Port. Math. 80 (2023), 107–132
DOI 10.4171/PM/2097

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Existence and boundary asymptotic behavior of strictly convex solutions for singular Monge–Ampère problems with gradient terms

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Abstract. In this paper, we study the existence as well as the boundary asymptotic behavior of strictly convex solutions for singular Monge–Ampère problems with gradient terms. The standard tools are Karamata regular variation theory and the sub-super-solution method. In order to apply these methods, we need to know the properties of the weight function b and the nonlinear term f. We find new structure conditions on b and f to overcome the difficulties due to the singularity of b and the gradient terms.

1. Introduction

Let Ω be a strictly convex, bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$. We consider the following singular Monge–Ampère problems:

$$\begin{cases} \det(D^2u) = b(x)f(-u) + |Du|^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \det(D^2u) = b(x)f(-u)(1 + |Du|^q) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\det(D^2u)$ is a Monge–Ampère operator, $0 < q < n$, $b \in C^\infty(\Omega)$ is a positive weight function in Ω, and $f \in C^\infty(0, +\infty)$ is positive and nonincreasing.

The Monge–Ampère equation is a class of fully nonlinear partial differential equation, which arises from fluid mechanics, geometric problems and other scientific fields. In the past years, there is an extensive research devoted to the study of Monge–Ampère equations by different methods, see [5, 7, 9, 11, 14, 19, 21, 29, 31, 33, 37, 38, 40–44] and the references therein.

2020 Mathematics Subject Classification. Primary 35J66; Secondary 35J60, 35J96.
Keywords. Singular Monge–Ampère equations, gradient terms, strictly convex solutions, existence and boundary asymptotic behavior.

Moreover, we notice that a type of singular elliptic boundary value problem has received much attention, for example, see previous studies [8,13,23,24,26,39] and the references cited therein. In [30], the existence question for Monge–Ampère problem

$$\begin{cases} \det(D^2u) = u^{-(n+2)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

was studied by Loewner and Nirenberg when $n = 2$. Cheng and Yau [6] considered problem (1.3) in the case of $n \geq 2$ and also obtained existence results.

Using the sub-super-solution method, Lazer and McKenna [27] proved that the Monge–Ampère problem

$$\begin{cases} \det(D^2u) = b(x)u^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

admits a unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, where $\gamma > 1$ and $b \in C^\infty(\bar{\Omega})$ is positive. Besides, they proved that there exist two negative constants c_1 and c_2 such that u satisfies

$$c_1 d^\beta(x) \leq u \leq c_2 d^\beta(x) \quad \text{in } \Omega,$$

where $\beta = \frac{n+1}{n+\gamma}$ and $d(x) = \text{dist}(x, \partial\Omega)$.

In [32], Mohammed considered a more general Monge–Ampère problem

$$\begin{cases} \det(D^2u) = b(x)f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

and proved the existence of a solution when b satisfies

$$b(x) \leq Cd(x)^{\delta-n-1} \tag{1.6}$$

for some positive constants C and $0 < \delta < n - 1$. Note that (1.6) allows $b(x)$ to be singular near $\partial\Omega$.

Recently, Sun and Feng [14] (where k-Hessian equation was studied, it reduces to the Monge–Ampère equation when $k = n$), Li and Ma [28] applied the Karamata regular variation theory and the sub-super-solution method to analyze the boundary asymptotic behavior of convex solutions of (1.5). They covered a more general weight function b by using the following condition:

(**b**) there exist constants $C_2 > C_1 > 0$ such that

$$C_1\theta^{n+1}(d(x)) \leq b(x) \leq C_2\theta^{n+1}(d(x)) \quad \text{near } \partial\Omega, \tag{1.7}$$

for some $\theta(t) \in C^1(0, a)$ satisfying

$$\lim_{t\to 0+} \left(\frac{\Theta(t)}{\theta(t)}\right)' = D_\theta \in [0, +\infty),$$

where $\Theta(t) = \int_0^t \theta(s)ds$.

Remark 1.1. It is clear that $\theta(t) = t^\alpha$ satisfies

$$\lim_{t\to 0^+}\left(\frac{\Theta(t)}{\theta(t)}\right)' = \frac{1}{1+\alpha}.$$

When $\det(D^2u)$ is replaced by Δu or the p-Laplacian $\Delta_p u$ in (1.1) and (1.2), similar questions involving the gradient terms $|Du|$ have been extensively studied. For example, Brezis and Turner [4] studied the existence of positive solutions for the following problem:

$$\begin{cases} Lu = g(x,u,Du), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where L is a linear elliptic operator. Ghergu and Rădulescu [18] established several results related to existence, nonexistence or bifurcation of positive solutions for the boundary value problem

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) - |Du|^a, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $K \in C^{0,\gamma}(\bar{\Omega})$, $0 < \gamma < 1$, $0 < a \le 2$. Dupaigne, Ghergu and Rădulescu [10] considered

$$\begin{cases} -\Delta u \pm p(d(x))g(u) = \lambda f(x,u) + \mu|Du|^a, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $p(d(x))$ is a positive weight with possible singular behavior on the boundary of Ω. For general p-Laplacian problems with gradient terms, please see [1, 2, 16, 17, 34, 36]. Generally, the variational method cannot be directly applied because the gradient terms usually destroy the variational structure. The existing methods mainly involve the topological degree and sub-super-solution argument.

In [12], Feng, Sun, and Zhang studied the existence and boundary behavior for Monge–Ampère equations with nonlinear gradient terms of the form

$$\begin{cases} \det(D^2u) = b(x)f(-u) + g(|Du|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \det(D^2u) = b(x)f(-u)(1 + g(|Du|)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $b \in C^\infty(\bar{\Omega})$ is positive in Ω and satisfies **(b)** with $D_\theta \in [0,1]$, f is positive, decreasing on $(0,+\infty)$, $\lim_{s\to 0^+} f(s) = \infty$, and there exist positive d and c_1 such that $f(u) < \frac{c_1}{u^d}$, $g \in C^\infty(0,\infty)$ is positive and nondecreasing on $(0,\infty)$, and there exist a constant c_g and $0 \le q < n$ such that $g(x) \le c_g x^q$. They obtained interesting results on the existence and boundary asymptotic behavior of strictly convex solutions.

In this paper, we focus on the case $g(|Du|) = |Du|^q$ $(0 \le q < n)$ in (1.8) and (1.9) but allow much more general behaviors of $b(x)$ and f. We will construct some new functions with special structure to overcome the difficulties due to singularity of b and the gradient terms. Even in the case $b \in C^\infty(\bar{\Omega})$ we get a structure condition better than that in [12]. In fact, in this paper we will use a new technique to study problem (1.1) and (1.2), which is completely different from that used in [12]. Especially, the condition on b is weaker than **(b)**, see Remark 1.2 for detail.

We will apply the sub-super-solution method and Karamata regular variation theory to show the existence and boundary asymptotic behavior of solutions for problems (1.1) and (1.2).

Firstly, we suppose that f satisfies

(f) $f : (0, \infty) \to (0, \infty)$ is of class C^∞ and nonincreasing.

For the study of the existence and the boundary asymptotic behavior of solutions, let us introduce a function Φ defined by

$$\Phi(t) = \int_0^t [(n + 1 - \mu)F(s)]^{-\frac{1}{n+1-\mu}} ds, \quad t \in [0, a], \tag{1.10}$$

where

$$\mu = \begin{cases} 0, & \text{corresponding to problem (1.1)}, \\ q, & \text{corresponding to problem (1.2)}, \end{cases} \tag{1.11}$$

$$F(t) = \int_t^a f(s)ds, \tag{1.12}$$

and a is a positive constant. Since $\Phi' > 0$, the inverse function exists. Let ϕ be the inverse of Φ.

Next we discuss the conditions on b. To achieve this goal, we first introduce a few notations.

Let

$$P(\tau) = \int_\tau^1 p(t)dt,$$

where $p \in C^1(0, \infty)$ is a positive function satisfying $p'(t) < 0$ and $\lim_{t \to 0^+} p(t) = \infty$. Such a function p is said to be of class $\mathcal{P}_{\text{finite}}$ if

$$\int_{0+} [P(\tau)]^{\frac{1}{n-\mu}} d\tau < \infty, \tag{1.13}$$

where μ is defined in (1.11).

We suppose that b satisfies

($\mathbf{b}_1$) $b \in C^\infty(\overline{\Omega})$ is positive in $\overline{\Omega}$, or

($\mathbf{b}_2$) $b \in C^\infty(\Omega)$ is positive in Ω, singular near $\partial\Omega$, and there exists a function p of class $\mathscr{P}_{\text{finite}}$ such that

$$k_2 p(d(x)) \le b(x) \le k_1 p(d(x))$$

near $\partial\Omega$, where $k_1 > k_2 > 0$.

Remark 1.2. Condition ($\mathbf{b}_2$) here is weaker than [28, 32] (where $q = 0$), where $p(t)$ was required to satisfy

$$\int_{0+} [p(\tau)]^{\frac{1}{n+1}} d\tau < \infty. \tag{1.14}$$

For example, letting $p(t) = t^{-n-1}(-\ln t)^{-\beta}, 0 < t < t_0 < 1, \beta \in (n, n+1]$, then p satisfies

$$\int_{0+} [P(\tau)]^{\frac{1}{n}} d\tau < \infty,$$

but

$$\int_{0+} [p(\tau)]^{\frac{1}{n+1}} d\tau = \infty.$$

The existence results are as follows.

Theorem 1.1. *Let* (**f**) *and* ($\mathbf{b}_1$) *hold. Then problem* (1.1) *(or* (1.2)*) admits a strictly convex solution.*

Theorem 1.2. *Let* (**f**) *and* ($\mathbf{b}_2$) *hold. Then problem* (1.1) *(or* (1.2)*) admits a strictly convex solution.*

The next theorem provides an asymptotic behavior of strictly convex solutions close to the boundary. To state the theorem, we need to introduce two constants I_0 and J_0.

Let

$$I(s) = \frac{\Phi''(s)\Phi(s)}{(\Phi'(s))^2}. \tag{1.15}$$

Suppose that $\lim_{s\to 0^+} I(s)$ exists and is denoted by I_0. Set

$$\omega(t) = \int_0^t [(n-\mu)P(s)]^{\frac{1}{n-\mu}} ds, \tag{1.16}$$

and

$$J(s) = -\frac{\omega(s)\omega''(s)}{(\omega'(s))^2}. \tag{1.17}$$

Suppose that $\lim_{s\to 0^+} J(s)$ exists and is denoted by J_0.

Theorem 1.3. *Suppose that f satisfies* (**f**)*,* $\lim_{s\to 0^+} f(s) = \infty$ *and such that* I_0 *is well defined and* $I_0 \neq 0$*. Suppose that* b *satisfies* ($\mathbf{b}_2$) *and such that* J_0 *exists and* $J_0 \neq 0$*. Then there exist* ξ_1, ξ_2 *such that for any strictly convex solution* u *to problem* (1.1) *it holds*

$$-\phi\big(\xi_1[\omega(d(x))]^{\frac{n}{n+1}}\big) \le u(x) \le -\phi\big(\xi_2[\omega(d(x))]^{\frac{n}{n+1}}\big) \quad \text{near } \partial\Omega. \tag{1.18}$$

And there exist η_1, η_2 *such that for any strictly convex solution* u *to problem* (1.2) *it holds*

$$-\phi\big(\eta_1[\omega(d(x))]^{\frac{n-q}{n+1-q}}\big) \le u(x) \le -\phi\big(\eta_2[\omega(d(x))]^{\frac{n-q}{n+1-q}}\big) \quad \text{near } \partial\Omega. \tag{1.19}$$

Corollary 1.1. *In Theorem 1.3, if* Ω *is a ball of radius* R*, and* $\lim_{d(x)\to 0} \frac{b(x)}{p(d(x))} = \bar{k}$*, then there exists* ξ_0 *such that for any strictly convex solution* u *to problem* (1.1)*, it holds*

$$\lim_{\substack{x\in\Omega,\\ d(x)\to 0}} \frac{u(x)}{\phi[(\omega(d(x)))^{\frac{n}{n+1}}]} = \xi_0^{1-I_0}. \tag{1.20}$$

And there exists η_0 *such that for any strictly convex solution* u *to problem* (1.2)*, it holds*

$$\lim_{\substack{x\in\Omega,\\ d(x)\to 0}} \frac{u(x)}{\phi[(\omega(d(x)))^{\frac{n-q}{n+1-q}}]} = \eta_0^{1-I_0}. \tag{1.21}$$

Remark 1.3. Comparing with the previous articles, such as [12], the main features of this paper are as follows.

(i) Our condition on f is weaker than that of [12], no assumption of the form $f(u) < \frac{c_1}{u^d}$ for $u \in (0, +\infty)$ and for some positive constants c_1, d are assumed. Our technique depends on the construction of the function Φ in (1.10) which has a relation with a gradient term and a special structure condition (1.15). However, in [12], the author did not define the function Φ and the structure condition (1.15).

(ii) The conditions on b are also different. We study not only the case $b \in C^\infty(\bar{\Omega})$ (considered in [12]) but also the case b is singular near $\partial\Omega$. We solve the difficulty of singularity by constructing a special structure condition (1.17) which is not used in [12].

(iii) We use different methods from [12] to deal with gradient terms.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results to be used in the subsequent sections. Section 3 is devoted to proving Theorem 1.1 and Theorem 1.2, Section 4 is devoted to proving Theorem 1.3 and Corollary 1.1.

2. Preliminary results

In this section, we shall give some lemmas and definitions for the convenience of later use.

Lemma 2.1 ([7, Proposition 2.1]). *Let $u \in C^2(\Omega)$ be such that the matrix $(u_{x_i x_j})$ is invertible for $x \in \Omega$, and let h be a C^2 function defined on an interval containing the range of u. Then*

$$\det(D^2 h(u)) = \det(D^2 u)\big\{[h'(u)]^n + [h'(u)]^{n-1} h''(u)(\nabla u)^T B(u) \nabla u\big\}, \qquad (2.1)$$

where A^T denotes the transpose of the matrix A, $B(u)$ denotes the inverse of the matrix $(u_{x_i x_j})$, and

$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})^T.$$

Moreover, when $u(x) = d(x)$, we have

$$\det(D^2 h(u)) = (-h'(u))^{n-1} h''(u) \prod_{i=1}^{n-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})}, \quad x \in \Omega_{\delta_1},$$

where $\Omega_{\delta_1} = \{x \in \Omega : 0 < d(x) < \delta_1\}$, $\bar{x} \in \partial\Omega$ is the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$ and $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at $\bar{x}$.

Lemma 2.2 ([22, Lemma 2.1]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $u, v \in C^2(\Omega)$ are strictly convex. If*

(1) $\psi(x, z, p) \geq \phi(x, z, p)$, $\forall (x, z, p) \in (\Omega \times \mathbb{R} \times \mathbb{R}^n)$;

(2) $\det(D^2 u) \geq \psi(x, u, Du)$ *and* $\det(D^2 v) \leq \phi(x, v, Dv)$ *in* Ω;

(3) $u \leq v$ *on* $\partial\Omega$;

(4) $\psi_z(x, z, p) > 0$ *or* $\phi_z(x, z, p) > 0$,

then $u \leq v$ in Ω.

The following interior estimate for derivatives of smooth solutions of Monge–Ampère equations is a special case of [15, Theorem A.42].

Lemma 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega \in C^\infty$. Let $f \in C^\infty(\bar{\Omega} \times [0, \infty) \times \mathbb{R}^n)$ with $f(x, u, p) > 0$ for $(x, u, p) \in \bar{\Omega} \times [0, \infty) \times \mathbb{R}^n$. Let $u \in C^\infty(\bar{\Omega})$ be a convex solution of the Dirichlet problem*

$$\begin{cases} \det(D^2 u) = f(x, u, Du), & x \in \Omega, \\ u(x) = c \in R, & x \in \partial\Omega. \end{cases} \qquad (2.2)$$

Let Ω' be a subdomain of Ω with $\bar{\Omega}' \subset \Omega$ and $k \geq 1$ be an integer. Then there exists a positive constant C which depends only on k, a, b, bounds for the derivatives of $f(x, u, p)$ and $\operatorname{dist}(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^k(\bar{\Omega}')} \leq C.$$

The existence result below is a special case of [20, Theorem 1.3].

Lemma 2.4 ([20, Theorem 1.3]). *The equation of the form*

$$\begin{cases} \det(D^2u) = \varphi(x,u,Du), & x \in \Omega, \\ u(x) = \phi(x) \in C^\infty, & x \in \partial\Omega, \end{cases}$$

where $\varphi(x,u,p)$ is a positive C^∞ function for $x \in \bar{\Omega}$, $u \le \max\phi$, $p \in \mathbb{R}^n$ and $\phi \in C^\infty(\partial\Omega)$, admits a strictly convex solution $u \in C^\infty(\bar{\Omega})$, if there exists a subsolution $\underline{u} \in C^2(\bar{\Omega})$ such that $u = \phi$ on $\partial\Omega$ and which satisfies

$$\det(\underline{u}_{ij}) \ge \varphi(x,\underline{u},D\underline{u}), \quad \forall x \in \Omega.$$

If $\varphi_u \ge 0$, then this solution is unique.

Next we prove some properties of I_0 and J_0 by the Karamata regular variation theory, which was introduced and established by Karamata in 1930, and it is a basic tool in stochastic processes (see [3, 25, 35]).

Definition 2.1. A positive measurable function f defined on $(0,a)$, for some constant $a > 0$, is called *regularly varying at zero* with index ρ, written $f \in \mathrm{RVZ}_\rho$, if for each $\xi > 0$ and some $\rho \in R$,

$$\lim_{s\to 0^+} \frac{f(\xi s)}{f(s)} = \xi^\rho.$$

In particular, when $\rho = 0$, f is called *slowly varying at zero*.

Clearly, if $f \in \mathrm{RVZ}_\rho$, then $L(s) = \frac{f(s)}{s^\rho}$ is slowly varying at zero.

Definition 2.2. A positive measurable function f defined on $(0,a)$, for some constant $a > 0$, is called *rapidly varying at zero*,

$$\begin{aligned} &\text{if } \lim_{s\to 0^+} f(s) = \infty, && \text{and for each } \rho > 1, && \lim_{s\to 0^+} f(s)s^\rho = \infty, \\ &\text{or if } \lim_{s\to 0^+} f(s) = 0, && \text{and for each } \rho > 1, && \lim_{s\to 0^+} f(s)s^{-\rho} = 0. \end{aligned}$$

Proposition 2.1 (Representation theorem). *A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = \psi(s)\exp\Big(\int_s^{a_1} \frac{y(\tau)}{\tau}d\tau\Big), \quad s \in (0,a_1)$$

for some $0 < a_1 < a$, where the functions ψ and y are measurable and for $s \to 0^+$, $y(s) \to 0$ and $\psi(s) \to c_0$ with $c_0 > 0$.

On the other hand, for $(0,a_1)$ one says that

$$\hat{L}(s) = c_0 \exp\Big(\int_s^{a_1} \frac{y(\tau)}{\tau}d\tau\Big)$$

is normalized slowly varying at zero *and*

$$f(s) = s^{\rho}\widehat{L}(s), \quad s \in (0, a_1)$$

is normalized regularly varying at zero *with index* ρ *and write* $f \in \mathrm{NRVZ}_{\rho}$.

Proposition 2.2. *A function* $f \in \mathrm{RVZ}_{\rho}$ *belongs to* NRVZ_{ρ} *if and only if*

$$f \in C^1(0, a_1), \text{ for some } a_1 > 0 \text{ and } \lim_{s \to 0^+} \frac{sf'(s)}{f(s)} = \rho.$$

Proposition 2.3 (Asymptotic behavior). *If a function* L *is slowly varying at zero, then for* $a > 0$ *and* $t \to 0^+$,

(1) $\int_0^t s^{\rho}L(s)ds \cong (1+\rho)^{-1}t^{1+\rho}L(t)$, *for* $\rho > -1$;

(2) $\int_t^a s^{\rho}L(s)ds \cong (-1-\rho)^{-1}t^{1+\rho}L(t)$, *for* $\rho < -1$.

Lemma 2.5. *Let* f *satisfy* (**f**) *and such that* I_0 *is well defined, then* $0 \le I_0 \le 1$.

Proof. In order to estimate I_0, by integrating (1.15) from 0 to v we have

$$\begin{aligned}
\int_0^v I(s)ds &= \int_0^v \frac{\Phi''(s)\Phi(s)}{(\Phi'(s))^2}ds = \int_0^v \frac{\Phi(s)}{(\Phi'(s))^2}d\Phi'(s) \\
&= \frac{\Phi(s)}{\Phi'(s)}\bigg|_0^v - \int_0^v \frac{(\Phi'(s))^3 - 2\Phi(s)\Phi'(s)\Phi''(s)}{(\Phi'(s))^3}ds \\
&= \frac{\Phi(v)}{\Phi'(v)} - v + 2\int_0^v I(s)ds.
\end{aligned}$$

At the same time, it follows from (1.10) that $\Phi'(v) > 0$, so we have

$$0 \le \lim_{v \to 0^+} \frac{\Phi(v)}{v\Phi'(v)} = 1 - \lim_{v \to 0^+} \frac{\int_0^v I(t)ds}{v} = 1 - \lim_{v \to 0^+} I(v) = 1 - I_0,$$

i.e. $I_0 \le 1$, therefore, we get $0 \le I_0 \le 1$. ∎

Lemma 2.6. *Let* f *satisfy* (**f**) *and such that* I_0 *is well defined, we have*

(1) $I_0 \in (0, 1)$ *if and only if* $F \in \mathrm{NRVZ}_{p+1}$ *with* $p < -1$. *In this case,* $f \in \mathrm{RVZ}_p$;

(2) *if* $I_0 = 1$, *then* F *is rapidly varying at zero;*

(3) *if* $I_0 = 0$, *then* F *is slowly varying at zero.*

Proof. (1)

$$\begin{aligned}
\lim_{s \to 0^+} \frac{\frac{1}{\Phi'(s)}}{s(\frac{1}{\Phi'(s)})'} &= -\lim_{s \to 0^+} \frac{\frac{1}{\Phi'(s)}\Phi(s)}{s\frac{\Phi''(s)}{(\Phi'(s))^2}\Phi(s)} \\
&= -\frac{1}{I_0}\lim_{s \to 0^+} \frac{\frac{\Phi(s)}{\Phi'(s)}}{s}
\end{aligned}$$

$$= -\frac{1}{I_0} \lim_{s \to 0+} \frac{(\Phi'(s))^2 - \Phi(s)\Phi''(s)}{(\Phi'(s))^2}$$
$$= \frac{I_0 - 1}{I_0}. \tag{2.3}$$

By Proposition 2.2, $\frac{1}{\Phi'(s)} \in \mathrm{NRV}_{I_0/(I_0-1)}$ and $F \in \mathrm{NRVZ}_{(n+1-\mu)I_0/(I_0-1)}$ by Proposition 2.1. Denote $(n+1-\mu)I_0/(I_0-1)$ by $p+1$, then $p+1 < 0$, i.e. $p < -1$.

Now, we assume that $F \in \mathrm{NRVZ}_{p+1}$ with $p < -1$, then $\frac{1}{\Phi'(s)} \in \mathrm{NRV}_{(p+1)/(n+1-\mu)}$, and

$$\lim_{s \to 0+} \frac{s\left(\frac{1}{\Phi'(s)}\right)'}{\frac{1}{\Phi'(s)}} = \frac{p+1}{n+1-\mu}, \qquad \frac{1}{\Phi'(s)} = s^{\frac{p+1}{n+1-\mu}} \widehat{L}(s), \quad \forall s \in (0, a_1),$$

where $\widehat{L}(s)$ is normalized slowly varying at zero. By Proposition 2.3,

$$\begin{aligned}
I_0 &= -\lim_{s \to 0+} \left(\frac{1}{\Phi'(s)}\right)' \Phi(s) \\
&= -\lim_{s \to 0+} \frac{s\left(\frac{1}{\Phi'(s)}\right)'}{\frac{1}{\Phi'(s)}} \lim_{s \to 0+} \frac{\frac{1}{\Phi'(s)}}{s} \Phi(s) \\
&= -\frac{p+1}{n+1-\mu} \lim_{s \to 0+} s^{\frac{p+\mu-n}{n+1-\mu}} \widehat{L}(s) \int_0^s \tau^{-\frac{p+1}{n+1-\mu}} \widehat{L}(\tau)^{-1} d\tau \\
&= -\frac{p+1}{n+1-\mu} \lim_{s \to 0+} s^{\frac{p+\mu-n}{n+1-\mu}} \widehat{L}(s) \left(1 - \frac{p+1}{n+1-\mu}\right)^{-1} s^{1-\frac{p+1}{n+1-\mu}} \widehat{L}(s)^{-1}.
\end{aligned}$$

It follows that $I_0 = \frac{p+1}{p-n+\mu} < 1$. In this case,

$$F(s) = s^{p+1} \widehat{L}(s), \quad \forall 0 < s < a_1.$$

Taking the derivative with respect to s,

$$f(s) = s^p[(p+1) + y(s)]\widehat{L}(s), \quad \forall 0 < s < a_1,$$

where $y(s) \to 0$ as $s \to 0^+$. It follows from Definition 2.1 that $f \in \mathrm{RVZ}_p$.

(2) If $I_0 = 1$, by (2.3) we have

$$\lim_{s \to 0+} \frac{sF'(s)}{F(s)} = -\infty.$$

Then, for an arbitrary $M > 1$, there exists $l = l(M) > 0$ small enough such that

$$\frac{F'(s)}{F(s)} < -\frac{M+1}{s}, \quad \forall 0 < s < l.$$

Integrating the above inequality with respect to s, we obtain

$$\ln F(l) - \ln F(s) < -(M+1)(\ln l - \ln s), \quad \forall 0 < s < l.$$

Therefore, we get

$$\frac{F(l)}{F(s)} < \left(\frac{l}{s}\right)^{-(M+1)}, \quad \forall 0 < s < l,$$

i.e.

$$F(s)s^M > \frac{F(l)l^{M+1}}{s}, \quad \forall 0 < s < l.$$

Let $s \to 0^+$ and then we see that F is rapidly varying at zero by Definition 2.2.

(3) If $I_0 = 0$, we see

$$\lim_{s\to 0^+} \frac{s\left(\frac{1}{\Phi'(s)}\right)'}{\frac{1}{\Phi'(s)}} = 0.$$

Let

$$\frac{s\left(\frac{1}{\Phi'(s)}\right)'}{\frac{1}{\Phi'(s)}} := y(s), \quad \forall s > 0,$$

i.e.

$$\frac{\left(\frac{1}{\Phi'(s)}\right)'}{\frac{1}{\Phi'(s)}} := \frac{y(s)}{s}, \quad \forall s > 0. \tag{2.4}$$

Integrating (2.4) from s to a_1, we have

$$\frac{1}{\Phi'(s)} = c_0 \exp\left(-\int_s^{a_1} \frac{y(\tau)}{\tau} d\tau\right), \quad s \in (0, a_1),$$

where $c_0 = \frac{1}{\Phi'(a_1)}$.

By $\lim_{s\to 0^+} y(s) = 0$, we have that for each $\varepsilon > 1$,

$$\frac{\frac{1}{\Phi'(\varepsilon s)}}{\frac{1}{\Phi'(s)}} = \exp\left(\int_s^{\varepsilon s} \frac{y(\tau)}{\tau} d\tau\right) = \exp\left(\int_1^{\varepsilon} \frac{y(sv)}{v} dv\right) \to 1 \quad \text{as } s \to 0^+.$$

Then $\frac{1}{\Phi'(s)}$ is slowly varying at zero, it follows that F is slowly varying at zero. ■

Lemma 2.7. *Let $p \in \mathcal{P}_{\text{finite}}$ be defined as in Section 1 and such that J_0 is well defined. We have*

(1) *$J_0 \geq 0$;*

(2) *$J_0 \in (0, \infty)$ if and only if $P \in \mathrm{NRVZ}_{r+1}$ with $-1 > r > -n - 1 + \mu$, where $r + 1 = -(n-\mu)J_0/(J_0+1)$. In this case, $p \in \mathrm{RVZ}_r$.*

(3) *If $J_0 = 0$, then P is slowing varying at zero.*

(4) *If $J_0 = \infty$, then $P \in \mathrm{NRVZ}_{-n+\mu}$. In this case, $p \in \mathrm{RVZ}_{-n+\mu-1}$.*

Proof. The proof is similar to Lemma 2.6. So we omit it. ■

3. Proof of Theorem 1.1 and Theorem 1.2

In this section, we will give the proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. By [5, Theorem 1.1], there exists $u_0 \in C^\infty(\bar{\Omega})$, which is the unique strictly convex solution to

$$\det(D^2 u_0) = 1 \quad \text{in } \Omega, \qquad u_0 = 1 \quad \text{on } \partial\Omega.$$

Let $z(x) := 1 - u_0(x)$. Then $z(x) > 0$ in Ω and it is the unique strictly concave solution to

$$(-1)^n \det(D^2 z) = 1 \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

Since ϕ is the inverse of Φ, we have

$$\phi'(t) = \big[(n+1-\mu)F(\phi(t))\big]^{\frac{1}{n+1-\mu}}$$

and

$$\phi''(t) = -\big[(n+1-\mu)F(\phi(t))\big]^{\frac{\mu-n+1}{n+1-\mu}} f(\phi(t)).$$

We hence can easily get

$$(\phi'(t))^{n-\mu-1}\phi''(t) = -f(\phi(t)) \quad \text{and} \quad \frac{\phi'(t)}{\phi''(t)} = -\frac{\big[(n+1-\mu)F(\phi(t))\big]^{\frac{n-\mu}{n+1-\mu}}}{f(\phi(t))}. \tag{3.2}$$

Let $v = -c\phi(z)$, combining with (3.1), (3.2) and Lemma 2.1, then

$$\begin{aligned}
\det(D^2 v) &= \det(D^2 z)\big[v'(z)^n + v'(z)^{n-1} v''(z)(\nabla z)^T B(z)\nabla z\big] \\
&= \det(D^2 z)\big[(-c)^n \phi'^n + (-c)^n \phi'^{n-1}\phi''(\nabla z)^T B(z)\nabla z\big] \\
&= -c^n \phi'^{n-1}\phi''\Big[-\frac{\phi'}{\phi''} - (\nabla z)^T B(z)\nabla z\Big] \\
&= c^n f(\phi(z))\phi'^{\mu}\Big[\frac{[(n+1-\mu)F(\phi(z))]^{\frac{n-\mu}{n+1-\mu}}}{f(\phi(z))} - (\nabla z)^T B(z)\nabla z\Big].
\end{aligned}$$

Let

$$\Delta = \frac{[(n+1-\mu)F(\phi(z))]^{\frac{n-\mu}{n+1-\mu}}}{f(\phi(z))} - (\nabla z)^T B(z)\nabla z.$$

Since $(z_{x_i x_j})$ is negative definite for $x \in \bar{\Omega}$, so is its inverse $B(z)$. Since $|\nabla z| > 0$ near $\partial\Omega$, we obtain

$$-(\nabla z)^T B(z)\nabla z > 0 \quad \text{for } x \in \bar{\Omega} \text{ near } \partial\Omega.$$

For $x \in \Omega$,

$$\frac{[(n+1-\mu)F(\phi(z))]^{\frac{n-\mu}{n+1-\mu}}}{f(\phi(z))} > 0$$

and it is bounded away from 0 for $x \in \Omega$ outside any neighborhood of $\partial\Omega$. Hence there exists $M_1 > 0$ depending on c such that

$$\Delta > M_1.$$

For problem (1.1), we have $\mu = 0$. It follows that

$$\begin{aligned}
&\det(D^2 v) - b(x) f(-v) - |Dv|^q \\
&\quad = c^n f(\phi(z))\Delta - b(x) f(-v) - c^q \phi'^q |Dz|^q \\
&\quad = c^q f(\phi(z))\Big[c^{n-q}\Delta - \frac{b(x)}{c^q}\frac{f(-v)}{f(-v/c)} \\
&\qquad\qquad - \frac{[(n+1-q)F(\phi(z))]^{\frac{q}{n+1-q}}}{f(\phi(z))}|Dz|^q\Big]
\end{aligned} \tag{3.3}$$

Let

$$\Delta^* = c^{n-q}\Delta - \frac{b(x)}{c^q}\frac{f(-v)}{f(-v/c)} - \frac{[(n+1-q)F(\phi(z))]^{\frac{q}{n+1-q}}}{f(\phi(z))}|Dz|^q.$$

Since f is nonincreasing, $\frac{f(-v)}{f(-v/c)} \le 1$ for large c, combing this with the fact that $b \in C(\overline{\Omega})$ is positive, we can easily get that $\Delta^* > 0$ when c is large enough, so we have

$$\det(D^2 v) > b(x) f(-v) + |Dv|^q.$$

By Lemma 2.4, it follows that problem (1.1) admits a strictly convex solution.

For problem (1.2), $\mu = q$, we have

$$\begin{aligned}
&\det(D^2 v) - b(x) f(-v)(1 + |Dv|^q) \\
&\quad = c^n f(\phi(z))\phi'^q\Delta - b(x) f(-v)(1 + c^q\phi'^q|Dz|^q) \\
&\quad = c^q f(-v)\phi'^q\Big[c^{n-q}\Delta\frac{f(-v/c)}{f(-v)} - b(x)\Big(\frac{1}{c^q\phi'^q} + |Dz|^q\Big)\Big]
\end{aligned} \tag{3.4}$$

Similar to the proof for (1.1) we can get

$$\det(D^2 v) > b(x) f(-v)(1 + |Dv|^q),$$

i.e. problem (1.2) admits a strictly convex solution. The proof of Theorem 1.1 is completed. ∎

Proof of Theorem 1.2. Since b satisfies **(b$_2$)**, it is obvious that b satisfies the condition of [38, Theorem 1.5], then there is a strictly concave function $z_0(x)$ satisfying

$$(-1)^n \det(D^2 z_0) = b(x) \quad \text{in } \Omega, \qquad z_0 = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

Let $v = -c\phi(z_0)$, combining with (3.2), (3.5) and Lemma 2.1, then

$$\begin{aligned}\det(D^2 v) &= \det(D^2 z_0)\big[v'(z_0)^n + v'(z_0)^{n-1} v''(z_0)(\nabla z_0)^T B(z_0)\nabla z_0\big]\\ &= \det(D^2 z_0)\big[(-c)^n \phi'^n + (-c)^n \phi'^{n-1}\phi''(\nabla z_0)^T B(z_0)\nabla z_0\big]\\ &= -c^n \phi'^{n-1}\phi'' b(x)\Big[-\frac{\phi'}{\phi''} - (\nabla z_0)^T B(z_0)\nabla z_0\Big]\\ &= c^n b(x) f(\phi(z_0))\phi'^{\mu}\Big[\frac{[(n+1-\mu)F(\phi(z_0))]^{\frac{n-\mu}{n+1-\mu}}}{f(\phi(z_0))}\\ &\qquad - (\nabla z_0)^T B(z_0)\nabla z_0\Big].\end{aligned}$$

We first consider problem (1.1).

Similar to the proof in Theorem 1.1 we obtain that there exists $c_1 > 0$ large enough such that

$$\det(D^2 v) > b(x) f(-v) + |Dv|^q,$$

i.e. $\underline{v} = -c_1\phi(z_0)$ is a subsolution of (1.1).

Moreover, there exists $c_2 > 0$ small enough such that

$$\det(D^2 v) < b(x) f(-v) + |Dv|^q,$$

i.e. $\overline{v} = -c_2\phi(z_0)$ is a supersolution of (1.1).

Now, we define

$$\Omega_j = \Big\{x \in \Omega : \underline{v} < -\frac{1}{j}\Big\} \quad \text{for } j = 1, 2, \ldots$$

Clearly

$$\overline{\Omega}_n \subset \Omega_{n+1}$$

and

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Consider

$$\begin{cases} \det(D^2 u) = b(x) f(-u) + |Du|^q, & x \in \Omega_j, \\ \qquad u = -\dfrac{1}{j}, & x \in \partial\Omega_j. \end{cases} \tag{3.6}$$

Combining the fact that

$$\det(D^2 \underline{v}) \geq b(x) f(-\underline{v}) + |D\underline{v}|^q \quad \text{in } \Omega,$$

and

$$\underline{v} = -\frac{1}{j} \quad \text{on } \partial\Omega_j$$

with Lemma 2.4 we get that (3.6) has a strictly convex solution u_j.

Moreover, by Lemma 2.2, we have

$$\underline{v} \le u_j \quad \text{in } \Omega_j. \tag{3.7}$$

Since

$$u_j = \underline{v} \le u_{j+1} \quad \text{on } \partial\Omega_j,$$

using Lemma 2.2 again, we get

$$u_j \le u_{j+1} \quad \text{in } \Omega_j. \tag{3.8}$$

At the same time, we observe that

$$\overline{v} = -c_2\phi(z_0) = \frac{c_2}{c_1}\underline{v} = -\frac{c_2}{c_1}\frac{1}{j} \ge -\frac{1}{j} \quad \text{on } \partial\Omega_j$$

for $\frac{c_2}{c_1} \ll 1$. Then by Lemma 2.2,

$$u_j \le \overline{v} \quad \text{in } \Omega_j \tag{3.9}$$

$\forall x \in \Omega$, we can choose a positive constant j_0 so that $x \in \Omega_{j_0}$. From (3.7)–(3.9), we have

$$\underline{v} \le u_j \le u_{j+1} \le \overline{v} \quad \text{in } \Omega_{j_0}$$

for any $j \ge j_0$. Thus,

$$u(x) = \lim_{j\to\infty} u_j(x).$$

By Lemma 2.3, we have

$$\|u_j\|_{C^\infty(\bar{\Omega})_{j_0}} \le C,$$

where C is a positive constant depending only on $n, q, b, f, \overline{v}, \underline{v}$. Hence, the convergence is uniform in every compact subset of Ω and $u \in C(\Omega)$, which implies

$$\underline{v} \le u(x) \le \overline{v}.$$

This shows that u is a strictly convex solution of (1.1).

The proof of the existence of solutions of (1.2) is similar to (1.1), so we omit it. Above all, the proof of Theorem 1.2 is finished. ∎

4. Proof of Theorem 1.3

For $\delta > 0$, we set

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}, \quad \Gamma_\delta = \{x \in \Omega : d(x) = \delta\},$$

where $d(x) = \inf_{y\in\partial\Omega} |x - y|$. When Ω is C^∞-smooth, we choose $\delta_1 > 0$ such that $d \in C^\infty(\Omega_{\delta_1})$(see [19, Lemmas 14.16 and 14.17]).

Let $\bar{x} \in \partial\Omega$ be the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_i(\bar{x})$ $(i = 1, 2, \ldots, n-1)$ be the principle curvature of $\partial\Omega$ at $\bar{x}$, we can choose a coordinate system such that

$$\begin{aligned} Dd(x) &= (0, 0, \ldots, 1), \\ D^2d(x) &= \operatorname{diag}\Big[\frac{-\kappa_1(\bar{x})}{1 - d(x)\kappa_1(\bar{x})}, \ldots, \frac{-\kappa_{n-1}(\bar{x})}{1 - d(x)\kappa_{n-1}(\bar{x})}, 0\Big]. \end{aligned}$$

Since

$$\lim_{d(x)\to 0} \prod_{i=1}^{n-1} (1 - d(x)\kappa_i(\bar{x})) = 1,$$

we can choose ε small such that for $x \in \Omega_{\delta_\varepsilon}$(where δ_ε is corresponding to ε),

$$1 - \varepsilon < \prod_{i=1}^{n-1} (1 - d(x)\kappa_i(\bar{x})) < 1 + \varepsilon.$$

For convenience, we set

$$M^* = \max_{\bar{x}\in\partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\bar{x}), \quad m^* = \min_{\bar{x}\in\partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\bar{x}). \tag{4.1}$$

Thus, we have

$$\frac{m^*}{1+\varepsilon} \le \prod_{i=1}^{n-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \le \frac{M^*}{1-\varepsilon}, \quad x \in \Omega_{\delta_\varepsilon}.$$

Proof of Theorem 1.3. We first consider problem (1.1). In this case $\mu = 0$, we get

$$\phi'(t)^{n-1}\phi''(t) = -f(\phi(t)) \quad \text{and} \quad \omega'(t)^{n-1}\omega''(t) = -p(t). \tag{4.2}$$

For an arbitrary $\varepsilon \in (0, 1/2)$, let

$$\begin{aligned} \bar{\xi}_\varepsilon &= \Big[\frac{k_1(1+2\varepsilon)}{m^*(\frac{n}{n+1})^n\big[\frac{n}{n+1}\frac{1}{J_0} + \frac{1}{n+1}\frac{1}{I_0}\frac{1}{J_0} + \frac{1}{I_0}\big]}\Big]^{\frac{1}{n+1}}, \\ \underline{\xi}_\varepsilon &= \Big[\frac{k_2(1-2\varepsilon)}{M^*(\frac{n}{n+1})^n\big[\frac{n}{n+1}\frac{1}{J_0} + \frac{1}{n+1}\frac{1}{I_0}\frac{1}{J_0} + \frac{1}{I_0}\big]}\Big]^{\frac{1}{n+1}}, \end{aligned}$$

where k_1, k_2 are given in ($\mathbf{b}_2$). By Lemma 2.5 and Lemma 2.7 we know $0 \le I_0 \le 1$, $0 \le J_0 \le +\infty$. Combining this with the condition of Theorem 1.3, we have that $\bar{\xi}_\varepsilon$ and $\underline{\xi}_\varepsilon$ are well defined.

From Lemma 2.6 and Lemma 2.7 and the definition of $\bar{\xi}_\varepsilon$ and $\underline{\xi}_\varepsilon$ we see that

$$\lim_{t\to 0+} \frac{[n\tilde{p}(t)]^{\frac{n+1}{n}}}{p(t)\int_0^t (n\tilde{p}(\tau))^{\frac{1}{n}}\,d\tau} = \frac{1}{J_0},$$

$$\lim_{s\to 0+} \frac{[(n+1)F(s)]^{\frac{n}{n+1}}}{f(s)\Phi(s)} = \frac{1}{I_0},$$

$$\bar{\xi}_\varepsilon^{n+1}\frac{m^*}{k_1}\Big(\frac{n}{n+1}\Big)^n\Big[\frac{n}{n+1}\frac{1}{J_0} + \frac{1}{n+1}\frac{1}{I_0}\frac{1}{J_0} + \frac{1}{I_0}\Big] - (1+\varepsilon) = \varepsilon,$$

$$\underline{\xi}_\varepsilon^{n+1}\frac{M^*}{k_2}\Big(\frac{n}{n+1}\Big)^n\Big[\frac{n}{n+1}\frac{1}{J_0} + \frac{1}{n+1}\frac{1}{I_0}\frac{1}{J_0} + \frac{1}{I_0}\Big] - (1-\varepsilon) = -\varepsilon.$$

Define

$$\bar{u}_\varepsilon = -\phi\big(\underline{\xi}_\varepsilon[\omega(d(x))]^{\frac{n}{n+1}}\big), \quad \underline{u}_\varepsilon = -\phi\big(\bar{\xi}_\varepsilon[\omega(d(x))]^{\frac{n}{n+1}}\big), \quad x \in \Omega_{\delta_\varepsilon},$$

where δ_ε is sufficiently small such that

$$\begin{aligned}
&\bar{\xi}_\varepsilon^{n+1}\frac{m^*}{k_1}\Big(\frac{n}{n+1}\Big)^n\Big[\frac{n}{n+1}\frac{1}{J(d(x))} + \frac{1}{n+1}\frac{1}{I(-\underline{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(-\underline{u}_\varepsilon)}\Big] \\
&\quad - (1+\varepsilon) > 0, \\
&\underline{\xi}_\varepsilon^{n+1}\frac{M^*}{k_2}\Big(\frac{n}{n+1}\Big)^n\Big[\frac{n}{n+1}\frac{1}{J(d(x))} + \frac{1}{n+1-q}\frac{1}{I(-\bar{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(-\bar{u}_\varepsilon)}\Big] \\
&\quad - (1-\varepsilon) < 0
\end{aligned}$$

hold. Then, by (2.1) and (4.1), we have

$$\begin{aligned}
&\det(D^2\underline{u}_\varepsilon) - b(x)f(-\underline{u}_\varepsilon) - |D\underline{u}_\varepsilon|^q \\
&= \Big[\bar{\xi}_\varepsilon\Big(\frac{n}{n+1}\Big)\omega^{\frac{-1}{n+1}}\omega'\phi'\Big]^{n-1}\Big[-\bar{\xi}_\varepsilon^2\Big(\frac{n}{n+1}\Big)^2\omega^{\frac{-2}{n+1}}\omega'^2\phi'' \\
&\quad + \bar{\xi}_\varepsilon\frac{n}{(n+1)^2}\omega^{\frac{-2-n}{n+1}}\omega'^2\phi' - \bar{\xi}_\varepsilon\frac{n}{n+1}\omega^{\frac{-1}{n+1}}\omega''\phi'\Big] \\
&\quad \cdot \prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - b(x)f(-\underline{u}_\varepsilon) - |D\underline{u}_\varepsilon|^q
\end{aligned}$$

$$
\begin{aligned}
&= \bar{\xi}_\varepsilon^{n+1}\Big(\frac{n}{n+1}\Big)^n \omega'^{n-1}\omega''\phi'^{n-1}\phi'' \\
&\quad \cdot\Big[-\frac{n}{n+1}\frac{\omega'^2}{\omega''\omega}+\frac{1}{n+1}\frac{\phi'}{\bar{\xi}_\varepsilon\omega^{\frac{n}{n+1}}\phi''}\frac{\omega'^2}{\omega''\omega}-\frac{\phi'}{\bar{\xi}_\varepsilon\omega^{\frac{n}{n+1}}\phi''}\Big] \\
&\quad \cdot\prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})}-b(x)f(-\underline{u}_\varepsilon)-|D\underline{u}_\varepsilon|^q \\
&\geq \frac{m^*}{k_1(1+\varepsilon)}\bar{\xi}_\varepsilon^{n+1}\Big(\frac{n}{n+1}\Big)^n f(-\underline{u}_\varepsilon)b(x) \\
&\quad \cdot\Big[\frac{n}{n+1}\frac{1}{J(d(x))}+\frac{1}{n+1}\frac{1}{I(-\underline{u}_\varepsilon)}\frac{1}{J(d(x))}+\frac{1}{I(-\underline{u}_\varepsilon)}\Big] \\
&\quad -b(x)f(-\underline{u}_\varepsilon)-|D\underline{u}_\varepsilon|^q \\
&= \frac{1}{1+\varepsilon}b(x)f(-\underline{u}_\varepsilon)\Big\{\frac{m^*}{k_1}\bar{\xi}_\varepsilon^{n+1}\Big(\frac{n}{n+1}\Big)^n\Big[\frac{n}{n+1}\frac{1}{J(d(x))} \\
&\quad +\frac{1}{n+1}\frac{1}{I(-\underline{u}_\varepsilon)}\frac{1}{J(d(x))}+\frac{1}{I(-\underline{u}_\varepsilon)}\Big]-(1+\varepsilon)-\frac{(1+\varepsilon)|D\underline{u}_\varepsilon|^q}{b(x)f(-\underline{u}_\varepsilon)}\Big\}. \qquad (4.3)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\lim_{d(x)\to 0^+}\frac{(1+\varepsilon)|D\underline{u}_\varepsilon|^q}{b(x)f(-\underline{u}_\varepsilon)} \\
&\quad \leq \lim_{d(x)\to 0^+}\frac{(1+\varepsilon)\bar{\xi}_\varepsilon^q(\frac{n}{n+1})^q\omega^{\frac{-q}{n+1}}\omega'^q\phi'^q}{k_2\omega'^{n-1}\omega''\phi^{n-1}\phi''} \\
&\quad = \lim_{d(x)\to 0^+}\bar{\xi}_\varepsilon^{q+1}\frac{1+\varepsilon}{k_2}\Big(\frac{n}{n+1}\Big)^q\cdot\frac{1}{I(-\bar{u}_\varepsilon)}\frac{1}{J(d(x))}\cdot\frac{\omega^{2n+1-q}}{\omega'^{n+1-q}\phi'^{n-q}} \\
&\quad = 0,
\end{aligned}
$$

we can choose smaller δ_ε such that

$$
\det(D^2\underline{u}_\varepsilon)-b(x)f(-\underline{u}_\varepsilon)-|D\underline{u}_\varepsilon|^q\geq 0,
$$

which means $\underline{u}_\varepsilon$ is a subsolution to problem (1.1) in $\Omega_{\delta_\varepsilon}$.

On the other hand, we have

$$
\begin{aligned}
&\det(D^2\bar{u}_\varepsilon)-b(x)f(-\bar{u}_\varepsilon)-|D\bar{u}_\varepsilon|^q \\
&\quad = \Big[\underline{\xi}_\varepsilon\Big(\frac{n}{n+1}\Big)\omega^{\frac{-1}{n+1}}\omega'\phi'\Big]^{n-1}\Big[-\underline{\xi}_\varepsilon^2\Big(\frac{n}{n+1}\Big)^2\omega^{\frac{-2}{n+1}}\omega'^2\phi'' \\
&\qquad +\underline{\xi}_\varepsilon\frac{n}{(n+1)^2}\omega^{\frac{-2-n}{n+1}}\omega'^2\phi'-\underline{\xi}_\varepsilon\frac{n}{n+1}\omega^{\frac{-1}{n+1}}\omega''\phi'\Big] \\
&\qquad \cdot\prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})}-b(x)f(-\bar{u}_\varepsilon)-|D\bar{u}_\varepsilon|^q
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M^*}{k_2(1-\varepsilon)} \underline{\xi}_\varepsilon^{n+1} (\frac{n}{n+1})^n f(-\overline{u}_\varepsilon) b(x) \\
&\quad \cdot \Big[\frac{n}{n+1} \frac{1}{J(d(x))} + \frac{1}{n+1} \frac{1}{I(-\overline{u}_\varepsilon)} \frac{1}{J(d(x))} + \frac{1}{I(-\overline{u}_\varepsilon)}\Big] - b(x) f(-\overline{u}_\varepsilon) \\
&= \frac{b(x) f(-\overline{u}_\varepsilon)}{1-\varepsilon} \Big\{ \frac{M^*}{k_2} \underline{\xi}_\varepsilon^{n+1} \Big(\frac{n}{n+1}\Big)^n \\
&\quad \cdot \Big[\frac{n}{n+1} \frac{1}{J(d(x))} + \frac{1}{n+1-q} \frac{1}{I(-\overline{u}_\varepsilon)} \frac{1}{J(d(x))} + \frac{1}{I(-\overline{u}_\varepsilon)}\Big] - (1-\varepsilon) \Big\} \\
&\leq 0,
\end{aligned}
$$

i.e. $\overline{u}_\varepsilon$ is a supersolution to problem (1.1) in $\Omega_{\delta_\varepsilon}$.

Let $v(x) = -d(x)$. Then we can choose a sufficiently large constant $M_0 > 0$ such that

$$
u + M_0 v \leq \overline{u}_\varepsilon \quad \text{on } \Gamma_{\delta_\varepsilon}.
$$

Since

$$
u = v = \overline{u}_\varepsilon = 0 \quad \text{on } \partial\Omega,
$$

and

$$
\begin{aligned}
\det(D^2(u + M_0 v)) &\geq \det(D^2 u) = b(x) f(-u) + |Du|^q \\
&\geq b(x) f(-(u + M_0 v)) + |D(u + M_0 v)|^q,
\end{aligned}
$$

then it follows from Lemma 2.4 that

$$
u + M_0 v \leq \overline{u}_\varepsilon \quad \text{in } \Omega_{\delta_\varepsilon},
$$

which implies

$$
\frac{u}{-\phi\big(\underline{\xi}_\varepsilon [\omega(d(x))]^{\frac{n}{n+1}}\big)} \geq 1 + \frac{M_0 d(x)}{-\phi\big(\underline{\xi}_\varepsilon [\omega(d(x))]^{\frac{n}{n+1}}\big)} \quad \text{in } \Omega_{\delta_\varepsilon}.
$$

Since $d(x) \to 0$ ($\varepsilon \to 0$), by L'Hospital's rule we have

$$
\lim_{d(x)\to 0} \frac{M_0 d(x)}{-\phi\big(\underline{\xi}_\varepsilon [\omega(d(x))]^{\frac{n}{n+1}}\big)} = \lim_{d(x)\to 0} \frac{(n+1) M_0}{-n \underline{\xi}_\varepsilon \phi' \omega' \omega^{-\frac{1}{n+1}}} = 0,
$$

therefore, we can obtain

$$
\lim_{d(x)\to 0} \inf_{x\in\Omega} \frac{u(x)}{-\phi\big(\xi_1 [\omega(d(x))]^{\frac{n}{n+1}}\big)} \geq 1,
$$

where

$$
\xi_1 = \Bigg[\frac{k_2}{M^* (\frac{n}{n+1})^n \big[\frac{n}{n+1} \frac{1}{J_0} + \frac{1}{n+1} \frac{1}{I_0} \frac{1}{J_0} + \frac{1}{I_0}\big]}\Bigg]^{\frac{1}{n+1}}.
$$

Similarly, we can get

$$\lim_{d(x)\to 0}\sup_{x\in\Omega}\frac{u(x)}{-\phi\big(\xi_2[\omega(d(x))]^{\frac{n}{n+1}}\big)}\le 1,$$

where

$$\xi_2=\left[\frac{k_1}{m^*\big(\frac{n}{n+1}\big)^n\big[\frac{n}{n+1}\frac{1}{J_0}+\frac{1}{n+1}\frac{1}{I_0}\frac{1}{J_0}+\frac{1}{I_0}\big]}\right]^{\frac{1}{n+1}}.$$

It follows that (1.18) holds.

Now, we consider problem (1.2). In this case, $\mu=q$, and we can get

$$(\phi'(t))^{n-q-1}\phi''(t)=-f(\phi(t)),\quad \omega'(t)^{n-1-q}\omega''(t)=-p(t). \tag{4.4}$$

Let

$$\bar{\xi}_\varepsilon=\left[\frac{k_1(1+2\varepsilon)}{m^*\big(\frac{n-q}{n+1-q}\big)^{n-q}\big[\frac{n-q}{n+1-q}\frac{1}{J_0}+\frac{1}{n+1-q}\frac{1}{I_0}\frac{1}{J_0}+\frac{1}{I_0}\big]}\right]^{\frac{1}{n+1-q}},$$

$$\underline{\xi}_\varepsilon=\left[\frac{k_2\big(1-2\varepsilon\big)}{M^*\big(\frac{n-q}{n+1-q}\big)^{n-q}\big[\frac{n-q}{n+1-q}\frac{1}{J_0}+\frac{1}{n+1-q}\frac{1}{I_0}\frac{1}{J_0}+\frac{1}{I_0}\big]}\right]^{\frac{1}{n+1-q}},$$

where k_1, k_2 are given in Theorem 1.3. From Lemma 2.6 and Lemma 2.7 and the definition of $\bar{\xi}_\varepsilon$ and $\underline{\xi}_\varepsilon$ we see that

$$\lim_{t\to 0}\frac{[(n-q)\tilde{p}(t)]^{\frac{n+1-q}{n-q}}}{p(t)\int_0^t((n-q)\tilde{p}(\tau))^{\frac{1}{n-q}}d\tau}=\frac{1}{J_0},$$

$$\lim_{s\to 0}\frac{[(n+1-q)F(s)]^{\frac{n-q}{n+1-q}}}{f(s)\Phi(s)}=\frac{1}{I_0},$$

$$\bar{\xi}_\varepsilon^{n+1-q}\frac{m^*}{k_1}\Big(\frac{n-q}{n+1-q}\Big)^n\Big[\frac{n-q}{n+1-q}\frac{1}{J_0}+\frac{1}{n+1-q}\frac{1}{I_0}\frac{1}{J_0}+\frac{1}{I_0}\Big]-(1+\varepsilon)=\varepsilon,$$

$$\underline{\xi}_\varepsilon^{n+1-q}\frac{M^*}{k_2}\Big(\frac{n-q}{n+1-q}\Big)^n\Big[\frac{n-q}{n+1-q}\frac{1}{J_0}+\frac{1}{n+1-q}\frac{1}{I_0}\frac{1}{J_0}+\frac{1}{I_0}\Big]-(1-\varepsilon)=-\varepsilon.$$

Define

$$\overline{u}_\varepsilon=-\phi\big(\underline{\xi}_\varepsilon[\omega(d(x))]^{\frac{n-q}{n+1-q}}\big),\quad \underline{u}_\varepsilon=-\phi\big(\bar{\xi}_\varepsilon[\omega(d(x))]^{\frac{n-q}{n+1-q}}\big),$$

combining with (2.1) and (4.2), we have

$$
\begin{aligned}
&\det(D^2\underline{u}_\varepsilon) - b(x)f(-\underline{u}_\varepsilon)(1+|D\underline{u}_\varepsilon|^q)\\
&= \Big[\bar{\xi}_\varepsilon\Big(\frac{n-q}{n+1-q}\Big)\omega^{\frac{-1}{n+1-q}}\omega'\phi'\Big]^{n-1}\Big[-\bar{\xi}_\varepsilon^2\Big(\frac{n-q}{n+1-q}\Big)^2\omega^{\frac{-2}{n+1-q}}\omega'^2\phi''\\
&\quad + \bar{\xi}_\varepsilon\frac{n-q}{(n+1-q)^2}\omega^{\frac{-2-n+q}{n+1-q}}\omega'^2\phi' - \bar{\xi}_\varepsilon\frac{n-q}{n+1-q}\omega^{\frac{-1}{n+1-q}}\omega''\phi'\Big]\\
&\quad \cdot\prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - b(x)f(-\underline{u}_\varepsilon)(1+|D\underline{u}_\varepsilon|^q)\\
&= \bar{\xi}_\varepsilon^{n+1}\Big(\frac{n-q}{n+1-q}\Big)^n\omega^{\frac{-q}{n+1-q}}\omega'^{n-1}\omega''\phi'^{n-1}\phi''\\
&\quad \cdot\Big[-\frac{n-q}{n+1-q}\frac{\omega'^2}{\omega''\omega} + \frac{1}{n+1-q}\frac{\phi'}{\bar{\xi}_\varepsilon\omega^{\frac{n-q}{n+1-q}}\phi''}\frac{\omega'^2}{\omega''\omega} - \frac{\phi'}{\bar{\xi}_\varepsilon\omega^{\frac{n-q}{n+1-q}}\phi''}\Big]\\
&\quad \cdot\prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - b(x)f(-\underline{u}_\varepsilon)(1+|D\underline{u}_\varepsilon|^q)\\
&\ge \frac{m^*}{k_1(1+\varepsilon)}\bar{\xi}_\varepsilon^{n+1-q}\Big(\frac{n-q}{n+1-q}\Big)^{n-q} f(-\underline{u}_\varepsilon)b(x)|D\underline{u}_\varepsilon|^q\\
&\quad \cdot\Big[\frac{n-q}{n+1-q}\frac{1}{J(d(x))} + \frac{1}{n+1-q}\frac{1}{I(-\underline{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(-\underline{u}_\varepsilon)}\Big]\\
&\quad - b(x)f(-\underline{u}_\varepsilon)(1+|D\underline{u}_\varepsilon|^q)\\
&= \frac{1}{1+\varepsilon}b(x)f(-\underline{u}_\varepsilon)|D\underline{u}_\varepsilon|^q\Big\{\frac{m^*}{k_1}\bar{\xi}_\varepsilon^{n+1-q}\Big(\frac{n-q}{n+1-q}\Big)^{n-q}\\
&\quad \cdot\Big[\frac{n-q}{n+1-q}\frac{1}{J(d(x))} + \frac{1}{n+1-q}\frac{1}{I(-\underline{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(-\underline{u}_\varepsilon)}\Big]\\
&\quad - \Big[\frac{1+\varepsilon}{|D\underline{u}_\varepsilon|^q} + (1+\varepsilon)\Big]\Big\}\\
&\ge 0,
\end{aligned}
$$

which means $\underline{u}_\varepsilon$ is a subsolution to problem (1.2) in $\Omega_{\delta_\varepsilon}$.

On the other hand, we have

$$
\begin{aligned}
&\det(D^2\bar{u}_\varepsilon) - b(x)f(-\bar{u}_\varepsilon)(1+|D\bar{u}_\varepsilon|^q)\\
&\le \Big[\underline{\xi}_\varepsilon\Big(\frac{n-q}{n+1-q}\Big)\omega^{\frac{-1}{n+1-q}}\omega'\phi'\Big]^{n-1}\Big[-\underline{\xi}_\varepsilon^2\Big(\frac{n-q}{n+1-q}\Big)^2\omega^{\frac{-2}{n+1-q}}\omega'^2\phi''\\
&\quad + \underline{\xi}_\varepsilon\frac{n-q}{(n+1-q)^2}\omega^{\frac{-2-n+q}{n+1-q}}\omega'^2\phi' - \underline{\xi}_\varepsilon\frac{n-q}{n+1-q}\omega^{\frac{-1}{n+1-q}}\omega''\phi'\Big]\\
&\quad \cdot\prod_{i=1}^{n-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - b(x)f(-\bar{u}_\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{M^*}{k_2(1-\varepsilon)}\underline{\xi}_\varepsilon^{n+1-q}\left(\frac{n-q}{n+1-q}\right)^{n-q} f(-\overline{u}_\varepsilon)b(x)|D\overline{u}_\varepsilon|^q \\
&\quad \cdot\left[\frac{n-q}{n+1-q}\frac{1}{J(d(x))}+\frac{1}{n+1-q}\frac{1}{I(-\overline{u}_\varepsilon)}\frac{1}{J(d(x))}+\frac{1}{I(-\overline{u}_\varepsilon)}\right] \\
&\quad - b(x)f(-\overline{u}_\varepsilon) \\
&= \frac{1}{1-\varepsilon}b(x)f(-\overline{u}_\varepsilon)|D\overline{u}_\varepsilon|^q\left\{\frac{M^*}{k_2}\underline{\xi}_\varepsilon^{n+1-q}\left(\frac{n-q}{n+1-q}\right)^{n-q}\right. \\
&\quad \left.\cdot\left[\frac{n-q}{n+1-q}\frac{1}{J(d(x))}+\frac{1}{n+1-q}\frac{1}{I(-\overline{u}_\varepsilon)}\frac{1}{J(d(x))}+\frac{1}{I(-\overline{u}_\varepsilon)}\right]-(1-\varepsilon)\right\} \\
&\le 0,
\end{aligned}
$$

i.e. $\overline{u}_\varepsilon$ is a supersolution to problem (1.2) in $\Omega_{\delta_\varepsilon}$.

The remaining proof is similar to that above. So we omit it here. The proof of Theorem 1.3 is finished. ∎

Proof of Corollary 1.1. We only prove (1.20). The proof of (1.21) is similar to that of (1.20). So we omit it.

Since ϕ is the inverse of Φ, it follows from (1.10) that

$$
\phi'(t) = \big[(n+1-\mu)F(\phi(t))\big]^{\frac{1}{n+1-\mu}}.
$$

Then

$$
\begin{aligned}
\lim_{t\to 0+}\frac{t\phi'(t)}{\phi(t)} &= \lim_{t\to 0}\frac{t[(n+1-\mu)F(\phi(t))]^{\frac{1}{n+1-\mu}}}{\phi(t)} \\
&= \lim_{s\to 0+}\frac{\frac{\Phi(s)}{\Phi'(s)}}{s} = 1-\lim_{s\to 0+}\frac{\Phi(s)\Phi''(s)}{\Phi'^2(s)} = 1-I_0.
\end{aligned}
$$

It follows from Proposition 2.2 that $\phi\in \mathrm{NRVZ}_{1-I_0}$.

Combing this with Theorem 1.3 and Definition 2.1 we obtain

$$
\begin{aligned}
&\lim_{\substack{x\in\Omega,\\ d(x)\to 0}}\frac{u(x)}{\phi\big[(\omega(d(x)))^{\frac{n}{n+1}}\big]} \\
&\quad = \lim_{\substack{x\in\Omega,\\ d(x)\to 0}}\frac{u(x)}{\phi\big[\xi_0(\omega(d(x)))^{\frac{n}{n+1}}\big]}\frac{\phi\big[\xi_0(\omega(d(x)))^{\frac{n}{n+1}}\big]}{\phi\big[(\omega(d(x)))^{\frac{n}{n+1}}\big]} \\
&\quad = \xi_0^{1-I_0},
\end{aligned}
$$

where

$$
\xi_0 = \left\{\frac{\bar{k}R^{n-1}}{\left(\frac{n}{n+1}\right)^n\left[\frac{n}{n+1}\frac{1}{J_0}+\frac{1}{n+1}\frac{1}{J_0I_0}+\frac{1}{I_0}\right]}\right\}^{\frac{1}{n+1}}.
$$
∎

Acknowledgments. The authors are grateful to anonymous referees for their constructive comments and suggestions, which has greatly improved this paper.

Funding. This work is sponsored by Beijing Natural Science Foundation under Grant No. 1212003.

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Received 3 August 2022; revised 24 November 2022.

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