Port. Math. (Online first)
DOI 10.4171/PM/2091

A note on the real support of Silva–Hasumi–Morimoto ultrahyperfunctions

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Abstract. Recently, we have studied the relation between the singular spectrum of a class of tempered ultrahyperfunctions corresponding to proper convex cones and their expressions as boundary values of holomorphic functions. In this note, we added some more results on the singular spectrum of the tempered ultrahyperfunctions. More specifically, the "microlocalization" of a Silva–Hasumi–Morimoto ultrahyperfunction corresponding to an open proper convex cone is addressed. Following Nishiwada's approach to the analytic wave front set, we introduce the real singular spectrum of such an ultrahyperfunction and prove that it canonically projects onto Silva's real support.

1. Introduction

In 1958, Sebastião e Silva [19, 20] introduced the class of generalized functions called by him tempered ultradistributions, which are the Fourier image of L. Schwartz distributions of exponential type. Later, Hasumi [10] considered the global theory of tempered ultrahyperfunctions in the higher dimensional space, and Morimoto [12–14] (who coined the name tempered ultrahyperfunctions) localized the theory of tempered ultrahyperfunctions in the imaginary direction.

The interest in ultrahyperfunctions arose simultaneously with the growing interest in various classes of analytic functionals and various attempts to develop a theory of such functionals that would be analogous to the Schwartz theory of distributions. In the first decade of this century, a renewed interest in ultrahyperfunctions appeared in the Brüning–Nagamachi papers [1–3, 15] on quantum field theory with a fundamental length. In connection with this theme, an important topic in the spectral analysis of a quantum field theory with a fundamental length concerns the singularities of the ultrahyperfunctions.

2020 Mathematics Subject Classification. Primary 46F20; Secondary 46F12, 46F15, 81T05, 81T10.
Keywords. Singular spectrum, tempered ultrahyperfunctions.

Using the cohomological approach, Morimoto gave a clear description of the cotangential components of singularities of ultrahyperfunctions in [11] (the ultrahyperfunctions were called in [11] cohomological ultradistributions), not necessarily assumed to be tempered. Following Nishiwada [16, 17], in [9] (where many other references on ultrahyperfunctions can be found) the authors have shown an alternative way of describing the singularities of a class of tempered ultrahyperfunctions corresponding to proper convex cones in terms of generalized boundary values of holomorphic functions; namely, the singular spectrum of a tempered ultrahyperfunction u is characterized by the directions from which the boundary values can be taken in an analytic representation of u. In this article, without using cohomology, we added some more results on the singular spectrum of the tempered ultrahyperfunctions. More specifically, using the notion of real support introduced by Sebastião e Silva [19], we show that the canonical projection of the singular support of a tempered ultrahyperfunction coincides with its real support.

2. Tempered ultrahyperfunctions in very few words

To begin with, we shall recall very briefly the basic definition of tempered ultrahyperfunctions. Firstly, we shall consider the function

$$h_K(\xi) = \sup_{x\in K} \bigl|\langle \xi, x\rangle\bigr|, \quad \xi \in \mathbb{R}^n,$$

the indicator of K, where K is a compact set in $\mathbb{R}^n$. We have $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = [-k,k]^n$, $0 < k < \infty$, the indicator function $h_K(\xi)$ can be easily determined,

$$h_K(\xi) = \sup_{x\in K} \bigl|\langle \xi, x\rangle\bigr| = k|\xi|, \quad \xi \in \mathbb{R}^n, \quad |\xi| = \sum_{i=1}^{n} |\xi_i|.$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha \varphi(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n; K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\xi\in\mathbb{R}^n;\alpha\leqslant N} \bigl\{e^{h_K(\xi)} |D^\alpha \varphi(\xi)|\bigr\} < \infty, \quad N = 0, 1, 2, \ldots \tag{2.1}$$

The space $H_b(\mathbb{R}^n; K)$ equipped with the topology given by the seminorms (2.1) is a Fréchet space [10, 13]. If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leqslant h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n; K_2) \hookrightarrow H_b(\mathbb{R}^n; K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n; O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \ldots$

contained in O such that for each $i = 1, 2, \ldots, K_i \subset K^\circ_{i+1}$ (K°_{i+1} denotes the interior of K_{i+1}) and $O = \bigcup_{i=1}^{\infty} K_i$. Then the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ according to restriction mappings above, i.e.,

$$H(\mathbb{R}^n; O) = \lim_{K \subset O} \operatorname{proj} H_b(\mathbb{R}^n; K), \tag{2.2}$$

where K runs through the convex compact sets contained in O. By $H'(\mathbb{R}^n; O)$ we denote the dual space of $H(\mathbb{R}^n; O)$.

Proposition 2.1 (Hasumi [10, Proposition 3], Morimoto [13, Theorem 5]). *A distribution $V \in H'(\mathbb{R}^n; O)$ may be expressed as a finite order derivative of a continuous function of exponential growth*

$$V = D^\gamma_\xi [e^{h_K(\xi)} g(\xi)],$$

where $g(\xi)$ is a bounded continuous function.

In the space $\mathbb{C}^n$ of n complex variables $z_\kappa = x_\kappa + i y_\kappa$, $1 \leqslant \kappa \leqslant n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, i.e., Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \leqslant \mathsf{M}_{K,N}(\varphi)(1 + |z|)^{-N} \tag{2.3}$$

is valid. The best possible constants in (2.3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \inf\bigl\{\mathsf{M}_{K,N}(\varphi) \mid \sup_{z \in T(K)} (1 + |z|)^N |\varphi(z)| < \infty, N = 0, 1, 2, \ldots\bigr\}. \tag{2.4}$$

If $K_1 \subset K_2$ are two convex compact sets, we have that the canonical injection

$$\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1)) \tag{2.5}$$

is continuous.

Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, with topology defined by the seminorms (2.4), the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces

$$\mathfrak{H}(T(O)) = \lim_{K \subset O} \operatorname{proj} \mathfrak{H}_b(T(K)), \tag{2.6}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

Let K be a convex compact set in $\mathbb{R}^n$. Then the space $\mathfrak{H}(T(K))$ is characterized as an inductive limit

$$\mathfrak{H}(T(K)) = \liminf_{K_1 \supset K} \mathfrak{H}_b(T(K_1)), \tag{2.7}$$

where K_1 runs through the convex compact sets such that K is contained in the interior of K_1 and the inductive limit is taken following the restriction mappings (2.5).

The Fourier transformation is well defined on the space $H(\mathbb{R}^n; O)$. Further, if $\varphi \in H(\mathbb{R}^n; O)$, the Fourier transform of φ belongs to the space $\mathfrak{H}(T(O))$, for any open convex nonempty set $O \subset \mathbb{R}^n$. By the dual Fourier transform, $H'(\mathbb{R}^n; O)$ is topologically isomorphic to the space $\mathfrak{H}'(T(-O))$ [13].

Remark 1. We will put $\mathfrak{H} = \mathfrak{H}(\mathbb{C}^n) = \mathfrak{H}(T(\mathbb{R}^n))$ and, as usual, we shall denote the dual space of $\mathfrak{H}$ by $\mathfrak{H}'$.

Definition 2.2. A tempered ultrahyperfunction is by definition a continuous linear functional on $\mathfrak{H}$.

In this note, we are interested in the class of tempered ultrahyperfunctions corresponding to proper convex cones. Therefore, we start by recalling some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if C (unless specified otherwise, all cones will have their vertices at zero) is invariant under positive homoteties, i.e., if for all $\lambda > 0$, $\lambda C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it does not contain any straight line (observe that if C is a proper cone, it follows that if $y \in C$ and $y \neq 0$ then $-y \notin C$). A cone C' is called compact in C – we write $C' \Subset C$ – if the projection

$$\mathsf{pr}\bar{C}' \overset{\text{def}}{=} \bar{C}' \cap S^{n-1} \subset \mathsf{pr}C \overset{\text{def}}{=} C \cap S^{n-1},$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in the y-space, we associate with C a closed convex cone C^* in the ξ-space which is the set

$$C^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, y \rangle \geqslant 0, \forall y \in C\}.$$

The cone C^* is called the *dual cone* of C. As in Carmicheal [4, 5], we define the following.

Definition 2.3. Let C be a proper open convex cone with vertex at the origin, and let $C' \Subset C$. Let $B[0; r]$ denote a *closed* ball with center at the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. We define the tube domain by

$$T^{(C' \setminus (C' \cap B[0;r]))} = \{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in (C' \setminus (C' \cap B[0;r]))\}.$$

We are going to introduce a space of holomorphic functions which satisfy certain estimates according to Carmichael [4,5]. We want to consider the space consisting of holomorphic functions f such that

$$\big|f(z)\big| \leqslant \mathsf{M}(C')(1+|z|)^N e^{\sigma|y|}, \quad z = x+iy \in T^{(C'\setminus(C'\cap B[0;r]))}, \tag{2.8}$$

for all $\sigma > 0$, where $\mathsf{M}(C')$ is a constant depending on an arbitrary compact cone $C' \Subset C$ and N is a non-negative real number. The set of all functions f which for every cone $C' \Subset C$ are holomorphic in $T^{(C'\setminus(C'\cap B[0;r]))}$ and satisfy the estimate (2.8) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Remark 2. The space of functions $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ constitutes a generalization of the space $\mathfrak{A}_\omega^i$ of Sebastião e Silva [19] and the space $\mathfrak{A}_\omega$ of Hasumi [10] to arbitrary tubular radial domains in $\mathbb{C}^n$ (a tube domain is said to be radial if its base is a connected cone in $\mathbb{R}^n$).

We now shall introduce another space of holomorphic functions whose elements are analytic in a domain that is larger than $T^{(C'\setminus(C'\cap B[0;r]))}$ and has boundary values in $\mathbb{R}^n$. Let $B(0;r)$ denote an *open* ball with center at the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Let $T^{(C'\setminus(C'\cap B(0;r)))}$ denote the subset of $\mathbb{C}^n$ defined by

$$T^{(C'\setminus(C'\cap B(0;r)))} = \big\{x+iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in \big(C' \setminus (C' \cap B(0;r))\big)\big\}. \tag{2.9}$$

Definition 2.4. Let C be a proper open convex cone with vertex at the origin, and let $C' \Subset C$. Denote by $T(C')$ the tube domain

$$\big\{x+iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C'\big\}.$$

We say that the function $f = f(z)$ is in the space $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ if it is holomorphic in $T(C')$ and satisfies the estimate

$$\big|f(z)\big| \leqslant \mathsf{M}(C')(1+|z|)^N e^{\sigma|y|}, \quad z = x+iy \in T^{(C'\setminus(C'\cap B(0;r)))}. \tag{2.10}$$

Note that $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}} \subset \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ for any open convex cone C. Following Hasumi [10], we define the *kernel* of the mapping $f : \mathfrak{H} \to \mathbb{C}$ by $\boldsymbol{\Pi}$, where $\boldsymbol{\Pi}$ is the set of all pseudo-polynomials in one of the variables $z_1, \ldots, z_n$. We recall that a pseudo-polynomial is a function of the form

$$\sum_\alpha z_j^\alpha G_\alpha(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n),$$

where $G_\alpha(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ are functions in $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ with respect to $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$. Then, the function $f \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ belongs to the kernel $\boldsymbol{\Pi}$ if and only if $\langle f(z), \psi(x)\rangle = 0$, with $\psi \in \mathfrak{H}$ and $x = \operatorname{Re} z$. Put $\mathscr{U}_C = \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}/\boldsymbol{\Pi}$, that is, $\mathscr{U}_C$ is the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ by the set of pseudo-polynomials $\boldsymbol{\Pi}$.

Definition 2.5. The set $\mathscr{U}_C$ is the subspace of the tempered ultrahyperfunctions generated by $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ corresponding to a proper open convex cone with vertex at the origin $C \subset \mathbb{R}^n$.

Definition 2.6. We denote by $H'_{C^*}(\mathbb{R}^n; O)$ the subspace of $H'(\mathbb{R}^n; O)$ of distributions of exponential growth with support in the cone C^*,

$$H'_{C^*}(\mathbb{R}^n; O) = \left\{ V \in H'(\mathbb{R}^n; O) \mid \operatorname{supp}(V) \subseteq C^* \right\}. \tag{2.11}$$

3. Analytic representations

In this section, the boundary values of holomorphic functions are always considered in the distribution sense defined below. We say that $f \in \mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ has a boundary value $U = BV(f)$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C' \Subset C$, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U, \psi \rangle = \lim_{\substack{y \to 0, \\ y \in C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x)\,dx$$

exists.

For distributions $V \in H'_{C^*}(\mathbb{R}^n; O)$ we define the Fourier–Laplace transform by

$$f(z) = (2\pi)^{-n} \langle V, e^{i\langle \cdot, z \rangle} \rangle. \tag{3.1}$$

Concerning the Fourier–Laplace (3.1) we have the following:

Lemma 3.1. *To each compact subcone C' of C corresponds a positive real number r such that the function $f(z)$ in (3.1) is analytic in the truncated tube domain*

$$T^{(C' \setminus (C' \cap B[0;r]))},$$

where $B[0;r]$ is the closed ball in $\mathbb{R}^n$, centered at the origin, and with radius r.

Proof. See Carmichael [4, Theorem 2]. ∎

Lemma 3.2. *The function $f(z)$ in (3.1) is analytic in the tube domain*

$$T^C = \mathbb{R}^n + iC.$$

Proof. Just take $\Gamma = C$ and $K = \{0\}$ in [21, Proposition 6.6]. ∎

Example 1. If $u(x) = \frac{1}{x+i0^+}$ $(x \in \mathbb{R})$ is the distributional boundary value, then it corresponds to the proper cone $C = (0, +\infty)$ and we have

$$f(z) = \frac{1}{z} \quad (z \neq 0),$$

which is analytic on $T^C = \mathbb{R} + iC = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

Remark 3. In Lemma 3.2, we cannot assure the analyticity of $f(z)$ in the tube domain T^C if the cone C is not proper. For instance, take $u = \delta$ (Dirac's delta "function"). We have that $u = \delta$ corresponds to the cone $C = \mathbb{R}$, since, up to a multiplicative constant, we have $\widehat{\delta} = 1$, whose support is $C = \mathbb{R}$. Notice, in this case, that

$$f(z) = \frac{1}{z} \quad (z \neq 0),$$

is not analytic in the tube domain $T^C = \mathbb{R} + i\mathbb{R} = \mathbb{C}$!

Lemma 3.3. *Let C be a proper open convex cone with vertex at the origin. Let $V \in H'_{C^*}(\mathbb{R}^n; O)$. Then there exists a function $f(z) \in \mathscr{H}_c^{*o}$ such that $f(z) \to \mathscr{F}[V] \in \mathfrak{H}'$ in the weak topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.*

Proof. See Carmichael [4, Theorem 2]. ∎

4. The real singular spectrum

In the previous section, we have examined the tempered ultrahyperfunctions as the boundary values of holomorphic functions. Thus, in particular, one may want to know the set in which a given tempered ultrahyperfunction corresponding to an open proper convex cone with vertex at the origin fails to be analytic; such a set, which can be easily defined, is the so called *singular spectrum* of a tempered ultrahyperfunction, denoted by $S.S.(u)$. Following Nishiwada [17, Theorem 1.2], we shall introduce the singular spectrum of a tempered ultrahyperfunction corresponding to an open proper convex cone with vertex at the origin $C \subset \mathbb{R}^n$, using its analytical representation, $f(z)$, induced by the Fourier–Laplace transform (3.1). Precisely, we have the following.

Definition 4.1. Let $u \in \mathscr{U}_C$ be a tempered ultrahyperfunction generated by $f \in \mathscr{H}_c^{*o}$ corresponding to an open proper convex cone – with vertex at the origin. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$ be arbitrarily fixed, where $T^*\mathbb{R}^n$ stands for the cotangent bundle. Then, $(x_0, \xi_0) \notin S.S.(u) \subset T^*\mathbb{R}^n \setminus \{0\}$ if and only if there exists an open complex neighborhood $\widetilde{U}$ of x_0 in $\mathbb{R}^n$ and a cone $C' \Subset C$ in $\mathbb{R}^n$ such that the analytic representation f of u fulfills the following:

(a) f is continuous over $\widetilde{U}$;

(b) f is holomorphic in the tube-symmetric domain

$$T^{(-C' \setminus (-C' \cap B(0;r)))} \cup T^{(C' \setminus (C' \cap B(0;r)))},$$

where $T^{(C' \setminus (C' \cap B(0;r)))}$ is defined by (2.9).

The following result is immediate from the above definition:

Proposition 4.2. *Let $u \in \mathscr{U}_C$ be a tempered ultrahyperfunction corresponding to an open proper convex cone with vertex at the origin $C \subset \mathbb{R}^n$. Then, the following properties are valid:*

(a) *$S.S.(u)$ is a closed subset of $T^*\mathbb{R}^n \setminus \{0\}$;*

(b) *$S.S.(u)$ is a conic subset of $T^*\mathbb{R}^n \setminus \{0\}$ in the sense that*

$$(x_0, \xi_0) \in S.S.(u) \Longrightarrow (x_0, \lambda\xi_0) \in S.S.(u), \quad \forall \lambda > 0.$$

Proof. Item (a). Let (x_0, ξ_0) be an arbitrary point in the complement of

$$\big(T^*\mathbb{R}^n \setminus \{0\}\big) \setminus S.S.(u).$$

Let C and $\widetilde{U}$ as in Definition 4.1. Assume that

$$x_0' \in (\widetilde{U} \cap \mathbb{R}^n) \setminus \{x_0\} \quad \text{and} \quad \xi_0' \in (C^* \setminus \{0\}) \setminus \{\xi_0\}.$$

By [6, Lemma 1.2.2] there exists a positive α (depending on y) such that $\langle \xi_0', y\rangle \geqslant \alpha|\xi_0'||y|$ for all $\xi_0' \in C^* \setminus \{0\}$ and $y \in C'$. This implies that $\xi_0' \in C^* \setminus \{0\}$ satisfies Definition 4.1. Now, for $\widetilde{U}$ as in Definition 4.1, it is clear that the analytic representation f of the tempered ultrahyperfunction u satisfies the items (a) and (b) of the same definition. Thus, according to Definition 4.1, we have that $(x_0', \xi_0') \notin S.S.(u)$. Hence, $(\widetilde{U} \cap \mathbb{R}^n) \times C'$ is an open neighborhood of $(x_0, \xi_0) \in (T^*\mathbb{R}^n \setminus \{0\}) \setminus S.S.(u)$. As (x_0, ξ_0) is arbitrary, it follows that $(T^*\mathbb{R}^n \setminus \{0\}) \setminus S.S.(u)$ is open, that is, it follows that $S.S.(u)$ is closed on $T^*\mathbb{R}^n \setminus \{0\}$. This proves item (a).

Item (b). Our reasoning is based on reduction to absurd. Suppose that for some $(x_0, \xi_0) \in S.S.(u)$ there corresponds some $\lambda > 0$ such that

$$(x_0, \lambda\xi_0) \notin S.S.(u).$$

In this case, from the definition of $S.S.(u)$, we would be able to conclude the absurd that

$$(x_0, \xi_0) = \Big(x_0, \frac{1}{\lambda} \cdot (\lambda\xi_0)\Big) \notin S.S.(u),$$

because, due to the hypothesis $(x_0, \lambda\xi_0) \notin S.S.(u)$, we would conclude that $\lambda\xi_0$ is contained in a conical neighborhood C^* such that $\{x_0\} \times C^*$ does not intersect $S.S.(u)$ and from this we would conclude that

$$\Big(x_0, \frac{1}{\lambda} \cdot (\lambda\xi_0)\Big) \notin S.S.(u), \quad \forall \lambda > 0.$$

Item (b) follows. ∎

We also recall the notion of *real support* of a tempered ultrahyperfunction u, introduced by Sebastião e Silva [19] (see also Section 4.5 of the Master dissertation by Debrouwere [7], and [8] which contains a detailed study of generalizations of the Silva's real support that are applicable to more general ultrahyperfunctions).

Definition 4.3. Let $u \in \mathscr{U}_C$ be a tempered ultrahyperfunction corresponding to an open proper convex cone with vertex at the origin $C \subset \mathbb{R}^n$. Then, its real support $\mathrm{supp}_{\mathbb{R}}(u)$ is the complement in $\mathbb{R}^n$ of the largest open set $\Omega \subset \mathbb{R}^n$ for which the analytic representation $\mathscr{H}_c^{*o} \ni f(z)$ of u is analytic in the tube domain $\Omega + i\mathbb{R}^n \subset \mathbb{C}^n$.

Remark 4. Unlike distributions or hyperfunctions, tempered ultrahyperfunctions do not have a sound notion of singular support. As matter of fact, the notion carrier of a tempered ultrahyperfunction has a major drawback as far as uniqueness is concerned, since in some examples the least carrier of an analytical functional is not available. With this in mind, we must understand the reason for the real support in the previous definition to be confined to the real part of $\mathbb{C}^n$ (Figure 1 can help visualizing this situation for $x \in \mathrm{supp}_{\mathbb{R}}(u)$, in the case $n = 1$).

Theorem 4.4. *Let $u \in \mathscr{U}_C$ be a tempered ultrahyperfunction corresponding to an open proper convex cone with vertex at the origin $C \subset \mathbb{R}^n$. Let $\pi : T^*\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ be the canonical projection. Then, the following identity holds:*

$$\pi(S.S.(u)) = \mathrm{supp}_{\mathbb{R}}(u).$$

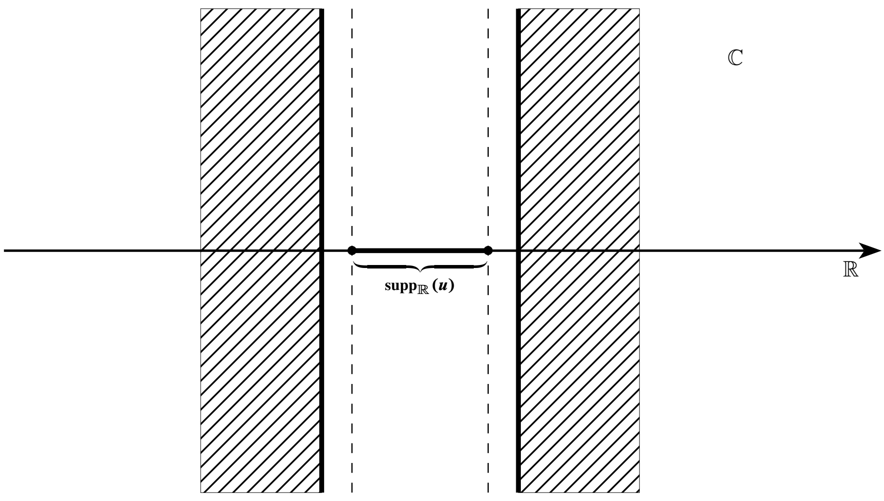

Figure 1. The real support $\mathrm{supp}_{\mathbb{R}}(u)$ of a tempered ultrahyperfunction u in the case $n = 1$.

Proof. Firstly, let $(x_0,\xi_0) \notin S.S.(u)$ and $\widetilde{U}$ and C' be as in Definition 4.1. By the continuous version of the Edge-of-the-Wedge theorem [18], we conclude that the analytic representation $f \in \mathscr{H}_c^{*o}$ of u is holomorphic in an open complex neighborhood W of x_0 which, without loss of generality, may be supposed to be contained in $\widetilde{U}$. Thus, f is holomorphic in the open set

$$W \cup T^{(-C'\setminus(-C'\cap B(0;r)))} \cup T^{(C'\setminus(C'\cap B(0;r)))} \subset \mathbb{C}^n,$$

from which we are able to select an open tube domain $\Omega + i\mathbb{R}^n$ $(x_0 \in \Omega \subset W)$ where f is holomorphic. In particular, we conclude that $x_0 \notin \operatorname{supp}_{\mathbb{R}}(u)$. Thus, we have proved that

$$(x_0,\xi_0) \notin S.S.(u) \Longrightarrow (x_0,\xi_0) \notin \operatorname{supp}_{\mathbb{R}}(u),$$

from which we conclude that

$$\operatorname{supp}_{\mathbb{R}}(u) \subset \pi(S.S.(u)).$$

Conversely, Let $(x_0,\xi_0) \in S.S.(u)$. By absurd, suppose that for all $x_0 \notin \operatorname{supp}_{\mathbb{R}}(u)$, then by the definition of real support, the analytic representation f of u is holomorphic in a tube domain of the type $\Omega + i\mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is open and contains x_0. In this case, taking $\widetilde{U} = \Omega + i\mathbb{R}^n$ and $C' \Subset C$ as in Definition 4.1 which implies the absurd conclusion that $(x_0,\xi_0) \notin S.S.(u)$. Thus, if $(x_0,\xi_0) \in S.S.(u)$, then $x_0 \in \operatorname{supp}_{\mathbb{R}}(u)$, that is, we must have

$$\pi(S.S.(u)) \subset \operatorname{supp}_{\mathbb{R}}(u).$$

The proof is complete. ■

Acknowledgments. The authors thank the referee whose careful reading and valuable remarks on the manuscript undoubtedly led to a better presentation of the paper.

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Received 28 October 2022.

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