} \equiv \quad & \text{(Definition 15)}\\
\left(((\Phi^{\mathfrak{t}} \to \Psi^{\mathfrak{t}}) \wedge (\Phi \to \Psi)_{\mathrm{c}})_{\mathrm{Bc}}(\underline{C},\underline{B};\underline{a},\underline{d})\right)^{\mathfrak{o}} \equiv \quad & \text{(Definition 21)}\\
((\Phi^{\mathfrak{t}} \to \Psi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{C},\underline{B};\underline{a},\underline{d}) \wedge ((\Phi \to \Psi)_{\mathrm{c}})_{\mathrm{Bc}}(;))^{\mathfrak{o}} \equiv \quad & \text{(Definition 21)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{B}\underline{a}\underline{d}\, (\Phi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{a};\underline{b}) \to (\Psi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{C}\underline{a};\underline{d})) \wedge ((\Phi \to \Psi)_{\mathrm{c}})^{\mathfrak{o}} \equiv \quad & \text{(Definition 18, Lemma 23)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{B}\underline{a}\underline{d}\, ((\Phi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{a};\underline{b}))^{\mathfrak{o}} \to ((\Psi^{\mathfrak{t}})_{\mathrm{Bc}}(\underline{C}\underline{a};\underline{d}))^{\mathfrak{o}} \wedge (\Phi \to \Psi) \equiv \quad & \text{(IH)}\\
(\tilde{\forall} \underline{b} \leq^* \underline{B}\underline{a}\underline{d}\, \Phi_{\mathrm{Bt}}(\underline{a};\underline{b}) \to \Psi_{\mathrm{Bt}}(\underline{C}\underline{a};\underline{d})) \wedge (\Phi \to \Psi) \equiv \quad & \text{(Definition 5)}\\
(\Phi \to \Psi)_{\mathrm{Bt}}(\underline{C},\underline{B};\underline{a},\underline{d}) \wedge (\Phi \to \Psi) \equiv \quad & \text{(Proposition 6)}\\
(\Phi \to \Psi)_{\mathrm{Bt}}(\underline{C},\underline{B};\underline{a},\underline{d}). \quad &
\end{aligned}$$

∎

8. Conclusion and future work

We introduced variants with t-truth and q-truth for the intuitionistic nonstandard bounded functional interpretation of the nonstandard extensional Heyting arithmetic. We presented four different approaches to prove its soundness theorem, the "standard" way, one using a detour through q-truth, one relying on the copies-only method and finally a method relying on the translations $\mathfrak{t}$ and $\mathfrak{o}$. For the reasons mentioned in the introduction we are convinced that these methods may be adapted without difficulty to prove the soundness of variants with truth for other functional interpretations. Let us briefly comment on why each of the new methods may have advantages.

(1) The method explored in Section 5 is useful because it is easier to hardwire q-truth than t-truth.

(2) The copies-only method developed in Section 6 has the obvious advantage of avoiding a lot of bureaucracy, allowing one to focus only on the copies instead of the whole interpreted formula.

(3) The method presented in Section 7 can be seen as a theoretical clarification of the fact that one can deal with interpretations with truth as blocs of independent processes. In this way, one can see this method as a sort of formalisation of the copies-only method.

Finally, let us say a few words concerning future work. It is well known that a characterisation theorem for interpretations with truth is a sort of "unobtainable goal". In spite of that we can maybe nevertheless separate certain principles. So we would like to pay attention to the characteristic principles of the version without truth and see if they are interpretable in the version with truth, as these give some idea of how powerful a theory is.

Recently, in [4] a parametrised interpretation for Heyting arithmetic was introduced. It is claimed that this parametrised interpretation should, in principle, be able to also deal with interpretations with truth. We believe that this is worth exploring as it might shed some extra light concerning the relation with linear logic, in analogy with what is done in [10].

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