Port. Math. (Online first)
DOI 10.4171/PM/2103

Published by EMS Press

Theoretical analysis of a discrete population balance model with sum kernel

Sonali Kaushik, Rajesh Kumar, and Fernando P. da Costa

Abstract. The Oort–Hulst–Safronov equation is a relevant population balance model. Its discrete form, developed by Pavel Dubovski, is the main focus of our analysis. The existence and density conservation are established for non-negative symmetric coagulation rates satisfying $V_{i,j} \leqslant i + j$, $\forall i, j \in \mathbb{N}$. Differentiability of the solutions is investigated for kernels with $V_{i,j} \leqslant i^\alpha + j^\alpha$ where $0 \leqslant \alpha \leqslant 1$ with initial conditions with bounded $(1+\alpha)$-th moments. The article ends with a uniqueness result under an additional assumption on the coagulation kernel and the boundedness of the second moment.

1. Introduction

The *coagulation* is defined as the process when clusters of mass i and j (i and j-mers) merge together to generate a $(i+j)$-mer. Coagulation processes have a plethora of real-world applications, including the collision of asteroids [15], red blood cell aggregation [14], helium bubble formation in nuclear materials [5], colloidal chemistry [1], formation of Saturn's rings [6], among many others.

This paper discusses a discrete model, i.e., a model for which the properties of the particles, namely size, are described by a discrete variable $i \in \mathbb{N}$, known as the Safronov–Dubovski (S-D) coagulation equation [8]. The equation seems to have been first proposed by Dubovski [10, 11] in 1999, long after the introduction of the continuous version, called the Oort–Hulst–Safronov (OHS) equation [13, 16], in the context of coagulation of particles in the celestial phenomena.

The S-D model is defined for $t \in [0,\infty)$ as

$$\frac{d\psi_i(t)}{dt} = \psi_{i-1}(t)\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(t) - \psi_i(t)\sum_{j=1}^{i} jV_{i,j}\psi_j(t) - \sum_{j=i}^{\infty} V_{i,j}\psi_i(t)\psi_j(t), \qquad i \in \mathbb{N}, \tag{1.1}$$

2020 Mathematics Subject Classification. Primary 34A12; Secondary 35Q70, 45K05, 47J35.
Keywords. Discrete population balance model, Safronov–Dubovski coagulation equation, Oort–Hulst–Safronov equation, existence of solutions, conservation of mass, differentiability.

where the first sum on the right-hand side is defined to be identically zero when $i = 1$. The focus of our study is on the basic properties (existence, uniqueness, density conservation, differentiability) of solutions $\psi_i(t)$ to the initial value problems defined by this system with initial conditions

$$\psi_i(0) = \psi_{0i} \geqslant 0. \tag{1.2}$$

From a physical point of view, the equations in system (1.1) are the rate equations describing the time-dependent behavior of the concentrations of clusters of sizes (or masses) i, (i-clusters, for short) in a system of particles whose dynamics can be described as follows: (a) particles of size $i \geqslant 2$ are produced when a $(i-1)$-cluster is struck by a particle of size $j \leqslant i-1$: the result of this collision is that the smaller j-cluster is pulverized into j particles of size 1, each of which attaches itself to different particles of size $i-1$ to form particles of size i; this is described mathematically by the first expression in the right part of (1.1)

$$\psi_{i-1}(t) \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j(t);$$

(b) particles of size i are destroyed either by being impacted by smaller clusters and thus growing to clusters of size $i+1$, by the mechanism just described, resulting in the second term on the right-hand side of (1.1)

$$-\psi_i(t) \sum_{j=1}^{i} j V_{i,j} \psi_j(t),$$

or by being themselves the smaller clusters in a collision with a larger j-cluster, in this case it is the i-cluster which is pulverized into a number i of 1-clusters which will then attach to j-clusters to produce $(j+1)$-clusters, which results in the last term in the right-hand side of (1.1)

$$-\sum_{j=i}^{\infty} V_{i,j} \psi_i(t) \psi_j(t).$$

The parameters $V_{i,j}$ for $i \neq j$, called the coagulation kernel, are the rate constants for the reaction between clusters of sizes i and j, and are assumed to be time-independent, non-negative, and symmetric, i.e., $V_{i,j} = V_{j,i}$. As discussed in [10], the rate $V_{i,i}$ is equal to half of the collision's rate for the particles of size i.

An extremely useful tool in the mathematical study of coagulation problems is the moments of the solutions. The r-th moment of a solution $\psi = (\psi_i)$ of (1.1) is defined by

$$\mu_r(\psi(\cdot)) = \mu_r(\cdot) := \sum_{i=1}^{\infty} i^r \psi_i(\cdot). \tag{1.3}$$

Putting $r = 0$ gives the zeroth moment, denoted as $\mu_0(\cdot)$, which has the physical interpretation of the total number of particles per unit volume. Taking $r = 1$ in (1.3) we get the first moment, $\mu_1(\cdot)$, which can be physically interpreted as (proportional to) the mass of the system per unit volume. We expect the mass to be a conserved quantity, i.e., $\mu_1(t) = \mu_1(0)$, for kernels with slowly increasing rate of coagulation. Though the physical relevance of the second moment has not been much discussed in the literature, it can be interpreted as the energy dissipated in the process [9]. In some cases, it is convenient to consider moments (1.3) with weight sequences other than i^r. In those cases, like in Section 4, where a more general sequence $g = (g_i)$ is needed, we denote, when required, the corresponding moment by μ_g.

In [11], Dubovski derived this model and calculated the propagation of the coagulation front. The author discovered the connection between the violation of mass conservation law and the value of coagulation front escaping to infinity. Bagland [2] established that the solution for the S-D model, when

$$\lim_{j\to\infty} \frac{V_{i,j}}{j} = 0, \qquad i, j \geqslant 1,\ \psi_0 \in L_1,$$

exists for $t \in [0, \infty)$. Davidson [8], in 2014, presented a global existence theorem, mass conservation result, and uniqueness theorem for three types of kernels, namely

$$\begin{aligned} jV_{i,j} &\leqslant M, && \text{for } j \leqslant i; \\ V_{i,j} &\leqslant C_V h_i h_j, && \text{with } \frac{h_i}{i} \to 0 \quad \text{as} \quad i \to \infty; \end{aligned}$$

and

$$V_{i,j} \leqslant C_V, \qquad \forall i, j, \text{ for some } C_V > 0.$$

Mass is proven to be conserved for $V_{i,j} \leqslant C_V i^{1/2} j^{1/2}$ and the solution is shown to be unique in the third case, i.e., the bounded kernel is considered. In general, for large classes of kernels such as the product kernel, mass is not a conserved quantity, see [10] for the continuous OHS equation. This phenomenon is a consequence of a part of the mass of the cluster distribution (ψ_i) being transported into larger and larger values of i, and a part of it being lost, in finite time, to the limit $i \to \infty$, physically interpreted as an infinite size cluster, or gel, in a process called *gelation*. One can find results on gelation for coagulation-type models in, for instance, [4, Chapter 9] and [7].

In this paper, we study the existence of mass conserving solutions to the initial value problem (1.1)–(1.2) with rate kernels satisfying $V_{i,j} \leqslant c \times (i + j)$, $\forall i, j \in \mathbb{N}$, for some positive constant c, and initial condition with finite mass. Actually, by dividing the equations by c and redefining $V_{i,j}$ and the time scale t, we can consider $c = 1$. This will be done hereinafter in order to turn the expressions a little bit simpler. To establish the regularity of the solutions we need to consider a balance between the class of kernels satisfying the growth condition $V_{i,j} \leqslant i^\alpha + j^\alpha$, for $\alpha \in [0, 1]$ and all $i, j \in \mathbb{N}$, and the initial condition with some finite higher moment. The uniqueness result is also established for restrictive classes of kernels, namely those that in addition to satisfying $V_{i,j} \leqslant i + j$, also satisfy $V_{i,j} \leqslant C_V \min\{i^\eta, j^\eta\}$, $0 \leqslant \eta \leqslant 2$, $\forall i, j \in \mathbb{N}$, for some $C_V > 0$. The boundedness of a higher moment in finite time plays a significant role in proving uniqueness. Let us now define some basic notation and notions that are needed throughout.

The set of finite mass sequences is defined by

$$X = \{z = (z_k) : \|z\| < \infty\}, \tag{1.4}$$

with

$$\|z\| := \sum_{k=1}^{\infty} k|z_k|. \tag{1.5}$$

It is clear that $(X, \|\cdot\|)$ is a Banach space. In our analysis, we will mainly consider its non-negative cone

$$X^+ = \{\psi = (\psi_i) \in X : \psi_i \geqslant 0\}. \tag{1.6}$$

Definition 1.1. The solution $\psi = (\psi_i)$ of the initial value problem (1.1)–(1.2) on $[0, T)$, where $0 < T \leqslant \infty$, is a function $\psi : [0, T) \to X^+$ with the following properties:

(a) ψ_i is continuous, for every i.

(b) $\int_0^t \sum_{j=1}^{\infty} V_{i,j}\psi_j(s)ds < \infty$ for every i and all $0 \leqslant t < T$.

(c) For all i and $t \in [0, T)$ the mild (integrated) version of (1.1)–(1.2) holds,

$$\psi_i(t) = \psi_{0i} + \int_0^t \Big(\psi_{i-1}\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j - \psi_i \sum_{j=1}^{i} jV_{i,j}\psi_j - \psi_i \sum_{j=i}^{\infty} V_{i,j}\psi_j\Big)(s)ds, \tag{1.7}$$

where the first sum on the right-hand side is defined to be zero if $i = 1$.

The article is organized in six sections. The second section discusses the preliminary results required to establish the main results of the work. Section 3 deals with the existence of solutions and its corollary. Further, in Section 4, density conservation is shown for all the solutions of the given equation and the regularity result is proved in Section 5. Finally, the statement and proof of the uniqueness theorem are part of Section 6.

2. A finite-dimensional truncation

Our general approach in this paper consists in considering a finite n-dimensional truncation of (1.1) and, after obtaining appropriate *a priori* estimates for its solutions, passing to the limit $n \to \infty$ and getting corresponding results for (1.1).

In this section, we introduce a truncated system of the S-D model and study some useful results about the moments of its solutions. The finite n-dimensional truncated system for the equation (1.1) that we shall consider corresponds to assuming that no particles with size larger than n can exist initially or be formed by the dynamics. Thus, for the phase variable $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ the system is

$$\frac{d\psi_i}{dt} = \Psi_i^n(\psi), \qquad \text{for } 1 \leqslant i \leqslant n, \tag{2.1}$$

where

$$\Psi_1^n(\psi) := -V_{1,1}\psi_1^2 - \psi_1 \sum_{j=1}^{n-1} V_{1,j}\psi_j \tag{2.2}$$

$$\Psi_i^n(\psi) := \psi_{i-1} \sum_{j=1}^{i-1} jV_{i-1,j}\psi_j - \psi_i \sum_{j=1}^{i} jV_{i,j}\psi_j - \psi_i \sum_{j=i}^{n-1} V_{i,j}\psi_j, \quad \text{for } 2 \leqslant i \leqslant n-1, \tag{2.3}$$

$$\Psi_n^n(\psi) := \psi_{n-1} \sum_{j=1}^{n-1} jV_{n-1,j}\psi_j. \tag{2.4}$$

From what was stated above the initial conditions of interest are

$$\psi_i(0) = \psi_i^0 \geqslant 0, \qquad \text{for } 1 \leqslant i \leqslant n. \tag{2.5}$$

It can be observed here that we have truncated the last sum up to $n-1$, not n. This was done to make sure that the truncation conserves mass, which will be beneficial in proving the existence result. The fact that solutions to (2.1)–(2.5) exist and are unique is an obvious consequence of the Picard–Lindelöf theorem, because the right-hand side of (2.1) is a polynomial vector field. That the solutions with non-negative initial data are also non-negative for later times is a result that can also be established by the standard technique of adding a positive ε to the right-hand side of all equations in (2.1), proving that the positive cone $\mathbb{R}^{n+}$ is invariant for the resulting system, and using the uniform convergence in compact time intervals of its solutions ψ^ε to solutions ψ of (2.1) as $\varepsilon \to 0$ (see, e.g., [12, Theorem III-4-5]).

As was already pointed out in the introduction in the analysis of coagulation-type systems estimates about the time evolution of moments of solutions are of paramount importance. In particular, the establishment of uniform in n estimates of the n-truncated system will provide the necessary control to pass to the limit as $n \to \infty$ and to draw conclusions about the solutions to the full, infinite-dimensional, coagulation system. In a way analogous to the r-th moments defined in (1.3), we consider the quantities

$$\mu_g^n(t) := \sum_{i=1}^{n} g_i \psi_i(t). \tag{2.6}$$

The following result on the evolution of μ_g^n will be relevant for the argument in the proof of the existence theorem in the next section.

Lemma 2.1. *Let $\psi = (\psi_i)_{i\in\{1,\dots,n\}}$ be a solution of (2.1)–(2.5) defined in an open interval I containing 0. Let $g = (g_i)$ be a real sequence. Then*

$$\frac{d\mu_g^n}{dt} = \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}(i g_{j+1} - i g_j - g_i)V_{i,j}\psi_i\psi_j. \tag{2.7}$$

Proof. By (2.2)–(2.4) we can write

$$\begin{aligned}
\frac{d\mu_g^n}{dt} &= \sum_{i=1}^{n} g_i \Psi_i^n(\psi) \\
&= \Big(-g_1 V_{1,1}\psi_1^2 - g_1\psi_1\sum_{j=1}^{n-1} V_{1,j}\psi_j\Big) \\
&\quad + \sum_{i=2}^{n-1} g_i\Big(\psi_{i-1}\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j - \psi_i\sum_{j=1}^{i} jV_{i,j}\psi_j - \psi_i\sum_{j=i}^{n-1} V_{i,j}\psi_j\Big) \\
&\quad + g_n\psi_{n-1}\sum_{j=1}^{n-1} jV_{n-1,j}\psi_j.
\end{aligned}$$

Rewriting the right-hand side by collecting together the first and fourth terms, the second and fifth terms, and the third and sixth we obtain

$$\frac{d\mu_g^n}{dt} = \sum_{i=1}^{n-1}\sum_{j=1}^{i}(g_{i+1} - g_i) jV_{i,j}\psi_i\psi_j - \sum_{i=1}^{n-1}\sum_{j=i}^{n-1} g_i V_{i,j}\psi_i\psi_j,$$

and now altering the order of variables in the first equation and using the symmetry of the rate coefficients, $V_{j,i} = V_{i,j}$, we finally conclude (2.7). ∎

For $g = (i^p)$ it is clear from (2.7) that, for all $t \in I$,

$$\frac{d\mu_0^n(t)}{dt} \leqslant 0 \tag{2.8}$$

and thus $\mu_0^n(t) \leqslant \mu_0^n(0)$, for all $t \in I \cap \{t \geqslant 0\}$. This *a priori* bound implies that nonnegative solutions of the truncated systems (2.1)–(2.4) are globally defined forward in time, i.e., $I \supset [0, +\infty)$. We also immediately conclude that, for all $t \geqslant 0$,

$$\frac{d\mu_1^n(t)}{dt} = 0, \tag{2.9}$$

which means that solutions to the truncated system conserve mass. For further reference, this is stated in the next lemma.

Lemma 2.2. *Solutions to Cauchy problems for the truncated systems* (2.1)–(2.4) *are globally defined forward in time and conserve mass, i.e., satisfy*

$$\mu_1^n(t) = \mu_1^n(0), \qquad \forall t \geqslant 0. \tag{2.10}$$

For the existence proof let us consider $\nu_m^n(t)$ defined as in [3] by

$$\nu_m^n(t) := \sum_{i=m}^{n} i\psi_i^n(t), \tag{2.11}$$

where $\psi^n = (\psi_1^n, \dots, \psi_n^n)$ is a solution of the n-dimensional truncated system (2.1)–(2.4). From these expressions we immediately obtain

$$\begin{aligned}\frac{d\nu_m^n(t)}{dt} = \Big(&\sum_{i=m}^{n-1}\sum_{j=1}^{i} jV_{i,j}\psi_i^n\psi_j^n + m\psi_{m-1}^n\sum_{j=1}^{m-1} jV_{m-1,j}\psi_j^n \\ &- \sum_{i=m}^{n-1}\sum_{j=i}^{n-1} iV_{i,j}\psi_i^n\psi_j^n\Big)(t).\end{aligned} \tag{2.12}$$

Assume now $2m < n$ and consider the function $\kappa_m^n(\cdot)$ defined by

$$\kappa_m^n(t) := \sum_{i=m}^{2m} i\psi_i^n + 2m\sum_{i=2m+1}^{n} \psi_i^n, \tag{2.13}$$

where, again, $\psi^n = (\psi_1^n, \dots, \psi_n^n)$ is a solution of the n-dimensional truncated system. Then, after a few algebraic manipulations, we get

$$\begin{aligned}\frac{d\kappa_m^n(t)}{dt} &= \sum_{i=m}^{2m} i\psi_i^n + 2m\sum_{i=2m+1}^{n} \psi_i^n \\ &= m\psi_{m-1}^n(t)\sum_{j=1}^{m-1} jV_{m-1,j}\psi_j^n(t) + \sum_{i=m}^{2m-1}\psi_i^n(t)\sum_{j=1}^{i} jV_{i,j}\psi_j^n(t) \\ &\quad - \sum_{i=m}^{2m}\sum_{j=i}^{n-1} i\psi_i^n(t)V_{i,j}\psi_j^n(t) - 2m\sum_{i=2m+1}^{n-1}\sum_{j=i}^{n-1} V_{i,j}\psi_i^n(t)\psi_j^n(t).\end{aligned} \tag{2.14}$$

Finally, so that we can take $n \to \infty$ (or, eventually, only on a subsequence $n_k \to \infty$), we make use of the following lemma.

Lemma 2.3. *Take $\psi_0 = (\psi_{0i}) \in X^+$ and, for each $n \in \mathbb{N}$, consider $\psi_0^n \in X^+$ defined by $\psi_0^n = (\psi_{01}, \psi_{02}, \dots, \psi_{0n}, 0, 0, \dots)$ and let it be identified with the point of $\mathbb{R}^n$ obtained by discarding the j-th components, for $j > n$. Let ψ^n be the solution of the n-dimensional truncated system (2.1)–(2.4) when $V_{i,j} \leqslant i + j$ with initial condition $\psi^n(0) = \psi_0^n$ such that (2.10) holds, then ψ^n is relatively compact in $C([0, T])$.*

Proof. Using the truncated system (2.2)–(2.4) and the mass conservation of this system, (2.10), it can be shown that there exists a constant $\widetilde{C} > 0$ such that, for all $n \geqslant i \geqslant 1$,

$$\sup_{t \geqslant 0} \left(\psi_i^n(t) + \left| \frac{d\psi_i^n(t)}{dt} \right| \right) \leqslant \widetilde{C} \mu_1(0)^2.$$

Thus, the Ascoli–Arzelà theorem gives the intended result. ∎

Now, we have gathered all the required information to proceed with the existence results.

3. Existence result for the Cauchy problem

We can now prove the first main result of the paper: the existence of solutions to the Cauchy problem (1.1)–(1.2).

Theorem 3.1. *Let, $V_{i,j}$ be non-negative, symmetric, and satisfy $V_{i,j} \leqslant i + j$, for all i, j, and let $\psi_0 = (\psi_{0i}) \in X^+$. Then there exists a non-negative solution of (1.1)–(1.2) defined in $[0, \infty)$.*

Proof. Let n be an arbitrarily fixed positive integer and let ψ_0^n be defined as in the statement of Lemma 2.3. As we stated above, following (2.5), the initial value problem (2.1)–(2.5) has a unique solution, $\psi^n = (\psi_i^n)_{1 \leqslant i \leqslant n}$, which is globally defined, non-negative and, by Lemma 2.2, density conserving. By defining $\psi_i^n(t) = 0$ when $i > n$ we can consider $\psi^n(t)$ as an element of X^+, for all t, and thus

$$\|\psi^n(t)\| = \sum_{i=1}^{\infty} i \psi_i^n(t) = \sum_{i=1}^{n} i \psi_i^n(t) = \sum_{i=1}^{n} i \psi_{0i}^n = \sum_{i=1}^{n} i \psi_{0i} \leqslant \sum_{i=1}^{\infty} i \psi_{0i} = \|\psi_0\|. \tag{3.1}$$

By Lemma 2.3 and (2.10) for each i, there exists a subsequence of ψ^n (not relabeled) and a function $\psi_i : [0, \infty) \to \mathbb{R}$ of bounded variation on each subset of $[0, \infty)$, such that $\psi_i^n(t)$ converges to $\psi_i(t)$ as n approaches ∞, for every $t \in \mathbb{R}^+$. Thus for all $t \geqslant 0$,

$$\psi_i(t) \geqslant 0 \qquad \text{and} \qquad \|\psi(t)\| \leqslant \|\psi_0\|. \tag{3.2}$$

Our goal is to prove that this limit function ψ is a mild solution of the initial value problem (1.1)–(1.2), i.e., fulfills the conditions in Definition 1.1. This will be done by passing to the limit $n \to \infty$ in the integrated version of the truncated problem (2.1)–(2.5), namely

$$\psi_i^n(t) = \psi_{0i} + \int_0^t \Big(\psi_{i-1}^n(s) \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j^n(s) - \psi_i^n(s) \sum_{j=1}^{i} j V_{i,j} \psi_j^n(s) - \psi_i^n(s) \sum_{j=i}^{n-1} V_{i,j} \psi_j^n(s)\Big) ds. \quad (3.3)$$

To do this, and also to satisfy condition (b) in Definition 1.1, we need to prove that, for every fixed $i \in \mathbb{N}$, $T \geqslant 0$, and $\varepsilon > 0$, there exists m and N_0, with $N_0 > m \geqslant i$, such that, for all $n > N_0$,

$$\int_0^T \nu_m^n(t) dt \leqslant \varepsilon, \quad (3.4)$$

where ν_m^n was defined in (2.11). This can be achieved by integrating (2.12) in $[0, t]$ and using (2.14) to yield

$$\begin{aligned}
\nu_m^n(t) &= \nu_m^n(0) + \int_0^t \Big(\sum_{i=m}^{n-1} \sum_{j=1}^{i} j V_{i,j} \psi_i^n(s) \psi_j^n(s) + m \psi_{m-1}^n \sum_{j=1}^{m-1} j V_{m-1,j} \psi_j^n \\
&\qquad - \sum_{i=m}^{n-1} \sum_{j=i}^{n-1} i V_{i,j} \psi_i^n(s) \psi_j^n(s)\Big) ds \\
&= \nu_m^n(0) + \kappa_m^n(t) - \kappa_m^n(0) \\
&\quad + \int_0^t \Big(\psi_{2m}^n(s) \sum_{j=1}^{2m} j V_{2m,j} \psi_j^n(s) + \sum_{i=2m+1}^{n-1} \sum_{j=1}^{i} j V_{i,j} \psi_i^n(s) \psi_j^n(s) \\
&\qquad - \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} i V_{i,j} \psi_i^n(s) \psi_j^n(s) \\
&\qquad + 2m \sum_{2m+1}^{n-1} \sum_{j=i}^{n-1} V_{i,j} \psi_i^n(s) \psi_j^n(s)\Big) ds.
\end{aligned}$$

Some algebraic manipulations of the second double sum above provide

$$\begin{aligned}
\sum_{i=2m+1}^{n-1} \sum_{j=1}^{i} j V_{i,j} \psi_i^n(s) \psi_j^n(s) &= \sum_{i=2m+1}^{n-1} \sum_{j=1}^{2m} j V_{i,j} \psi_i^n(s) \psi_j^n(s) \\
&\quad + \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} i V_{j,i} \psi_j^n(s) \psi_i^n(s).
\end{aligned}$$

Substituting this into the above expression for $\nu_m^n(t)$ gives

$$\nu_m^n(t) = \nu_m^n(0) + \kappa_m^n(t) - \kappa_m^n(0) + \int_0^t \Big(\sum_{i=2m}^{n-1} \sum_{j=1}^{2m} j V_{i,j} \psi_i^n(s) \psi_j^n(s) + 2m \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} V_{i,j} \psi_i^n(s) \psi_j^n(s) \Big) ds. \tag{3.5}$$

By (3.1), (3.2), and the pointwise convergence of ψ_i^n to ψ_i we conclude that

$$\forall t \in [0,T], \forall \varepsilon > 0, \forall p > \tfrac{4\|\psi_0\|}{\varepsilon}, \exists N_0 : \forall n,\, n > N_0 \implies$$
$$\sum_{i=1}^{\infty} |\psi_i^n(t) - \psi_i(t)| = \sum_{i=1}^{p-1} |\psi_i^n(t) - \psi_i(t)| + \sum_{i=p}^{\infty} |\psi_i^n(t) - \psi_i(t)| < \frac{\varepsilon}{2} + \frac{2}{p}\|\psi_0\| < \varepsilon,$$

which enables us to take $n \to \infty$ in the definition of $\kappa_m^n(t)$ in (2.13) and yields

$$\kappa_m^n(t) \xrightarrow[n\to\infty]{} \sum_{i=m}^{2m} i\psi_i(t) + 2m \sum_{i=2m+1}^{\infty} \psi_i(t) =: \kappa_m(t) \leqslant \sum_{i=m}^{\infty} i\psi_i(t), \tag{3.6}$$

and so $\lim_{m\to\infty} \kappa_m(t) = 0$ and $|\kappa_m(t)| \leqslant \|\psi_0\|$, for all $t \in [0,T]$. Therefore, for every $\varepsilon > 0$, there exist M, N_0 with $N_0 > M$, such that, for all $m > M$, $n > N_0$ and $n \geqslant 2m+1$, we have

$$\kappa_m^n(t) \leqslant \tfrac{1}{3}\varepsilon, \tag{3.7}$$

and

$$\kappa_m^n(0) \leqslant \tfrac{1}{3}\varepsilon. \tag{3.8}$$

By (3.7) and (3.8) and using the assumption $V_{i,j} \leqslant i + j$, we can estimate the right-hand side of (3.5) as follows (redefining)

$$\begin{aligned}\nu_m^n(t) &\leqslant \varepsilon + \int_0^t \Big(\sum_{i=2m}^{n-1} \sum_{j=1}^{2m} j(i+j) \psi_i^n(s) \psi_j^n(s) + 2m \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} (i+j) \psi_i^n(s) \psi_j^n(s) \Big) ds \\ &\leqslant \varepsilon + \int_0^t \Big(2 \sum_{i=2m}^{n-1} \sum_{j=1}^{2m} ij\, \psi_i^n(s) \psi_j^n(s) + 4m \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} i\psi_i^n(s) \psi_j^n(s) \Big) ds\end{aligned}$$

$$\leqslant \varepsilon + \int_0^t \Big(2\sum_{i=m}^{n} i\psi_i^n(s)\sum_{j=1}^{n} j\psi_j^n(s) + 4\sum_{i=m}^{n} i\psi_i^n(s)\sum_{j=i}^{n} j\psi_j^n(s)\Big)ds$$
$$\leqslant \varepsilon + 6\|\psi_0\| \int_0^t \nu_m^n(s)ds.$$

Hence, thanks to Gronwall's lemma, we get, for all $t \in [0, T]$,

$$\nu_m^n(t) \leqslant k_1\varepsilon \tag{3.9}$$

where $k_1 = e^{6\|\psi_0\|T}$, which implies that for all $\varepsilon > 0$, there exist M, N_0 with $N_0 > M$, such that, for all $m > M$, $n > N_0$ and $n \geqslant 2m + 1$,

$$\int_0^t \nu_m^n(s)ds \leqslant \varepsilon k_1 T, \qquad \text{for all } t \in [0, T]. \tag{3.10}$$

Since, $\psi_i^n(t)$ is pointwise convergent to $\psi_i(t)$, the above expression entails that, for all $\varepsilon > 0$, there exists M such that, for all $m > M$, we have

$$\int_0^T \sum_{i=m}^{\infty} i\psi_i(t)dt \leqslant \varepsilon.$$

Hence, when $V_{i,j} \leqslant i + j$, for all $i \geqslant 1$,

$$\int_0^T \sum_{j=1}^{\infty} V_{i,j}\psi_j(t)dt < \infty, \tag{3.11}$$

thus establishing (b) in Definition 1.1.

Now, for every fixed i, take $n > i$ sufficiently large and, for any ℓ such that $i < \ell < n - 1$, write (3.3) as

$$\Big|\psi_i^n(t) - \psi_i(0) - \int_0^t \Big(\psi_{i-1}^n(s)\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j^n(s)$$
$$- \psi_i^n(s)\sum_{j=1}^{i} jV_{i,j}\psi_j^n(s) - \psi_i^n(s)\sum_{j=i}^{\ell} V_{i,j}\psi_j^n(s)\Big)ds\Big|$$
$$= \psi_i^n(s)\int_0^t \sum_{j=\ell+1}^{n-1} V_{i,j}\psi_j^n(s)ds$$
$$\leqslant 2\|\psi_0\| \int_0^t \nu_{\ell+1}^n(s)ds.$$

Thus, from (3.10), for all $\varepsilon > 0$, there exists M such that, for all $\ell + 1 > M$ and all n sufficiently large, the right-hand side can be bounded above by $2\varepsilon\|\psi_0\|k_1T$.

Considering that each sum on the left-hand side has a fixed and finite number of terms, that $\psi_j^n(s) \to \psi_j(s)$ pointwise as $n \to \infty$, and each of the three terms inside the integral is bounded by $2\|\psi_0\|^2$, we can use the dominated convergence theorem and take $n \to \infty$ to conclude that, for every $\varepsilon > 0$, there exists M such that, for all $\ell \geqslant M$, we have

$$\Big|\psi_i(t) - \psi_i(0) - \int_0^t \Big(\psi_{i-1}(s)\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(s) - \psi_i(s)\sum_{j=1}^{i} jV_{i,j}\psi_j(s) - \psi_i(s)\sum_{j=i}^{\ell} V_{i,j}\psi_j(s)\Big)ds\Big| \leqslant 2\varepsilon\|\psi_0\|k_1T.$$

Hence, by the arbitrariness of ε, we can let $\ell \to \infty$ and conclude that $\psi = (\psi_i)$ satisfy (1.7), which completes the proof. ■

Next we establish that the subsequence ψ^{n_k} of solutions of the truncated system which converges to the solution ψ of (1.1)–(1.2) actually does so in the strong topology of X, uniformly for t in compact subsets of $[0,\infty)$.

Corollary 3.2. *Let ψ^{n_k} be the pointwise convergent subsequence of solutions to (2.3)–(2.5). Then, $\psi^{n_k} \to \psi$ in X uniformly on compact subsets of* $[0,\infty)$.

Proof. We prove that $\psi_i^{n_k}(t) \to \psi_i(t)$ for each i, uniformly on the compact subsets of $[0,\infty)$. To simplify notation, denote n_k simply by n. Let

$$\xi_m^n(t) := e^{-t}\Big(\mu_1^n(t) - \sum_{i=1}^{m-1} i\psi_i^n(t) + (2m+2)\mu_1^n(0)^2\Big). \tag{3.12}$$

Now, differentiating (3.12) gives

$$\frac{d\xi_m^n(t)}{dt} = e^{-t}\Big(\frac{d\mu_1^n(t)}{dt} - \frac{d}{dt}\sum_{i=1}^{m-1} i\psi_i^n(t)\Big) - e^{-t}\Big(\mu_1^n(t) - \sum_{i=1}^{m-1} i\psi_i^n(t) + (2m+2)\mu_1^n(0)^2\Big) \tag{3.13}$$

where

$$\frac{d}{dt}\sum_{i=1}^{m-1} i\psi_i^n(t) = \sum_{i=1}^{m-2}\sum_{j=1}^{i} jV_{i,j}\psi_i^n\psi_j^n - \sum_{i=1}^{m-1}\sum_{j=i}^{n-1} iV_{i,j}\psi_i^n\psi_j^n - (m-1)\psi_{m-1}^n\sum_{j=1}^{m-1} jV_{m-1,j}\psi_j^n. \tag{3.14}$$

Using the above expression, Lemma 2.2 and $V_{i,j} \leqslant i + j$ in (3.13), one can obtain

$$\frac{d\xi_m^n(t)}{dt} \leqslant e^{-t}\Big(\sum_{i=1}^{m-1}\sum_{j=i}^{n-1} j(i+j)\psi_i^n\psi_j^n + (m-1)\psi_{m-1}^n\sum_{j=1}^{m-1} j(m-1+j)\psi_j^n - (2m+2)\mu_1^n(0)^2\Big).$$

Some simplifications guarantee that

$$\frac{d\xi_m^n(t)}{dt} \leqslant 0, \qquad n \geqslant m,\ t \in [0,T].$$

Hence, $\xi_m^n(t) \to \xi_m(t)$, uniformly for t on compact subsets of $[0,T)$, where

$$\xi_m(t) := e^{-t}\Big(\mu_1(t) - \sum_{i=1}^{m-1} i\psi_i(t) + (2m+2)\mu_1(0)^2\Big).$$

Let, $K \subset [0,\infty)$ be compact and $t_n \to t$ in K, then

$$\lim_{n\to\infty}\|\psi^n(t_n)\| = \lim_{n\to\infty}\sum_{i=1}^{\infty} i\psi_i^n(t_n) = \sum_{i=1}^{\infty} i\psi_i(t) = \|\psi(t)\|$$

which ensures that $\|\psi^n\| \to \|\psi\|$, in $C(K,X)$. ■

4. All solutions conserve density

In this section, we prove that, under the assumption on the rate coefficients we have been using, all solutions of (1.1)–(1.2) conserve density.

Let $\psi = (\psi_i) \in X^+$ be a solution of (1.7) in $[0,T]$. Multiplying each equation in (1.7) by g_i and adding from $i = 1$ to n, we have, after some algebraic manipulations, for all $t \in [0,T]$,

$$\sum_{i=1}^{n} g_i\psi_i(t) - \sum_{i=1}^{n} g_i\psi_{0i} = \int_0^t \sum_{i=1}^{n}\sum_{j=i}^{n}(jg_{i+1} - jg_i - g_j)V_{i,j}\psi_i(s)\psi_j(s)ds - \int_0^t \sum_{j=1}^{n}\sum_{i=n+1}^{\infty} g_j V_{i,j}\psi_i(s)\psi_j(s)ds - \int_0^t g_{n+1}\psi_n(s)\sum_{j=1}^{n} jV_{n,j}\psi_j(s)ds. \tag{4.1}$$

We start by observing that, taking $g_i \equiv i$ in (4.1), we conclude that

$$\sum_{i=1}^{n} i\psi_i(t) \leqslant \sum_{i=1}^{n} i\psi_{0i} \leqslant \|\psi_0\|,$$

and, as this inequality is valid for all n, we can take the limit as $n \to \infty$ and conclude the *a priori* bound $\|\psi(t)\| \leqslant \|\psi_0\|$.

We now use (4.1) to prove that, under the assumed conditions on $V_{i,j}$, all solutions conserve density.

Theorem 4.1. *Let $V_{i,j} \leqslant i + j$ for all i and j. Let $\psi = (\psi_i) \in X^+$ be a solution of the Safronov–Dubovski equation (1.7). Then the total density of ψ is constant.*

Proof. Let $A \in \mathbb{N}$ be fixed, and consider the sequence $(g_i^A)_i \in \ell^\infty$ defined by

$$g_i^A = \min\{i, A\}. \tag{4.2}$$

Then

$$jg_{i+1}^A - jg_i^A - g_j^A = \begin{cases} -A, & \text{on } \{(i,j) : A \leqslant i \leqslant j \leqslant n\}, \\ 0, & \text{on } \{(i,j) : 1 \leqslant i \leqslant A-1 \text{ and } i \leqslant j \leqslant n\}, \end{cases}$$

and (4.1) becomes, for $n > A$,

$$\sum_{i=1}^{n} g_i^A \psi_i(t) - \sum_{i=1}^{n} g_i^A \psi_{0i} \tag{4.3}$$

$$= -\int_0^t A \sum_{j=A}^{n} \sum_{i=A}^{j} V_{i,j} \psi_i(s)\psi_j(s) ds \tag{4.4}$$

$$- \int_0^t \Big(\sum_{j=1}^{A} \sum_{i=n+1}^{\infty} jV_{i,j}\psi_i(s)\psi_j(s) + A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} V_{i,j}\psi_i(s)\psi_j(s) \Big) ds \tag{4.5}$$

$$- \int_0^t A\psi_n(s) \sum_{j=1}^{n} jV_{n,j}\psi_j(s) ds. \tag{4.6}$$

We first estimate the term in (4.4)

$$\begin{aligned} A \sum_{j=A}^{n} \sum_{i=A}^{j} V_{i,j}\psi_i\psi_j &\leqslant A \sum_{j=A}^{n} \psi_j \sum_{i=A}^{j} i\psi_i + A \sum_{j=A}^{n} j\psi_j \sum_{i=A}^{j} \psi_i \\ &= A \sum_{j=A}^{n} \frac{1}{j} j\psi_j \sum_{i=A}^{j} i\psi_i + A \sum_{j=A}^{n} j\psi_j \sum_{i=A}^{j} \frac{1}{i} i\psi_i \end{aligned}$$

$$\leqslant 2 \sum_{j=A}^{n} j\psi_j \sum_{i=A}^{n} i\psi_i$$

$$\leqslant 2 \sum_{j=A}^{\infty} j\psi_j \sum_{i=A}^{\infty} i\psi_i. \tag{4.7}$$

Thus, $\psi \in X^+$ implies that (4.7) converges to zero as $A \to \infty$. Furthermore, since (4.7) is bounded above by $2\|\psi_0\|^2$, the dominated convergence theorem implies that, for all $\varepsilon > 0$ there exists A_0 such that, for all $n > A \geqslant A_0$ the absolute value of (4.4) is smaller than $\frac{\varepsilon}{5}$.

Consider now (4.5). For the first double sum, observe that for $n > A$,

$$\begin{aligned}
\sum_{j=1}^{A} \sum_{i=n+1}^{\infty} jV_{i,j}\psi_i\psi_j &\leqslant \sum_{j=1}^{A} j\psi_j \sum_{i=n+1}^{\infty} i\psi_i + \sum_{j=1}^{A} j^2\psi_j \sum_{i=n+1}^{\infty} \psi_i \\
&\leqslant \|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i + \sum_{j=1}^{A} j^2\psi_j \frac{1}{n+1} \sum_{i=n+1}^{\infty} i\psi_i \\
&\leqslant \|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i + \frac{A}{A+1} \sum_{j=1}^{A} j\psi_j \sum_{i=n+1}^{\infty} i\psi_i \\
&\leqslant \|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i + \|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i \\
&\leqslant 2\|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i.
\end{aligned} \tag{4.8}$$

For the second double sum in (4.5) we have a similar estimate,

$$\begin{aligned}
A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} V_{i,j}\psi_i\psi_j &\leqslant A \sum_{j=A+1}^{n} \psi_j \sum_{i=n+1}^{\infty} i\psi_i + A \sum_{j=A+1}^{n} j\psi_j \sum_{i=n+1}^{\infty} \psi_i \\
&\leqslant \frac{A}{A+1} \sum_{j=A+1}^{n} j\psi_j \sum_{i=n+1}^{\infty} i\psi_i \\
&\quad + A \sum_{j=A+1}^{n} j\psi_j \frac{1}{n+1} \sum_{i=n+1}^{\infty} i\psi_i \\
&\leqslant \|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i + \frac{A}{n+1}\|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i \\
&\leqslant 2\|\psi_0\| \sum_{i=n+1}^{\infty} i\psi_i.
\end{aligned} \tag{4.9}$$

Thus, by (4.8) and (4.9) we conclude that the integrand function in (4.5) is bounded by $4\|\psi_0\|^2$ and converges pointwise to zero as $n \to \infty$, for each fixed A. Hence, again by the dominated convergence theorem we conclude that, as previously, for all $\varepsilon > 0$, there exists A_0 such that, for all $n > A_0$, the absolute value of the integral (4.5) is smaller than $\frac{\varepsilon}{5}$.

Finally, let us consider (4.6)

$$\begin{aligned} A\psi_n \sum_{j=1}^{n} jV_{n,j}\psi_j &\leqslant A\psi_n \sum_{j=1}^{n} j(n+j)\psi_j \\ &\leqslant 2An\psi_n \sum_{j=1}^{n} j\psi_j \\ &\leqslant 2\|\psi_0\| An\psi_n. \end{aligned} \tag{4.10}$$

Clearly, for each fixed A, (4.10) converges to zero as $n \to \infty$ and it is bounded above by $A\|\psi_0\|^2$, and so the dominated convergence theorem implies that, for every $\varepsilon > 0$, there exists $A_0 = A_0(\varepsilon)$ such that, for any fixed $A > A_0$, there exists $n_0 = n_0(\varepsilon, A)$ such that, for all $n > n_0 \vee A$, the absolute value of (4.6) is smaller than $\frac{\varepsilon}{5}$.

To estimate (4.3) observe that, for every $n > A$, we can write

$$\begin{aligned} \left|\sum_{i=1}^{n} g_i^A\psi_i(t) - \sum_{i=1}^{n} g_i^A\psi_{0i}\right| \geqslant &\left|\sum_{i=1}^{A} i\psi_i(t) - \sum_{i=1}^{A} i\psi_{0i}\right| \\ &- A\left|\sum_{i=A+1}^{n} \psi_i(t) - \sum_{i=A+1}^{n} \psi_{0i}\right|, \end{aligned}$$

and thus,

$$\begin{aligned} \left|\sum_{i=1}^{A} i\psi_i(t) - \sum_{i=1}^{A} i\psi_{0i}\right| \leqslant &\sum_{i=A+1}^{\infty} i\psi_i(t) + \sum_{i=A+1}^{\infty} i\psi_{0i} \\ &+ \left|\sum_{i=1}^{n} g_i^A\psi_i(t) - \sum_{i=1}^{n} g_i^A\psi_{0i}\right|. \end{aligned} \tag{4.11}$$

Now, for every $\varepsilon > 0$ there exists A_0 such that, for all $A > A_0$, each of the first two sums on the right-hand side of (4.11) can be made smaller than $\frac{\varepsilon}{5}$, and since the estimates of (4.4)–(4.6) obtained previously allow us to have the last term on the right-hand side of (4.11) smaller than $\frac{3}{5}\varepsilon$, we conclude that

$$\forall\varepsilon > 0,\ \exists A_0 : \forall A,\ A > A_0 \Longrightarrow \left|\sum_{i=1}^{A} i\psi_i(t) - \sum_{i=1}^{A} i\psi_{0i}\right| < \varepsilon,$$

which proves the result. ∎

5. Differentiability

This section is devoted to proving that the solution of the S-D model is first-order differentiable if the rate coefficients satisfy $V_{i,j} \leqslant i^\alpha + j^\alpha$ for $\alpha \in [0,1]$. This requires the boundedness of $(\alpha+1)$-moments of the solutions and an invariance result, which are proved below in Lemma 5.1 and Theorem 5.2, respectively.

Lemma 5.1. *Let the non-negative kernel $V_{i,j}$ satisfy $V_{i,j} \leqslant i^\alpha + j^\alpha$, for all $i, j \geqslant 1$, and for some fixed $0 \leqslant \alpha \leqslant 1$. For any $T \in (0,\infty)$, let ψ be a solution to (1.1)–(1.2) in $[0,T]$ with initial condition $\psi_0 \in X^+$. If the $(\alpha+1)$-moment of ψ_0, $\mu_{1+\alpha}(\psi_0)$, is bounded, then the $(\alpha+1)$-moment of $\psi(t)$ is also bounded for all $t \in [0,T]$.*

Proof. Let $\psi = (\psi_i)$ be a solution of (1.1)–(1.2) in $[0,T]$ with initial condition ψ_0. Multiplying each equation (1.7) by $i^{1+\alpha}$ and adding from $i=1$ to n, we have, for all $t \in [0,T]$,

$$\begin{aligned}\sum_{i=1}^{n} i^{1+\alpha}\psi_i(t) &+ \int_0^t \sum_{i=1}^{n}\sum_{j=i}^{\infty} i^{i+\alpha} V_{i,j}\psi_i(s)\psi_j(s)ds \\ &+ \int_0^t (n+1)^{1+\alpha}\psi_n(s)\sum_{j=1}^{n} jV_{n,j}\psi_j(s)ds \\ &= \sum_{i=1}^{n} i^{1+\alpha}\psi_{0i} + \int_0^t \sum_{i=1}^{n}\sum_{j=1}^{i}\big(j(i+1)^{1+\alpha} - ji^{1+\alpha}\big)V_{i,j}\psi_i(s)\psi_j(s)ds,\end{aligned} \tag{5.1}$$

and, due to non-negativity of solutions, this implies that

$$\begin{aligned}\sum_{i=1}^{n} i^{1+\alpha}\psi_i(t) \leqslant \sum_{i=1}^{n} i^{1+\alpha}\psi_{0i} \\ + \int_0^t \sum_{i=1}^{n}\sum_{j=1}^{i}\big(j(i+1)^{1+\alpha} - ji^{1+\alpha}\big)V_{i,j}\psi_i(s)\psi_j(s)ds.\end{aligned} \tag{5.2}$$

Since $\alpha \in [0,1]$ and $i \geqslant 1$ we have

$$(i+1)^{1+\alpha} - i^{1+\alpha} \leqslant (1+\alpha)i^\alpha + \frac{(1+\alpha)\alpha}{2!},$$

and so

$$\begin{aligned}j\big((i+1)^{1+\alpha} - i^{1+\alpha}\big)V_{i,j} \leqslant (1+\alpha)ji^{2\alpha} + (1+\alpha)j^{1+\alpha}i^\alpha + \frac{(1+\alpha)\alpha}{2!}ji^\alpha \\ + \frac{(1+\alpha)\alpha}{2!}j^{1+\alpha},\end{aligned}$$

from where, using $\sum_{i=1}^{n} i\psi_i(s) \leqslant \|\psi_0\|$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{i}\big(j(i+1)^{1+\alpha} - j i^{1+\alpha}\big)V_{i,j}\psi_i(s)\psi_j(s) \\
&\qquad \leqslant \frac{1}{2}(1+\alpha)\alpha\|\psi_0\|^2 + \frac{1}{2}(1+\alpha)(4+\alpha)\|\psi_0\|\sum_{i=1}^{n} i^{1+\alpha}\psi_i(s) \\
&\qquad \leqslant \|\psi_0\|^2 + 5\|\psi_0\|\sum_{i=1}^{n} i^{1+\alpha}\psi_i(s)
\end{aligned}$$

which, upon substitution in (5.2), gives

$$\sum_{i=1}^{n} i^{1+\alpha}\psi_i(t) \leqslant \sum_{i=1}^{n} i^{1+\alpha}\psi_{0i} + \|\psi_0\|^2 T + \int_0^t 5\|\psi_0\|\sum_{i=1}^{n} i^{1+\alpha}\psi_i(s)ds.$$

Hence, by Gronwall's lemma, we conclude that for all $t \in [0, T]$ and $n \geqslant 1$,

$$\begin{aligned}
\sum_{i=1}^{n} i^{1+\alpha}\psi_i(t) &\leqslant \Big(\sum_{i=1}^{n} i^{1+\alpha}\psi_{0i} + T\|\psi_0\|^2\Big)e^{5\|\psi_0\|t} \\
&\leqslant \big(\mu_{1+\alpha}(\psi_0) + T\|\psi_0\|^2\big)e^{5\|\psi_0\|t}, \qquad (5.3)
\end{aligned}$$

where the inequality (5.3) is due to the assumption about the boundedness of the $(1+\alpha)$-moment of the initial condition ψ_0. Since the right-hand side (5.3) does not depend on n we conclude that the same is valid in the limit $n \to \infty$, which proves the result. ∎

An important result regarding the evaluation of the higher moments of the solution is analyzed next.

Theorem 5.2. *Assume* (g_i) *be a real-valued non-negative sequence such that* $g_i = O(i^{\alpha+1})$. *Let* ψ *be a solution of* (1.1) *when* $V_{i,j} \leqslant i^\alpha + j^\alpha$, $\alpha \in [0, 1]$ *under the assumption that* $\mu_{\alpha+1}(\psi_0)$ *is bounded on some interval* $[0, T)$, *for* $0 < T \leqslant \infty$. *Let* $0 \leqslant t_1 < t_2 < T$. *If the following hypotheses hold*

$$\int_{t_1}^{t_2}\sum_{i=1}^{\infty}\sum_{j=1}^{i} j(g_{i+1} - g_i)V_{i,j}\psi_j(s)\psi_i(s)ds < \infty, \qquad \text{(H1)}$$

$$\int_{t_1}^{t_2}\sum_{i=1}^{\infty}\sum_{j=1}^{i} g_j V_{i,j}\psi_i(s)\psi_j(s)ds < \infty \qquad \text{(H2)}$$

then, for every $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=m}^{\infty} g_i \psi_i(t_2) - \sum_{i=m}^{\infty} g_i \psi_i(t_1) = &\int_{t_1}^{t_2} \sum_{i=m}^{\infty}\sum_{j=1}^{i}(jg_{i+1} - jg_i - g_j)V_{i,j}\psi_i(s)\psi_j(s)ds \\ &+ \delta_{m\geqslant 2}\int_{t_1}^{t_2} \sum_{i=m}^{\infty}\sum_{j=1}^{m-1} g_j V_{i,j}\psi_i(s)\psi_j(s)ds \\ &+ \delta_{m\geqslant 2}\int_{t_1}^{t_2} g_m\psi_{m-1}(s)\sum_{j=1}^{m-1} jV_{m-1,j}\psi_j(s)ds\end{aligned} \tag{5.4}$$

where $\delta_P = 1$ *if* P *holds, and is equal to zero otherwise.*

Proof. Take positive integers $m < n$. Multiplying each equation in (1.7) by g_i and summing over i from m to n, we obtain

$$\sum_{i=m}^{n} g_i \psi_i(t_2) - \sum_{i=m}^{n} g_i \psi_i(t_1)$$

$$= \int_{t_1}^{t_2} \sum_{i=m}^{n}\sum_{j=1}^{i}(jg_{i+1} - jg_i - g_j)V_{i,j}\psi_i(s)\psi_j(s)ds \tag{5.5}$$

$$+ \delta_{m\geqslant 2}\int_{t_1}^{t_2} \sum_{i=m}^{n}\sum_{j=1}^{m-1} g_j V_{i,j}\psi_i(s)\psi_j(s)ds \tag{5.6}$$

$$+ \delta_{m\geqslant 2}\int_{t_1}^{t_2} g_m\psi_{m-1}(s)\sum_{j=1}^{m-1} jV_{m-1,j}\psi_j(s)ds \tag{5.7}$$

$$- \int_{t_1}^{t_2} \sum_{i=m}^{n}\sum_{j=n+1}^{\infty} g_i V_{i,j}\psi_i(s)\psi_j(s)ds \tag{5.8}$$

$$- \int_{t_1}^{t_2} g_{n+1}\psi_n(s)\sum_{j=1}^{n} jV_{n,j}\psi_j(s)ds. \tag{5.9}$$

We need to prove that, as $n \to \infty$, the integrals in (5.8) and (5.9) converge to zero, and the other integrals converge to the corresponding ones on the right-hand side of (5.4).

Using (H2) by interchanging the order of summation and replacing i for j and then following (a) in Definition 1.1, one can obtain

$$\lim_{n\to\infty}\int_{t_1}^{t_2} \sum_{i=m}^{n}\sum_{j=n+1}^{\infty} g_i V_{i,j}\psi_i(s)\psi_j(s)ds = 0 \tag{5.10}$$

which proves the convergence of (5.8) to zero. Putting $g_i \equiv 1$ in (5.5)–(5.9) and using (H2), the terms (5.8) and (5.9) tend to zero as $n \to \infty$. Therefore,

$$\sum_{i=m}^{\infty} \psi_i(t_2) - \sum_{i=m}^{\infty} \psi_i(t_1) = \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{i} (-V_{i,j}) \psi_i(s) \psi_j(s) ds$$
$$+ \delta_{m \geqslant 2} \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{m-1} V_{i,j} \psi_i(s) \psi_j(s) ds$$
$$+ \delta_{m \geqslant 2} \int_{t_1}^{t_2} \psi_{m-1}(s) \sum_{j=1}^{m-1} j V_{m-1,j} \psi_j(s) ds. \quad (5.11)$$

For $p = 1, 2$, consider,

$$|g_{n+1}| \sum_{i=n+1}^{\infty} \psi_i(t_p) \leqslant C(n+1)^{\alpha+1} \sum_{i=n+1}^{\infty} \psi_i(t_p) \leqslant C \sum_{i=n+1}^{\infty} i^{\alpha+1} \psi_i(t_p)$$

for some $C \in \mathbb{R}^+$, and thus Lemma 5.1 guarantees that

$$\lim_{n \to \infty} |g_{n+1}| \sum_{i=n+1}^{\infty} \psi_i(t_p) = 0. \quad (5.12)$$

Replacing m by $n+1$ in (5.11), multiplying both sides by g_{n+1}, letting $n \to \infty$, and using (H2) together with (5.12) confirms that

$$\int_{t_1}^{t_2} g_{n+1} \psi_n(s) \sum_{j=1}^{n} j V_{n,j} \psi_j(s) ds \to 0. \quad (5.13)$$

By Definition 1.1, the boundedness of $\psi_i(t)$, (H1) and (H2), we conclude that, as $n \to \infty$,

$$\int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{i} (j g_{i+1} - j g_i - g_j) V_{i,j} \psi_i(s) \psi_j(s) ds$$
$$\to \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{i} (j g_{i+1} - j g_i - g_j) V_{i,j} \psi_i(s) \psi_j(s) ds \quad (5.14)$$

and

$$\int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{m-1} g_j V_{i,j} \psi_i(s) \psi_j(s) ds \to \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{m-1} g_j V_{i,j} \psi_i(s) \psi_j(s) ds. \quad (5.15)$$

Thus, using Definition 1.1 together with equations (5.10), (5.13)–(5.15) and the bounded convergence theorem, the result follows. ∎

Finally, the following proposition is discussed which is essential in showing that the solution of the S-D model is first-order differentiable.

Proposition 5.3. *Let* $\{V_{i,j}\}_{i,j\in\mathbb{N}}$ *be non-negative and* $V_{i,j} \leqslant i^\alpha + j^\alpha$, $0 \leqslant \alpha \leqslant 1$. *Let* $\psi = (\psi_i)$ *be a solution on some interval* $[0, T)$, *where* $0 < T \leqslant \infty$, *of the equation* (1.1) *with initial condition* f_0 *and having bounded* $\mu_{\alpha+1}(\psi_0)$. *Then, the series* $\sum_{j=1}^{i} jV_{i,j}\psi_j(t)\psi_i(t)$ *and* $\sum_{j=i}^{\infty} V_{i,j}\psi_j(t)\psi_i(t)$ *are absolutely continuous on the compact sub-intervals of* $[0, T)$.

Proof. Let (g_i) satisfy the conditions in the statement of Theorem 5.2. For (H1) to hold, proving the boundedness of the series $\sum_{i=1}^{\infty}\sum_{j=1}^{i} j(g_{i+1} - g_i)V_{i,j}\psi_i(t)\psi_j(t)$ is enough. Using the fact that $g_{i+1} - g_i = O(i^\alpha)$, we have the following:

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{i} j(g_{i+1} - g_i)V_{i,j}\psi_i\psi_j &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{i} Cji^\alpha V_{i,j}\psi_i\psi_j \\ &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{i} Cji^\alpha(i^\alpha + j^\alpha)\psi_i\psi_j \\ &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{i} C(ji^{\alpha+1} + i^\alpha j^{\alpha+1})\psi_i\psi_j\end{aligned}$$

for C being some positive constant. Hence, by Lemma 5.1 and Section 4, one can obtain

$$\sum_{i=1}^{\infty}\sum_{j=1}^{i} j(g_{i+1} - g_i)V_{i,j}\psi_i\psi_j \leqslant 2CN_{\alpha+1}\mu_1(0). \tag{5.16}$$

Thus, (H1) holds true. Further, to establish relation (H2), consider the expression

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{i} g_j V_{i,j}\psi_i(s)\psi_j(s) &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{i} C[j^{2\alpha+1} + i^\alpha j^{\alpha+1}]\psi_i\psi_j \\ &\leqslant 2CN_{\alpha+1}\mu_1(0)\end{aligned}$$

which is finite by Lemma 5.1. Therefore, all the hypotheses of Theorem 5.2 are satisfied for any $t_1, t_2 \in [0, T)$. Hence, considering $m = 1$ for $t \in [0, T)$, equation (5.4) implies the uniform convergence of the series $\sum_{i=1}^{\infty} g_i\psi_i(t)$. Since the series $\sum_{j=1}^{i} jV_{i,j}\psi_j(t)$ is bounded by this series, as $jV_{i,j} = O(i^{\alpha+1})$ when $j < i$, we conclude the uniform convergence of $\sum_{j=1}^{i} jV_{i,j}\psi_j(t)$. Further, the boundedness of $\psi_i(t)$ ensures the absolute continuity of $\sum_{j=1}^{i} jV_{i,j}\psi_j(t)\psi_i(t)$. Also, the series $\sum_{j=i}^{\infty} V_{i,j}\psi_j(t)$ is bounded by $\sum_{j=1}^{\infty} g_j\psi_j(t)$, which yields its uniform convergence. Finally, the boundedness of $\psi_i(t)$ gives the desired result. ■

Definition 1.1 (a), hypotheses (H1)–(H2) of Theorem 5.2, and Proposition 5.3 ensure that the solution f is differentiable in the classical sense in $[0, T)$.

6. Uniqueness

In this section, we discuss the uniqueness of solutions for the S-D model for a more restrictive class of kernels. It should be mentioned here that just with the assumption $V_{i,j} \leqslant i + j$ we were unable to prove a uniqueness result, and so a further assumption on the kernel was imposed.

Theorem 6.1. *Let the kernel $V_{i,j}$ be non-negative, symmetric, and satisfying the bounds $V_{i,j} \leqslant i + j$ and $V_{i,j} \leqslant C_V \min\{i^\eta, j^\eta\}$, with $0 \leqslant \eta \leqslant 2$, for all $i, j \in \mathbb{N}$ and some constant $C_V > 0$. If the assumptions of Lemma 5.1 hold, then the initial value problem (1.1)–(1.2) has a unique solution in X^+.*

Proof. We shall use an approach that revolves around defining a function (say $u(t)$) that is the difference between two solutions of the equation (1.1) (let ψ_i and ρ_i), both satisfying the initial condition (1.2). Furthermore, it makes use of the properties of the signum function such as

(P1) $\operatorname{sgn}(\mathscr{C}(t))\frac{d\mathscr{C}(t)}{dt} = \frac{d|\mathscr{C}(t)|}{dt}$,

(P2) $\operatorname{sgn}(a)\operatorname{sgn}(b) = \operatorname{sgn}(ab)$ and $|a| = a \times \operatorname{sgn}(a)$ for any real numbers a, b.

Let $u_i := \psi_i - \rho_i$. Our goal here is to obtain a differential inequality for the evolution of

$$u(t) := \sum_{i=1}^{\infty} |u_i(t)| = \sum_{i=1}^{\infty} \big|\psi_i(t) - \rho_i(t)\big| \tag{6.1}$$

and to use Gronwall's lemma to conclude that $u(t)$ stays identically zero if its initial value is zero, thus proving uniqueness.

Using the expression of the equation (1.1), we obtain

$$\begin{aligned}\frac{du_i(t)}{dt} &= \Big(\psi_{i-1}(t)\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(t) - \psi_i(t)\sum_{j=1}^{i} jV_{i,j}\psi_j(t) - \sum_{j=i}^{\infty} V_{i,j}\psi_i(t)\psi_j(t)\Big)\\ &\quad - \Big(\rho_{i-1}(t)\sum_{j=1}^{i-1} jV_{i-1,j}\rho_j(t) - \rho_i(t)\sum_{j=1}^{i} jV_{i,j}\rho_j(t) - \sum_{j=i}^{\infty} V_{i,j}\rho_i(t)\rho_j(t)\Big).\end{aligned}$$

Multiplying both sides by $\operatorname{sgn}(u_i(t))$, using (P1), then integrating both sides between $t = 0$ and an arbitrary $t > 0$, using $u_i(0) = 0$, and then summing over i from 1 to ∞ yields

$$\begin{aligned}u(t) = \int_0^t \sum_{i=1}^{\infty} \operatorname{sgn}(u_i(s))\Big(&\delta_{i\geqslant 2}\psi_{i-1}(s)\sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(s)\\ &- \psi_i(s)\sum_{j=1}^{i} jV_{i,j}\psi_j(s) - \sum_{j=i}^{\infty} V_{i,j}\psi_i(s)\psi_j(s)\Big)ds\end{aligned}$$

$$
-\int_0^t \sum_{i=1}^{\infty} \operatorname{sgn}(u_i(s))\Big(\delta_{i\geqslant 2}\rho_{i-1}(s)\sum_{j=1}^{i-1} jV_{i-1,j}\rho_j(s) \\
- \rho_i(s)\sum_{j=1}^{i} jV_{i,j}\rho_j(s) - \sum_{j=i}^{\infty} V_{i,j}\rho_i(s)\rho_j(s)\Big)ds.
$$

Replacing $i-1$ by i', then changing notation i' to i in the first and fourth sums, and finally using $\psi_i\psi_j - \rho_i\rho_j = \psi_i u_j + \rho_j u_i$, we get

$$
u(t) = \int_0^t \Bigg(\sum_{i=1}^{\infty}\Big(\operatorname{sgn}(u_{i+1})\sum_{j=1}^{i} jV_{i,j}(\psi_i u_j + \rho_j u_i) \\
- \operatorname{sgn}(u_i)\sum_{j=1}^{i} jV_{i,j}(\psi_i u_j + \rho_j u_i) \\
- \operatorname{sgn}(u_i)\sum_{j=i}^{\infty} V_{i,j}(\psi_i u_j + \rho_j u_i)\Big)\Bigg)(s)ds.
$$

Using (P2), one can obtain the following estimate:

$$
u(t) \leqslant \int_0^t \Bigg(\sum_{i=1}^{\infty}\sum_{j=1}^{i} jV_{i,j}(|u_j|\psi_i + |u_i|\rho_j) - \sum_{i=1}^{\infty}\sum_{j=1}^{i} jV_{i,j}\big(\operatorname{sgn}(u_i)u_j\psi_i + |u_i|\rho_j\big) \\
- \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} V_{i,j}\big(\operatorname{sgn}(u_i)u_j\psi_i + |u_i|\rho_j\big)\Bigg)(s)ds,
$$

and canceling the second terms in the first two double sums on the right-hand side, discarding the last term in the third double sum, and noting that $-\operatorname{sgn}(u_i)u_j \leqslant |u_j|$, we get the estimate

$$
u(t) \leqslant \int_0^t \Big(2\sum_{i=1}^{\infty}\sum_{j=1}^{i} jV_{i,j}|u_j|\psi_i + \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} V_{i,j}|u_j|\psi_i\Big)(s)ds.
$$

Finally, using the bounds of $V_{i,j}$, we can write

$$
u(t) \leqslant \int_0^t \Big(2\sum_{i=1}^{\infty}\sum_{j=1}^{i} j(i+j)|u_j|\psi_i + \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} C_V i^{\eta}|u_j|\psi_i\Big)(s)ds.
$$

Applying Lemma 5.1 with $\alpha = 1$, leads to

$$
u(t) \leqslant \int_0^t (4\mu_2 + C_V\mu_2)u(s)ds,
$$

where μ_2 is a bound on the second moment of ψ and ρ. The application of Gronwall's lemma enables us to conclude that $u(t) \equiv 0$, which implies that $\psi_i(t) = \rho_i(t)$, $\forall 0 \leqslant t \leqslant T$. Since T is arbitrary, we conclude the uniqueness of the solution to the initial value problem (1.1)–(1.2). ■

Funding. R. Kumar wishes to thank the Science and Engineering Research Board (SERB), Department of Science and Technology (DST), India, for the funding through the project MTR/2021/000866. The research of F. P. da Costa was partially supported by Fundação para a Ciência e a Tecnologia (Portugal) through projects CAMGSD UIDB/04459/2020 and UIDP/04459/2020. The authors claim no conflict of interest.

References

[1] M. Aizenman and T. A. Bak, Convergence to equilibrium in a system of reacting polymers. *Comm. Math. Phys.* **65** (1979), no. 3, 203–230 Zbl 0458.76062 MR 530150

[2] V. Bagland, Convergence of a discrete Oort–Hulst–Safronov equation. *Math. Methods Appl. Sci.* **28** (2005), no. 13, 1613–1632 Zbl 1079.65130 MR 2158221

[3] J. M. Ball and J. Carr, The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation. *J. Statist. Phys.* **61** (1990), no. 1-2, 203–234 Zbl 1217.82050 MR 1084278

[4] J. Banasiak, W. Lamb, and P. Laurençot, *Analytic methods for coagulation-fragmentation models, Volume II*. CRC Press, 2019 Zbl 1434.82002

[5] L. L. Bonilla, A. Carpio, J. C. Neu, and W. G. Wolfer, Kinetics of helium bubble formation in nuclear materials. *Phys. D* **222** (2006), no. 1-2, 131–140 Zbl 1139.82351 MR 2265774

[6] N. Brilliantov, P. Krapivsky, A. Bodrova, F. Spahn, H. Hayakawa, V. Stadnichuk, and J. Schmidt, Size distribution of particles in Saturn's rings from aggregation and fragmentation. *Proc. Natl. Acad. Sci. USA* **112** (2015), no. 31, 9536–9541

[7] F. P. da Costa, Mathematical aspects of coagulation-fragmentation equations. In *Mathematics of energy and climate change*, pp. 83–162, CIM Ser. Math. Sci. 2, Springer, Cham, 2015 MR 3644260

[8] J. Davidson, Existence and uniqueness theorem for the Safronov–Dubovski coagulation equation. *Z. Angew. Math. Phys.* **65** (2014), no. 4, 757–766 Zbl 1298.34029 MR 3238514

[9] P. B. Dubovskiĭ, *Mathematical theory of coagulation*. Lect. Notes Ser. 23, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994 MR 1290321

[10] P. B. Dubovskiĭ, A "triangle" of interconnected coagulation models. *J. Phys. A* **32** (1999), no. 5, 781–793 Zbl 0933.82028 MR 1674432

[11] P. B. Dubovskiĭ, Structural stability of disperse systems and finite nature of a coagulation front. *J. Exp. Theor. Phys.* **89** (1999), no. 2, 384–390

[12] P.-F. Hsieh and Y. Sibuya, *Basic theory of ordinary differential equations*. Universitext, Springer, New York, 1999 Zbl 0924.34001 MR 1697415

[13] H. C. van de Hulst, Generalization of some methods for solving an integral equation of the first kind. *Bull. Astronom. Inst. Netherlands* **10** (1946), 75–79 MR 23624

[14] A. Perelson and R. Samsel, Kinetics of red blood cell aggregation: an example of geometric polymerization. In *Kinetics of aggregation and gelation*, pp. 137–144, Elsevier, 1984

[15] S. Piotrowski, The collisions of asteroids. Astronomical Journal **57** (1952), 23

[16] V. Safronov, *Evolution of the protoplanetary cloud and formation of the earth and the planets.* Israel Program for Scientific Translations Jerusalem, 1972

Received 2 January 2023; revised 12 April 2023.

Sonali Kaushik
Birla Institute of Technology and Science, Pilani, Department of Mathematics, Pilani Campus, Rajasthan, 333031, India; sonalikaushik1234@gmail.com

Rajesh Kumar
Birla Institute of Technology and Science, Pilani, Department of Mathematics, Pilani Campus, Rajasthan, 333031, India; rajesh.kumar@pilani.bits-pilani.ac.in

Fernando P. da Costa
Departamento de Ciências e Tecnologia, Universidade Aberta, Lisboa; and Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal; fcosta@uab.pt