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Normal bundle of monomial curves: an application to rational curves

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Abstract. In this note, we give an application to the study of general rational curves in $\mathbb{P}^s(\mathbb{C})$ of the calculation of the splitting type of the normal bundle of any smooth monomial rational curve (i.e., embedded by monomial functions).

1. Introduction

In this paper, any degree d rational curve C in $\mathbb{P}^s(\mathbb{C})$ ($d > s \geq 3$) will be assumed smooth and nondegenerate. Such curves, up to projective transformations, are suitable projections of the rational normal curve Γ_d of degree d in $\mathbb{P}^d(\mathbb{C})$ from a projective linear space L of dimension $d - s - 1$. Let us call $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$ the morphism obtained in this way. The normal bundle of such curves splits as a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(\xi_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\xi_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\xi_{s-1})$ where ξ_i are suitable integers. In principle, one should calculate these integers for any chosen L.

In [2], the authors develop a general method to do this calculation. This method was previously used in [1] to get the splitting type of the restricted tangent bundle of C. However, while for the tangent bundle it is possible to get an easy formula (see [1, Theorem 3]), for the normal bundle this is not possible.

In [3] the authors gave a method for calculating the integers ξ_i when C is a smooth monomial curve, i.e., when the morphism $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$ is given by monomials of the same degree in two variables. In other setups, C is called "monomial" if its ideal in $\mathbb{P}^s(\mathbb{C})$ is generated by monomials. Here we do not consider the ideal of C and we focus on f; for instance, the standard twisted cubic in $\mathbb{P}^3(\mathbb{C})$ is a monomial curve according to our definition, but its ideal is not generated by monomials.

In [4], the authors study the moduli space of rational curves whose normal bundle has a fixed splitting type and, meanwhile, they get a very simple formula to calculate ξ_i for smooth monomial curves. Obviously the two methods give rise to the same

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integers (see the final part of [3, §5] and [4, Theorem 3.2]), but the two approaches are very different and we think that they are both useful for different aims.

Here we want to give a consequence of the possibility to get the splitting type of the normal bundle of rational monomial curves as in [3]. Our main theorem will be Theorem 3, however, it is not possible to state it without a background. In brief we can say that our strategy will be to associate a smooth monomial curve CA as above to any smooth rational curve C, satisfying mild assumptions, and to prove that $h^0(\mathbb{P}^1, f^*\mathcal{N}_C(-d-2-k)) \leq h^0(\mathbb{P}^1, f^*\mathcal{N}_{CA}(-d-2-k))$ for any $k \geq 2$ where $\mathcal{N}_C$ and $\mathcal{N}_{CA}$ are the normal bundles of C and CA in $\mathbb{P}^s(\mathbb{C})$, respectively. As the knowledge of this cohomology implies the knowledge of the numbers $c_i := \xi_i - d - 2$, we will get that the numbers c_i of C are bounded by the numbers c_i of CA (see Examples 4 and 5).

In Section 2, we fix notations and we recall the background. In Section 3, we associate a monomial curve CA to any smooth rational curve C having a suitable property and we prove our main theorem. In Section 4, we give our applications.

2. Notation and background material

For us, a rational curve $C \subset \mathbb{P}^s(\mathbb{C})$ will be the target of a morphism $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$. We will work always over $\mathbb{C}$. We will always assume that C is not contained in any hyperplane and that it is smooth. Let us put $d := \deg(C) > s \geq 3$. Let $\mathcal{I}_C$ be the ideal sheaf of C, then $\mathcal{N}_C := \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ as usual and, taking the differential of f, we get

$$0 \to \mathcal{T}_{\mathbb{P}^1} \to f^*\mathcal{T}_{\mathbb{P}^s} \to f^*\mathcal{N}_C \to 0$$

where $\mathcal{T}$ denotes the tangent bundle. Of course we can always write

$$\mathcal{T}_f := f^*\mathcal{T}_{\mathbb{P}^s} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(b_i + d + 2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(s-r)}(d+1),$$
$$\mathcal{N}_f := f^*\mathcal{N}_C = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i + d + 2)$$

for suitable integers $b_i \geq 0$ (see [1, (14)]) and $c_i \geq 0$ (see [2, Proposition 10] where we assumed $c_1 \geq \cdots \geq c_{s-1}$).

Every curve C is, up to a projective transformation, the projection to $\mathbb{P}^s$ of a d-Veronese embedding Γ_d of $\mathbb{P}^1$ in $\mathbb{P}^d := \mathbb{P}(V)$ from a $(d-s-1)$-dimensional projective space $L := \mathbb{P}(T)$ where V and T are vector spaces of dimension, respectively, $d+1$ and $e+1 := d-s$. For any vector $0 \neq v \in V$ let $[v]$ be the corresponding point in $\mathbb{P}(V)$. Of course we require that $L \cap \Gamma_d = \emptyset$ as we want that f is a morphism.

Let us denote by $U = \langle x, y\rangle$ a fixed 2-dimensional vector space such that $\mathbb{P}^1 = \mathbb{P}(U)$, then we can identify V with $S^d U$ (d-th symmetric power) in such a way that the rational normal degree d curve Γ_d can be considered as the set of pure tensors of degree d in $\mathbb{P}(S^d U)$ and the d-Veronese embedding is the map

$$\alpha x + \beta y \to (\alpha x + \beta y)^d, \qquad (\alpha : \beta) \in \mathbb{P}^1.$$

From now on, any degree d rational curve C will be determined (up to projective equivalences which are not important in our context) by the choice of a proper subspace $T \subset S^d U$ such that $\mathbb{P}(T) \cap \Gamma_d = \emptyset$.

By arguing in this way, the elements of a base of T can be thought as homogeneous, degree d polynomials in x, y. In [1, 2], the authors relate the polynomials of any base of T with the splitting type of $\mathcal{T}_f$ and $\mathcal{N}_f$. To describe this relation we need some additional definitions.

Let us indicate by $\langle \partial_x, \partial_y\rangle$ the dual space U^* of U, where ∂_x and ∂_y indicate the partial derivatives with respect to x and y.

Definition 1. Let T be any proper subspace of $S^d U$. Then

$$\partial T := \langle \omega(T) | \omega \in U^*\rangle,$$
$$\partial^{-1} T := \bigcap_{\omega \in U^*} \omega^{-1} T,$$
$$r(T) := \dim(\partial T) - \dim(T).$$

Note that Definition 1 allows to define also $\partial^k T$ and $\partial^{-k} T$ for any integer $k \geq 1$, by induction. Moreover, we can set $\partial^0 T := T$. Let us recall the following:

Theorem 1. *Let $T \subset S^d U$ be any proper subspace as above such that $\mathbb{P}(T) \cap \Gamma_d = \emptyset$. Then $r(T) \geq 1$ and there exist r polynomials $p_1, \ldots, p_r$ of degree $d + b_1, \ldots, d + b_r$ respectively, with $b_i \geq 0$ and $[p_i] \in \mathbb{P}^{d+b_i} \setminus \mathrm{Sec}^{b_i}(\Gamma_{d+b_i})$ for $i = 1, \ldots, r$, such that*

$$T = \partial^{b_1}(p_1) \oplus \partial^{b_2}(p_2) \oplus \cdots \oplus \partial^{b_r}(p_r)$$

and

$$\partial T = \partial^{b_1+1}(p_1) \oplus \partial^{b_2+1}(p_2) \oplus \cdots \oplus \partial^{b_r+1}(p_r).$$

Proof. It follows from [1, Theorem 1], because from our assumptions $S_T = 0$ in the notation of [1]. Recall that $\mathrm{Sec}^b(\Gamma_{d+b})$ is the variety generated by sets of $b + 1$ distinct points of Γ_{d+b}. ■

From the above decomposition of T it is possible to get directly the splitting type of $\mathcal{T}_f$ depending on the integers b_i (see [1, Theorem 3]), however, here we are interested in the splitting type of $\mathcal{N}_f$. To this aim the following Proposition is useful:

Proposition 1. *In the above notations, for any integer* $k \geq 0$*, let us call* $\varphi(k) := h^0(\mathbb{P}^1, \mathcal{N}_f(-d-2-k))$*. Then the splitting type of* $\mathcal{N}_f$ *is completely determined by* $\Delta^2[\varphi(k)] := \varphi(k+2) - 2\varphi(k+1) + \varphi(k)$.

Proof. We know that $\mathcal{N}_f(-d-2) = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i)$, so that we have only to determine the integers c_i. By definition, $\Delta^2[\varphi(k)]$ is exactly the number of integers c_i which are equal to k. Note that, by definition, $\varphi(k)$ is strictly decreasing. ∎

From Proposition 1 it follows that to know the splitting type of $\mathcal{N}_f$ it suffices to know $\varphi(k)$ for any $k \geq 0$.

Let us consider the linear operators

$$D_k : S^k U \otimes S^d U \to S^{k-1} U \otimes S^{d-1} U,$$

such that $D_k := \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$, and $D_k^2 : S^k U \otimes S^d U \to S^{k-2} U \otimes S^{d-2} U$. Of course, as $T \subset S^d U$, we can restrict D_k^2 to $S^k U \otimes T$ and we get a linear map $D^2_{k|S^k U \otimes T} : S^k U \otimes T \to S^{k-2} U \otimes \partial^2 T$; let us define

$$T_k := \ker(D^2_{k|S^k U \otimes T}).$$

Then we have the following:

Theorem 2. *In the above notations,*

$$\varphi(0) = d + e,$$
$$\varphi(1) = 2(e+1),$$
$$\varphi(2) = 3(e+1) - \dim(\partial^2 T),$$

and for any $k \geq 2$, $\varphi(k) = \dim(T_k)$.

Moreover, the number of integers c_i *such that* $c_i = 0$ *is* $d - 1 - \dim(\partial^2 T)$.

Proof. See [2, Theorem 1 and Proposition 11]; note that, for $k = 2$, there are two different ways to get $\varphi(2)$.

By Proposition 1 the number of integers c_i such that $c_i = 0$ is $\Delta^2[\varphi(0)] = d - 1 - \dim(\partial^2 T)$. ∎

In [3], a combinatorial formula is given to calculate $\varphi(k)$, for $k \geq 2$, when C is a monomial smooth rational curve, therefore we can assume that $\varphi(k)$ is known for any monomial smooth rational curve. Moreover, a method to determine the set $\{\xi_i\}$ is given in [3, Theorem 4 and Remark 2]. Let us recall this method: firstly decompose T as $T = T^1 \oplus T^2 \oplus \cdots \oplus T^q$ in such a way that $\partial^2 T = \partial^2 T^1 \oplus \partial^2 T^2 \oplus \cdots \oplus \partial^2 T^q$ for some $q \geq 1$; every T^j is called irreducible. Secondly: decompose every irreducible T^j, $j = 1, \ldots, q$ as explained in Theorem 1, getting the integers $b_1(j), \ldots, b_{r(j)}(j)$.

Thirdly: define $b_0(j) = b_{r(j)+1}(j) = -1$ for any $j = 1, \dots, q$ and consider the set $\{b_i(j) + b_{i+1}(j) + 2$ for $i = 0, \dots, r(j)$ and $j = 1, \dots, q\}$. This is the set of positive c_i, while the number of null c_i is given by Theorem 2. By recalling that

$$\sum_{j=1}^{q}[r(j) + 1] = \dim(\partial^2 T) - \dim(T)$$

we get a set of $s - 1$ integers $\{c_i\}$ and $\xi_i = c_i + d + 2$, $i = 1, \dots, s - 1$.

On the other hand, in [4], the authors give a very direct formula for calculating ξ_i when C is a monomial smooth rational curve of degree d (see [4, Theorem 3.2]). Such curve is the image of a map

$$f(x : y) = (x^{h_0} : x^{h_1} y^{d-h_1} : \dots : x^{h_i} y^{d-h_i} : \dots : x^{h_s} y^{d-h_s})$$

with $i = 0, \dots, s$ and $h_0 > h_1 > \dots > h_s \geq 0$. We require that this map is an embedding, hence it is necessary that: $h_0 = d$, $h_1 = d - 1$, $h_{s-1} = 1$, $h_s = 0$, (see [4, Lemma 3.1]) and $s \geq 3$. Then [4, Theorem 3.2] says that

$$\xi_i = d + h_{i-1} - h_{i+1} \qquad \text{for } i = 1, \dots, s - 1 \qquad \text{(Coskun–Riedl formula)}.$$

Of course the Coskun–Riedl formula gives the same integers ξ_i obtained by the method described in [3]; the interested reader can find a proof of this fact in that article.

3. Rational complete curves and main theorem

Let C be any smooth rational curve of degree d. The morphism $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$ is given by a $(s + 1, d + 1)$ matrix M of rank $s + 1$ such that

$$(x : y) \to M[x^d x^{d-1} y \cdots y^d]^t$$

where $[\cdots]^t$ denotes transposition. In other words, the parametric equations for C are

$$\begin{bmatrix} X_0 \\ X_1 \\ \cdots \\ X_s \end{bmatrix} = M \begin{bmatrix} x^d \\ x^{d-1} y \\ \cdots \\ y^d \end{bmatrix}.$$

As $\operatorname{rank}(M) = s + 1$ we can apply the Gauss elimination to M and we can transform it in a row echelon form. This is equivalent to multiply M on the left by a suitable non singular $(s + 1, s + 1)$ matrix, i.e., to change the projective coordinate system in $\mathbb{P}^s(\mathbb{C})$. By another change, if necessary, we can also assume that all pivots are 1.

The point $f(1:0) = (m_{1,1} : m_{2,1} : \cdots : m_{s+1,1})$ belongs to C, in particular $(m_{1,1} : m_{2,1} : \cdots : m_{s+1,1}) \neq (0 : 0 : \cdots : 0)$, hence we can assume that the first pivot is $m_{1,1} = 1$ and that $m_{i,1} = 0$ for $i \geq 2$, i.e., $f(1:0) = (1:0:\cdots:0)$. Let us consider the second column of M in the row echelon form (hence $m_{i,2} = 0$ for $i \geq 3$). If the second pivot would be not $m_{2,2} = 1$ then C would be singular at $(1 : 0 : \cdots : 0)$, but C is smooth, hence $m_{2,2} = 1$.

We give the following:

Definition 2. Let M be the above matrix. If $m_{s+1,j} = 0$ for $j = 1, \ldots, d$; $m_{s+1,d+1} = 1$; $m_{s,j} = 0$ for $j = 1, \ldots, d-1$ and $m_{s,d} = 1$, then we say that C is complete.

To any smooth rational curve C, whose associated matrix M is in a row echelon form as above, we can associate a monomial rational curve CA whose parametric equations are

$$\begin{bmatrix} X_0 \\ X_1 \\ \cdots \\ X_s \end{bmatrix} = M' \begin{bmatrix} x^d \\ x^{d-1}y \\ \cdots \\ y^d \end{bmatrix}$$

where M' is the matrix of the pivots of M, i.e., $M' := (m'_{i,j})$ is a matrix of type $(s+1, d+1)$ such that $m'_{i,j} = 1$ if and only if $m_{i,j} = 1$ is a pivot of M and $m'_{i,j} = 0$ otherwise. The meaning of the above definition is clarified by the following fact, easy to prove: if C is complete, then CA is smooth of degree d; while, in general, CA is smooth of degree $d' < d$, or singular of degree d.

Example 1. Here is a typical example of complete, smooth, rational curve C with $s = 5$ and $d = 9$, ($*$ denotes any complex number):

$$M = \begin{bmatrix} 1 & * & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In other words, putting $t := y/x$, the affine parametric equations of C are

$$\begin{aligned} X_0 &= 1 + *t + \cdots + *t^9, \\ X_1 &= t + *t^2 + \cdots + *t^9, \\ X_2 &= t^4 + *t^5 + \cdots + *t^9, \end{aligned}$$

$$X_3 = t^5 + *t^6 + \cdots + *t^9,$$
$$X_4 = t^8 + *t^9,$$
$$X_5 = t^9.$$

Then the affine parametric equations for CA are

$$X_0 = 1,$$
$$X_1 = t,$$
$$X_2 = t^4,$$
$$X_3 = t^5,$$
$$X_4 = t^8,$$
$$X_5 = t^9.$$

In practice: for any i, take the monomials in t of minimal degree appearing in the polynomials $X_i(t)$.

Example 2. Here is a typical example of a non complete, smooth, rational curve C with $s = 4$ and $d = 8$ such that CA is still smooth, ($*$ denotes any complex number, but there is at least a non zero number in the last column):

$$M = \begin{bmatrix} 1 & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}.$$

Now, putting $t := y/x$, the affine parametric equations of C are

$$X_0 = 1 + *t + \cdots + *t^8,$$
$$X_1 = t + *t^2 + \cdots + *t^8,$$
$$X_2 = t^4 + *t^5 + \cdots + *t^8,$$
$$X_3 = t^5 + *t^6 + \cdots + *t^8,$$
$$X_4 = t^6 + *t^7 + *t^8.$$

Then the affine parametric equations for the degree $d' = 6$ curve CA are

$$X_0 = 1,$$
$$X_1 = t,$$
$$X_2 = t^4,$$
$$X_3 = t^5,$$
$$X_4 = t^6.$$

Note that, as we want that CA is smooth, $m_{s,d'} = m_{s+1,d'+1} = 1$. In this example, $d' = \deg(CA) = 6 < \deg(C) = 8$.

We have the following:

Theorem 3. *Let C be a smooth, rational curve of degree d in $\mathbb{P}^s(\mathbb{C})$ and let us assume that CA is a smooth monomial rational curve of degree $d' \leq d$ associated to C as above. Let φ_C and φ_{CA} be, respectively, the functions defined by Proposition 1 for curves C and CA. Then, for any $k \geq 2$, $\varphi_C(k) \leq \varphi_{CA}(k)$.*

Proof. Firstly, let us assume that C is complete, hence $d' = d$, and let us consider the affine parametric equations af C as in the above examples. These equations define a regular map

$$f : \mathbb{A}^1 \to \mathbb{P}^s$$

as follows ($*$ denotes any complex number):

$$\begin{aligned} X_0 &= 1 + *t + \cdots + *t^d, \\ X_1 &= t + *t^2 + \cdots + *t^d, \\ &\vdots \\ X_i &= t^{p_i} + *t^{p_i+1} + \cdots + *t^d, \\ &\vdots \\ X_{s-1} &= t^{d-1} + *t^d, \\ X_s &= t^d. \end{aligned}$$

For any non zero complex number q let us define

(1) an isomorphism $\psi_q : \mathbb{A}^1 \to \mathbb{A}^1$,

$$\psi_q(t) = t/q;$$

(2) a rational curve C_q in $\mathbb{P}^s$ whose affine parametric equations are

$$\begin{aligned} X_0 &= 1 + q * t + \cdots + q^d * t^d, \\ X_1 &= t + q * t^2 + \cdots + q^{d-1} * t^d, \\ &\vdots \\ X_i &= t^{p_i} + q * t^{p_i+1} + \cdots + q^{d-p_i} * t^d, \\ &\vdots \\ X_{s-1} &= t^{d-1} + q * t^d, \\ X_s &= t^d \end{aligned}$$

defining a map

$$f_q : \mathbb{A}^1 \to \mathbb{P}^s;$$

(3) a linear isomorphism

$$F_q : \mathbb{P}^s \to \mathbb{P}^s$$

whose associated $(s+1, s+1)$ matrix is

$$\mathrm{diag}(1, q, \ldots, q^{p_i}, \ldots, q^{d-1}, q^d).$$

The definitions are given in order to get

$$F_q[f_q(\psi_q)] = f;$$

then we have that every curve C_q is projectively equivalent to C and they all have the same splitting type for the normal bundle in $\mathbb{P}^s$. Moreover, the smooth curve CA is obtained from C_q by letting $q \to 0$, hence, by semicontinuity, we have $\varphi_C(k) \leq \varphi_{CA}(k)$.

If C is not complete, but CA is still smooth, of degree $d' < d$, the above proof must be changed a little, taking into account that, in these cases,

$$X_s = t^{p_s} + *t^{p_s+1} + \cdots + *t^d$$

with $p_s = d'$, but the conclusion is the same. ∎

When C is complete there is another proof of the main theorem "by hands" without using any degeneration argument. We give here a sketch of it because we think that it is useful when one has to calculate the value $\varphi_C(k)$ to get the splitting type of $\mathcal{N}_f$ according to Proposition 1.

Let T_C and T_{CA} be the $(e+1)$-dimensional vector spaces determining C, and respectively CA, as explained in Section 2. Let us fix a monic monomial base for T_{CA}. By looking at the $(s+1, d+1)$ matrix M for C (in a row echelon form, with all pivots equal to 1) we see that a base for T_{CA} can be chosen by taking exactly the monomials in the string $\langle x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\rangle$ not corresponding to the $s+1$ pivots of the matrix.

It is possible to choose two corresponding bases: $\langle \tau_0, \tau_1, \ldots, \tau_e\rangle$ for T_C and $\langle \tilde{\tau}_0, \tilde{\tau}_1, \ldots, \tilde{\tau}_e\rangle$ for T_{CA}, such that $\mathrm{lt}(\tau_i) = \tilde{\tau}_i$ for $i = 0, \ldots, e$, where $\mathrm{lt}(\tau)$ denote the leading term of a polynomial $\tau \in \mathbb{C}[x, y]$ with respect to y.

For any $k \geq 2$, let us consider the generic element $\sum_{p=0}^{e} f_p \otimes \tau_p \in S^k U \otimes T_C$ and let us apply the operator D_k^2 to it. We get

$$D_k^2\Big[\sum_{p=0}^{e} f_p \otimes \tau_p\Big] = \sum_{p=0}^{e}(\partial_y\partial_y f_p \otimes \partial_x\partial_x\tau_p - 2\partial_x\partial_y f_p \otimes \partial_x\partial_y\tau_p + \partial_x\partial_x f_p \otimes \partial_y\partial_y\tau_p).$$

Now, let us consider all the degree $d-2$ monomials in $\mathbb{C}[x,y]$ involved by the $3(e+1)$ polynomials $\{\partial^2\tau_0, \partial^2\tau_1, \dots, \partial^2\tau_e\}$ generating $\partial^2 T_C$, i.e., $x^{d-2}, x^{d-3}y, \dots, x^{d-2-\beta_r}y^{\beta_r}$. We can write

$$D_k^2\Big[\sum_{p=0}^{e} f_p \otimes \tau_p\Big] = \sum_{q=0}^{\beta_r} A_q \otimes x^{d-2-q}y^q$$

so that $D_k^2[\sum_{p=0}^{e} f_p \otimes \tau_p] = 0$ if and only if $A_q = 0$ for $q \in [0, \beta_r]$.

Now, let us consider all the degree $d-2$ monomials in $\mathbb{C}[x,y]$ involved by the $3(e+1)$ monomials $\{\partial^2\tilde{\tau}_0, \partial^2\tilde{\tau}_1, \dots, \partial^2\tilde{\tau}_e\}$ generating $\partial^2 T_{CA}$. Thanks to our choice of bases $\langle \tau_0, \tau_1, \dots, \tau_e\rangle$ and $\langle \tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_e\rangle$ we have that

$$\{\partial^2\tilde{\tau}_0, \partial^2\tilde{\tau}_1, \dots, \partial^2\tilde{\tau}_e\} \subseteq \{x^{d-2}, x^{d-3}y, \dots, x^{d-2-\beta_r}y^{\beta_r}\}.$$

Let δ be the dimension of $\partial^2 T_{CA}$. Let us fix δ monic distinct monomials among $\{\partial^2\tilde{\tau}_0, \partial^2\tilde{\tau}_1, \dots, \partial^2\tilde{\tau}_e\}$ generating $\partial^2 T_{CA}$. These monomials are obviously independent and give rise to a base $\mathcal{B}$ for $\partial^2 T_{CA}$. Let us order this base $\mathcal{B}$ with respect to the ascending powers of y. Let us call

$$\begin{aligned} F_k &:= \ker(D^2_{k|S^kU\otimes T_{CA}}) \\ &= \Big\{\sum_{p=0}^{e} f_p \otimes \tilde{\tau}_p \in S^kU \otimes T_{CA} \mid D_k^2\Big[\sum_{p=0}^{e} f_p \otimes \tilde{\tau}_p\Big] = 0\Big\}. \end{aligned}$$

Obviously, the condition $D_k^2[\sum_{p=0}^{e} f_p \otimes \tilde{\tau}_p] = 0$ involves only the δ degree $d-2$ monomials belonging to $\mathcal{B}$. Let us define

$$\begin{aligned} E_k := \Big\{&\sum_{p=0}^{e} f_p \otimes \tau_p \in S^kU \otimes T_C \mid D_k^2\Big[\sum_{p=0}^{e} f_p \otimes \tau_p\Big] = \sum_{q=0}^{\beta_r} A_q \otimes x^{d-2-q}y^q \\ &\text{and } A_q = 0 \text{ only for the } \delta \text{ monomials } x^{d-2-q}y^q \text{ belonging to } \mathcal{B}\Big\}. \end{aligned}$$

Obviously $\varphi_C(k) = \dim[\ker(D^2_{k|S^kU\otimes T_C})] \le \dim(E_k)$.

To complete the proof of the theorem it is sufficient to prove that $\dim(E_k) \le \dim(F_k) = \varphi_{CA}(k)$. Note that E_k and F_k are both subspaces of $\mathbb{C}^{(e+1)(k+1)}$ and that this vector space is given by all the coefficients of the generic polynomials $f_p \in S^k(U)$, $p = 0, \dots, e$.

We have only δ relations defining E_k, one to one with the elements of $\mathcal{B}$. Every relation is of the following type and it does not depend on k:

$$\sum_{p=0}^{e}(a_p\partial_x\partial_x f_p + b_p\partial_x\partial_y f_p + c_p\partial_y\partial_y f_p) = 0, \qquad a_p, b_p, c_p \in \mathbb{C},$$

hence they give rise to a $(\delta, 3(e+1))$ matrix N of complex numbers which is the union of $e+1$ blocks of type $(\delta, 3)$, each one in a row echelon form due to the above choice for $\mathcal{B}$.

The δ relations defining E_k inside $\mathbb{C}^{(e+1)(k+1)}$ can be written in matricial form as

$$N\left[\partial_x\partial_x f_0 \quad \partial_x\partial_y f_0 \quad \partial_y\partial_y f_0 \quad \cdots \quad \partial_x\partial_x f_e \quad \partial_x\partial_y f_e \quad \partial_y\partial_y f_e\right]^t = 0. \qquad \text{(e)}$$

Note that the set of $k+1$ variables related to every polynomial f_p is distinct from the set of $k+1$ variables related to any other polynomial $f_{p'}$ if $p' \neq p$.

We can argue in the same way with the δ relations defining F_k inside $\mathbb{C}^{(e+1)(k+1)}$ getting an analogue matrix NA and a matrix relation analogous to (e),

$$NA\left[\partial_x\partial_x f_0 \quad \partial_x\partial_y f_0 \quad \partial_y\partial_y f_0 \quad \cdots \quad \partial_x\partial_x f_e \quad \partial_x\partial_y f_e \quad \partial_y\partial_y f_e\right]^t = 0. \qquad \text{(a)}$$

Note that NA is obtained from N simply by putting equal to zero every number appearing in N which is not a pivot in a single block. Moreover, $\delta \leq 3(e+1)$ (in fact, $\varphi_{CA}(2) = 3(e+1) - \delta \geq 0$) and therefore $\mathrm{rank}(N) = \mathrm{rank}(NA) = \delta$, being both the union of blocks in a row echelon form. Moreover, both matrices have the same pivots in the same position.

It follows that there exists a non singular upper triangular matrix Z of complex numbers, of order δ, such that $N' := ZN$, all complex numbers over the pivots of N are zero and the pivots of every block of N' are the same and in the same position with respect to N and hence NA (see the example below). Of course, E_k can be defined inside $\mathbb{C}^{(e+1)(k+1)}$ also by the δ relations

$$N'\left[\partial_x\partial_x f_0 \quad \partial_x\partial_y f_0 \quad \partial_y\partial_y f_0 \quad \cdots \quad \partial_x\partial_x f_e \quad \partial_x\partial_y f_e \quad \partial_y\partial_y f_e\right]^t = 0. \qquad \text{(ee)}$$

Now we can see that the dimension of E_k inside $\mathbb{C}^{(e+1)(k+1)}$ is the dimension of the vector space over $\mathbb{C}$ generated by the set $\mathcal{G}$ of coefficients of those polynomials among $\{\partial_x\partial_x f_0, \partial_x\partial_y f_0, \partial_y\partial_y f_0, \ldots, \partial_x\partial_x f_e, \partial_x\partial_y f_e, \partial_y\partial_y f_e\}$ such that in (ee) the corresponding columns of N' do not contain a pivot. The same is true for the dimension of F_k by considering (a) and NA, note that the quoted columns are the same for N' and NA hence the set $\mathcal{G}$ is the same.

If $k = 2$ the dimensions of E_2 and F_2 are exactly the number of such columns, i.e., $3(e+1) - \delta$, because the polynomials $\{\partial_x\partial_x f_0, \ldots, \partial_y\partial_y f_e\}$ have degree 0. If $k \geq 3$, to calculate $\dim(E_k)$ and $\dim(F_k)$ it is necessary to take into account all the relations among the elements of $\mathcal{G}$ arising from (ee) and (a). Of course, to prove that $\dim(E_k) \leq \dim(F_k)$, it is enough to prove that, passing from (ee) to (a), no new relations are introduced. It can be shown that this is true by a simple case by case examination.

In the following Example 3, we will illustrate how the above proof works. Applications of Theorem 3 will be explained later, in Examples 4 and 5.

Example 3. Let us consider a rational smooth curve C of degree 10 in $\mathbb{P}^6$ given by a matrix M as follows ($*$ denotes any complex number, blank denotes 0):

$$M = \begin{bmatrix} 1 & * & * & * & * & * & * & * & * & * & * \\ & 1 & * & * & * & * & * & * & * & * & * \\ & & 1 & * & * & * & * & * & * & * & * \\ & & & & & 1 & * & * & * & * & * \\ & & & & & & 1 & * & * & * & * \\ & & & & & & & & & 1 & * \\ & & & & & & & & & & 1 \end{bmatrix}$$

then $T_{CA} = \langle x^7y^3, x^6y^4, x^3y^7, x^2y^8\rangle$, $e + 1 = 4$; while T_C is generated by

$$\begin{aligned} \tau_0 &= *x^{10} + *x^9y + *x^8y^2 + x^7y^3, \\ \tau_1 &= *x^{10} + *x^9y + *x^8y^2 + x^6y^4, \\ \tau_2 &= *x^{10} + *x^9y + *x^8y^2 + *x^5y^5 + *x^4y^6 + x^3y^7, \\ \tau_3 &= *x^{10} + *x^9y + *x^8y^2 + *x^5y^5 + *x^4y^6 + x^2y^8. \end{aligned}$$

We have that $\partial^2 T_{CA} = \langle x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8\rangle$, $\delta = 8$ and the monomials involved by $\partial^2 T_C$ are: $x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8$. We have to forget x^8 and to consider the relations given by the other 8 monomials. The matrix $N = N^0 \cup N^1 \cup N^2 \cup N^3$ is of type $(8 = \delta, 12 = 3(e + 1))$ and it is the union of 4 submatrices of type $(8, 3)$ ($\natural_j$ denotes a non zero complex number),

$$N = \begin{bmatrix} \natural_1 & * & * & * & * & * & * & * & * & * & * & * \\ & \natural_2 & * & \natural_4 & * & * & * & * & * & * & * & * \\ & & \natural_3 & & \natural_5 & * & * & * & * & * & * & * \\ & & & & & \natural_6 & * & * & * & * & * & * \\ & & & & & & \natural_7 & * & * & * & * & * \\ & & & & & & & \natural_8 & * & \natural_{10} & * & * \\ & & & & & & & & \natural_9 & & \natural_{11} & * \\ & & & & & & & & & & & \natural_{12} \end{bmatrix},$$

$$N' = \begin{bmatrix} \natural_1 & & & * & * & & & & & * & * & \\ & \natural_2 & & \natural_4 & * & & & & & * & * & \\ & & \natural_3 & & \natural_5 & & & & & * & * & \\ & & & & & \natural_6 & & & & * & * & \\ & & & & & & \natural_7 & & & * & * & \\ & & & & & & & \natural_8 & & \natural_{10} & * & \\ & & & & & & & & \natural_9 & & \natural_{11} & \\ & & & & & & & & & & & \natural_{12} \end{bmatrix}.$$

On the other hand NA is the following:

$$\begin{bmatrix}
\natural_1 & & & & & & & & & & & \\
 & \natural_2 & & \natural_4 & & & & & & & & \\
 & & \natural_3 & & \natural_5 & & & & & & & \\
 & & & & & \natural_6 & & & & & & \\
 & & & & & & \natural_7 & & & & & \\
 & & & & & & & \natural_8 & & \natural_{10} & & \\
 & & & & & & & & \natural_9 & & \natural_{11} & \\
 & & & & & & & & & & & \natural_{12}
\end{bmatrix}.$$

From N' we get

$$\begin{bmatrix} \partial_x\partial_x f_0 \\ \partial_x\partial_y f_0 \\ \partial_y\partial_y f_0 \\ \partial_y\partial_y f_1 \\ \partial_x\partial_x f_2 \\ \partial_x\partial_y f_2 \\ \partial_y\partial_y f_2 \\ \partial_y\partial_y f_3 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \natural_{13} & \varepsilon & \zeta & \eta \\ & \natural_{14} & \vartheta & \iota \\ & & \rho & \lambda \\ & & \mu & \upsilon \\ & & \natural_{15} & \xi \\ & & & \natural_{16} \\ & & & \end{bmatrix} \begin{bmatrix} \partial_x\partial_x f_1 \\ \partial_x\partial_y f_1 \\ \partial_x\partial_x f_3 \\ \partial_x\partial_y f_3 \end{bmatrix}.$$

From NA we get analogous relations where every greek letter is zero.

Now, let us choose $k = 3$, so that $f_p = a_p x^3 + 3b_p x^2 y + 3c_p x y^2 + d_p y^3$ and $\partial_x\partial_x f_p = 6(a_p x + b_p y)$ and so on. In this case, $\mathscr{G} = \{a_1, b_1, c_1, a_3, b_3, c_3\}$. By dividing all polynomials by 6 we can write all the above relations as

$$\begin{bmatrix} a_0 & b_0 \\ b_0 & c_0 \\ c_0 & d_0 \\ c_1 & d_1 \\ a_2 & b_2 \\ b_2 & c_2 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \natural_{13} & \varepsilon & \zeta & \eta \\ & \natural_{14} & \vartheta & \iota \\ & & \rho & \lambda \\ & & \mu & \upsilon \\ & & \natural_{15} & \xi \\ & & & \natural_{16} \\ & & & \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \\ a_3 & b_3 \\ b_3 & c_3 \end{bmatrix}.$$

We get the following relations:

$$\begin{aligned}
a_0 &= \alpha a_1 + \beta b_1 + \gamma a_3 + \delta b_3, \\
b_0 &= \alpha b_1 + \beta c_1 + \gamma b_3 + \delta c_3, \\
b_0 &= \natural_{13} a_1 + \varepsilon b_1 + \zeta a_3 + \eta b_3, \\
c_0 &= \natural_{13} b_1 + \varepsilon c_1 + \zeta b_3 + \eta c_3, \\
c_0 &= \natural_{14} b_1 + \vartheta a_3 + \iota b_3,
\end{aligned}$$

$$
\begin{aligned}
d_0 &= \natural_{14} c_1 + \vartheta b_3 + \iota c_3,\\
c_1 &= \rho a_3 + \lambda b_3,\\
d_1 &= \rho b_3 + \lambda c_3,\\
a_2 &= \mu a_3 + \nu b_3,\\
b_2 &= \mu b_3 + \nu c_3,\\
b_2 &= \natural_{15} a_3 + \xi b_3,\\
c_2 &= \natural_{15} b_3 + \xi c_3,\\
c_2 &= \natural_{16} b_3,\\
d_2 &= \natural_{16} c_3,\\
c_3 &= d_3 = 0.
\end{aligned}
$$

It is easy to see that $\varphi_{CA}(3) = \varphi_C(3) = 0$ if $\natural_{15} \neq \natural_{16}$ and $\natural_{13} \neq \natural_{14}$. If $\natural_{15} = \natural_{16}$ but $\natural_{13} \neq \natural_{14}$ then $\varphi_{CA}(3) = \varphi_C(3) = 1$. If $\natural_{15} = \natural_{16}$ and $\natural_{13} = \natural_{14}$ then $\varphi_{CA}(3) = 2$ while for E_3 we have two generators with a relation at most, hence $\varphi_C(3) \leq 2$ and we have $\varphi_C(3) \leq \varphi_{CA}(3)$ in any case.

In general, to get $\varphi_C(3)$ we should know the exact values of the entries of M, but in Example 3 this is not important: the Coskun–Riedl formula proves that $\varphi_{CA}(3) = 0$ a priori. Therefore we can conclude that $\varphi_C(3) = 0$ for any curve C as above.

Remark 1. Unfortunately, it is not possible to get a good bound for $\varphi_C(k)$ from below: for any k, it is easy to count how many generators and relations are necessary to define $\ker(D^2_{k|S^k U \otimes T_C})$ inside $\mathbb{C}^{(e+1)(k+1)}$, but every relation can provide a big number of linear equations for $\ker(D^2_{k|S^k U \otimes T_C})$ and it is hard to determine a reasonable bound for the independent ones. On the other hand, if we consider all of them, we have that the bound from below becomes quickly a negative number, as k increases.

Remark 2. It is very natural to ask whether it is possible to extend the above sketched proof to curves C not complete, when CA is smooth of degree $d' < d$. However this is not possible. It is easy to give counterexamples.

4. Applications

The immediate application of Theorem 3 is the following:

Corollary 1. *Let C be a complete, smooth, rational curve of degree d in $\mathbb{P}^s(\mathbb{C})$ and let CA be the associated smooth rational monomial curve as before, with normal bundles $\mathcal{N}_C$ and $\mathcal{N}_{CA}$, respectively. Let $f_C : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$ and $f_{CA} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$*

be the related morphisms. Let $\varphi_C(k)$ and $\varphi_{CA}(k)$ be the two functions introduced in Section 2 for any integer $k \geq 0$. Then

(i) *if $\varphi_{CA}(k) = 0$ for $k \geq k_0$ (k_0 suitable integer) then $\varphi_C(k) = 0$ for $k \geq k_0$;*

(ii) *if $\Delta^2\varphi_{CA}(k) = 0$ for $k \geq k_0$ (k_0 suitable integer) then $\Delta^2\varphi_C(k) = 0$ for $k \geq k_0$;*

(iii) *assume that $f^*_{CA}\mathcal{N}_{CA} \simeq \mathcal{O}_{\mathbb{P}^1}(\xi'_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\xi'_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\xi'_{s-1})$ and let us put $\mu := \max\{\xi'_1, \ldots, \xi'_{s-1}\}$, then $f^*_C\mathcal{N}_C \simeq \mathcal{O}_{\mathbb{P}^1}(\xi_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\xi_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\xi_{s-1})$ with $\xi_i \leq \mu$ for any $i = 1, \ldots, s-1$;*

(iv) *the natural multiplication map*

$$H^0(C, \mathcal{O}_C(v-1)) \otimes H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)) \to H^0(C, \mathcal{O}_C(v))$$

is surjective for any integer $v \geq \mu - 1$.

Proof. (i) and (ii) follow directly by Theorem 3.

(iii) For a suitable integer $k_0 \gg 0$ it is surely true that $\Delta^2\varphi_{CA}(k) = 0$ for $k \geq k_0$; let us assume that k_0 is the minimal integer with this property. Recall that

$$f^*_{CA}\mathcal{N}_{CA}(-d-2) = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c'_i),$$

with $c'_1 \geq c'_2 \geq \cdots \geq c'_{s-1}$, and that $\Delta^2[\varphi_{CA}(k)]$ is exactly the number of integers c'_i which are equal to k. Hence, if $\Delta^2\varphi_{CA}(k) = 0$ for $k \geq k_0$, we have that $c'_1 = k_0 - 1$ and $\mu = k_0 + d + 1$. By (ii) we have that $\Delta^2\varphi_C(k) = 0$ for $k \geq k_0$. Recall that

$$f^*_C\mathcal{N}_C(-d-2) = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i),$$

with $c_1 \geq c_2 \geq \cdots \geq c_{s-1}$, and that $\Delta^2[\varphi_C(k)]$ is exactly the number of integers c_i which are equal to k. Hence $c_1 \leq k_0 - 1$ and $\xi_i = c_i + d + 2 \leq k_0 + d + 1 = \mu$ for any $i = 1, \ldots, s-1$.

(iv) For any integer $v \geq 1$, let us recall the following exact sequence due to Ein (see [5, Theorem 2.4]):

$$0 \to \mathcal{N}^*_C(v) \to \mathcal{O}_C(v-1) \otimes H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)) \to \mathcal{P}^1[\mathcal{O}_C(v)] \to 0$$

where $\mathcal{N}^*_C$ is the dual of $\mathcal{N}_C$ and $\mathcal{P}^1[\mathcal{O}_C(v)]$ denotes the principal parts bundle of $\mathcal{O}_C(v)$. If $h^1(C, \mathcal{N}^*_C(v)) = 0$ we have that

$$H^0(C, \mathcal{O}_C(v-1)) \otimes H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)) \to H^0(C, \mathcal{P}^1[\mathcal{O}_C(v)])$$

is surjective. On the other hand, $H^0(C, \mathcal{P}^1[\mathcal{O}_C(v)] \to H^0(C, \mathcal{O}_C(v))$ is always surjective (see [5, Proposition 2.3]). Hence the natural multiplication map is surjective if $h^1(C, \mathcal{N}^*_C(v)) = 0$.

By Serre duality $h^1(C, \mathcal{N}_C^*(v)) = h^0(C, \mathcal{N}_C(-v-2))$, so that $h^1(C, \mathcal{N}_C^*(v)) = 0$ if $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\xi_i - v - 2)) = 0$ for any $i = 1, \dots, s-1$, i.e., $\xi_i \le v + 1$ for any $i = 1, \dots, s-1$ and this is true if $v \ge \mu - 1$ by (iii). ∎

Now we give two examples of application of Theorem 3 to find bounds for the splitting type of rational curves. We will choose two monomial curves and we will find bounds for the values of the numbers c_i for all complete curves C whose associated curves CA are the chosen ones.

Example 4. Let us choose $d = 17$, $e = 7$, $s = d - e - 1 = 9$ and let CA be the projection to $\mathbb{P}^8(\mathbb{C})$ of the rational normal curve Γ_{17} from $L := \mathbb{P}^8(T_{CA})$ where $T_{CA} := \langle x^{15}y^2, x^{12}y^5, x^9y^8, x^8y^9, x^5y^{12}, x^4y^{13}, x^3y^{14}, x^2y^{15}\rangle$. CA is a monomial smooth rational curve and, by using the results of [3], it is easy to see that the function $\varphi_{CA}(k)$ has the following values for $k \ge 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\varphi_{CA}(k)$	24	16	8	4	2	0	0	0	$\cdots$

hence the string of integers c_i for CA is the following: $(4, 4, 2, 2, 1, 1, 1, 1)$.

Assume that CA is the associated monomial curve to a smooth rational curve C of degree 17 in $\mathbb{P}^8(\mathbb{C})$. Assume also that $\varphi_C(2) = \varphi_{CA}(2)$. By Theorem 3 we can say that the function $\varphi_C(k)$, a priori, has the following values for $k \ge 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\varphi_C(k)$	24	16	8	ε	η	0	0	0	$\cdots$

with $0 \le \varepsilon \le 4$ and $0 \le \eta \le 2$. Hence the function $\Delta^2\varphi_C(k)$ has the following values, for $k \ge 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\Delta^2\varphi_C(k)$	0	ε	$8 - 2\varepsilon + \eta$	$\varepsilon - 2\eta$	η	0	0	0	$\cdots$

As $\Delta^2\varphi_C(k) \ge 0$ we get $8 - 2\varepsilon + \eta \ge 0$ and $\varepsilon - 2\eta \ge 0$.

By considering all the constraints, we have that the possible strings of c_i for C are

$$(4, 4, 2, 2, 1, 1, 1, 1),$$
$$(4, 3, 3, 2, 1, 1, 1, 1),$$
$$(3, 3, 3, 3, 1, 1, 1, 1),$$
$$(4, 3, 2, 2, 2, 1, 1, 1),$$
$$(3, 3, 3, 2, 2, 1, 1, 1),$$
$$(4, 2, 2, 2, 2, 2, 1, 1),$$
$$(3, 3, 2, 2, 2, 2, 1, 1),$$
$$(3, 2, 2, 2, 2, 2, 2, 1),$$
$$(2, 2, 2, 2, 2, 2, 2, 2).$$

Note that, according to the sufficient condition stated in [4, Corollary 2.6], all above cases are possible.

Example 5. Let us choose $d = 17$, $e = 6$, $s = d - e - 1 = 10$ and let CA be the projection to $\mathbb{P}^8(\mathbb{C})$ of the rational normal curve Γ_{17} from $L := \mathbb{P}^8(T_{CA})$ where $T_{CA} := \langle x^{15}y^2, x^{12}y^5, x^9y^8, x^8y^9, x^4y^{13}, x^3y^{14}, x^2y^{15}\rangle$. CA is a monomial smooth rational curve and, by using the results of [3], it is easy to see that the function $\varphi_{CA}(k)$ has the following values for $k \geq 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\varphi_{CA}(k)$	23	14	6	2	0	0	0	0	$\cdots$

hence the string of integers c_i for CA is the following: $(3, 3, 2, 2, 1, 1, 1, 1, 0)$.

Assume that CA is the associated monomial curve to a smooth rational curve C of degree 17 in $\mathbb{P}^9(\mathbb{C})$. Assume also that $\varphi_C(2) = \varphi_{CA}(2)$. By Theorem 3 we can say that the function $\varphi_C(k)$, a priori, has the following values for $k \geq 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\varphi_C(k)$	23	14	6	ε	0	0	0	0	$\cdots$

with $0 \leq \varepsilon \leq 2$. Hence the function $\Delta^2\varphi_C(k)$ has the following values for $k \geq 0$:

k	0	1	2	3	4	5	6	7	$\cdots$
$\Delta^2\varphi_C(k)$	1	$2+\varepsilon$	$6-2\varepsilon$	ε	0	0	0	0	$\cdots$

The possible strings of c_i for C are

$$(3, 3, 2, 2, 1, 1, 1, 1, 0),$$
$$(3, 2, 2, 2, 2, 1, 1, 1, 0),$$
$$(2, 2, 2, 2, 2, 2, 1, 1, 0).$$

Note that, according to the sufficient condition stated in [4, Corollary 2.6], all above cases are possible.

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