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A remark on the logarithmic decay of the damped wave and Schrödinger equations on a compact Riemannian manifold

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Abstract. In this paper, we consider a compact Riemannian manifold (M, g) of class $C^1 \cap W^{2,\infty}$ and the damped wave or Schrödinger equations on M, under the action of a damping function $a = a(x)$. We establish the following fact: if the measure of the set $\{x \in M;\, a(x) \neq 0\}$ is strictly positive, then the decay in time of the associated energy is at least logarithmic.

1. Introduction

Consider a compact Riemannian manifold (M, g) of class $C^1 \cap W^{2,\infty}$, possibly with boundaries ∂M, endowed with a Lipschitz metric g. Denote $\mathrm{d}_g x$ (or simply $\mathrm{d}x$) the volume element in M associated to the metric $g \in C^0 \cap W^{1,\infty}$ and write vol_g the associated volume on M. Let Δ be the Laplace–Beltrami operator in (M, g). Recall that in local coordinates we may write

$$\Delta = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j). \tag{1.1}$$

In this note we are interested in the evolution of respectively the wave equation and the Schrödinger equation under the influence of a damping term localised via a function $0 \le a(x), a \in L^\infty(M)$ non trivial ($\int_M a(x)\,\mathrm{d}x > 0$) which consequently may be supported in a small subset of M, namely E. We shall briefly recall these models.

The damped wave equation in M under the damping $a\partial_t u$ corresponds to the initial value problem

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = 0, & \mathbb{R}_+ \times M, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & M, \end{cases} \tag{1.2}$$

where (u_0, u_1) is a given initial condition in the natural energy space $\mathscr{H} = H^1(M) \times L^2(M)$. If $\partial M \neq \emptyset$, we impose the boundary conditions

$$u|_{\partial M} = 0 \quad \text{(Dirichlet condition)} \quad \text{or} \quad \partial_\nu u|_{\partial M} = 0 \quad \text{(Neumann condition)}. \tag{1.3}$$

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The energy associated to (1.2) is as usual

$$\mathcal{E}_w(t,u_0,u_1) = \int_M |\partial_t u(t)|^2 \,\mathrm{d}x + \int_M |\nabla_x u(t)|^2 \,\mathrm{d}x, \qquad t \geq 0, \tag{1.4}$$

defined globally as $u \in C^0(\mathbb{R}_+; H^1(M)) \cap C^1(\mathbb{R}_+; L^2(M))$.

Further, the second model we are interested in is the initial value problem for the Schrödinger equation under the action of the damping $a = a(x)$, i.e.,

$$\begin{cases} i\partial_t \psi + \Delta\psi + ia(x)\psi = 0, & \mathbb{R}_+ \times M, \\ \psi|_{t=0} = \psi_0, & M, \end{cases} \tag{1.5}$$

for a given $\psi_0 \in L^2(M;\mathbb{C})$. Again, if $\partial M \neq \emptyset$, we impose the boundary conditions (1.3). The energy associated to (1.5) is

$$\mathcal{E}_S(t,\psi_0) = \int_M |\psi(t)|^2 \,\mathrm{d}x. \tag{1.6}$$

In this note we prove that if $E \subset M$ is any measurable set with $\mathrm{vol}_g(E) > 0$, the energy functionals $\mathcal{E}_w$ and $\mathcal{E}_S$ decay at least logarithmically in time. This is the content of Theorems 1 and 2 below.

1.1. Main results

Since we assume $a \geq 0$, $\int_M a(x)\,\mathrm{d}x > 0$, we deduce that there exists $n > 0$ such that the set

$$F_n = \left\{x \in M;\, a(x) > \frac{1}{n}\right\}$$

has positive measure. As a consequence, with $\alpha = \frac{1}{n}$, $\beta = \|a\|_{L^\infty}$, $F = F_n$, we get that the damping function $a = a(x)$ satisfies

$$\alpha 1_F(x) \leq a(x) \leq \beta, \qquad \text{for almost all } x \in M, \tag{1.7}$$

with $F \subset M$ of positive measure, not necessarily open. Our main result for the wave equation is the following.

Theorem 1. *Let $a \geq 0$, $\int_M a(x)\,\mathrm{d}x > 0$. Then, there exists a constant $C = C(F) > 0$ such that for every*

$$(u_0,u_1) \in \begin{cases} (H^2(M) \cap H_0^1(M)) \times H_0^1(M), & \textit{with Dirichlet boundary conditions}, \\ H^2(M) \times H^1(M), & \textit{otherwise}, \end{cases} \tag{1.8}$$

the solution to the associated damped wave equation (1.2) satisfies

$$\mathcal{E}_w(t,u_0,u_1) \leq \frac{C}{\log(2+t)^2}(\|u_0\|^2_{H^2(M)} + \|u_1\|^2_{H^1(M)}), \qquad t \geq 0, \tag{1.9}$$

where $\mathcal{E}_w$ is the energy defined in (1.4).

See Section 1.2 for notation. In the case of the Schrödinger equation we obtain the next analogous result.

Theorem 2. *Let* $a \geq 0$, $\int_M a(x)\,\mathrm{d}x > 0$. *Then, there exists a constant* $C = C(F) > 0$ *such that for every*

$$\psi_0 \in \begin{cases} H^2(M) \cap H_0^1(M), & \text{with Dirichlet boundary conditions}, \\ H^2(M), & \text{otherwise}, \end{cases} \tag{1.10}$$

the solution to the associated Schrödinger equation (1.5) *satisfies*

$$\mathcal{E}_S(t, \psi_0) \leq \frac{C}{\log(2+t)^4} \|\psi_0\|_{H^2(M)}^2, \qquad t \geq 0, \tag{1.11}$$

where $\mathcal{E}_S$ *is the energy defined in* (1.6).

The strategy of the proof combines the spectral inequalities obtained in [8] (see Theorem 3 below) with a sharp characterisation of the logarithmic decay of energy from [3, 4] (see Theorem 4 in Section 2.1). We give some details concerning these results in Section 2.

1.2. Notation and setting

As mentioned above, the volume induced by g on M is defined as

$$\mathrm{vol}_g(A) = \int_M 1_A(x)\,\mathrm{d}_g x,$$

for every Borel set $A \subset M$. In the case of the Euclidean flat space $\mathbb{R}^d$, we simply write $|A|$ for the d-dimensional Lebesgue measure of a given Borel set $A \subset \mathbb{R}^d$. In both cases we denote $B(x, r)$ the ball of radius $r > 0$ centred at a point x.

As M is compact, the Laplace–Beltrami operator on M defined in (1.1) has compact resolvent. Let $(e_k)_{k\in\mathbb{N}}$ be the family of L^2-normalised eigenfunctions of $-\Delta$, with eigenvalues $\lambda_k^2 \to +\infty$ and satisfying

$$-\Delta e_k = \lambda_k^2 e_k,$$
$$e_k|_{\partial M} = 0 \quad \text{(Dirichlet condition)} \quad \text{or} \quad \partial_\nu e_k|_{\partial M} = 0 \quad \text{(Neumann condition)}.$$

Recall that $(e_k)_{k\in\mathbb{N}}$ is a Hilbert basis of $L^2(M)$ endowed with the usual inner product $\langle\cdot,\cdot\rangle$. Moreover, the usual Sobolev norms on M can be defined using the spectral basis $(e_k)_{k\in\mathbb{N}}$ as follows:

$$\|f\|_{H^s(M)}^2 = \sum_{k\in\mathbb{N}} (1 + \lambda_k^{2s}) |\langle f, e_k\rangle|^2, \qquad f \in H^s(M),$$

for every $s \in \mathbb{R}$.

1.3. Previous work

1.3.1. Decay of damped waves. The study of the decay rates for (1.2) has been addressed by the seminal works [2] and [11]. These works establish an intimate relation between the rate decay of the energy and the support of the damping function. Under the geometric control condition of [2] one can expect an exponential decay, as shown for instance in [2, 7, 11]. On the other hand, when the support of a does not satisfy a geometric control condition, the decay rate of the associated damped wave equation may be slower than exponential. We can find examples in the literature of polynomial decay [1, 6, 14] or even logarithmic [7, 10–12].

Under some hypothesis on the geometry of the manifold, such as the assumption that the manifold is compact and hyperbolic (negative curvature), it is possible to expect exponential decay in some (positive) Sobolev spaces as soon as a is smooth and nonzero (cf. [9]).

In this paper we establish the following fact: if $|\{x \in M; a(x) \neq 0\}| > 0$, then the decay is at least logarithmic. We do not make any assumption on the curvature of the manifold.

1.3.2. Spectral inequalities. In the framework described in Section 1.2, Given a small subset $E \subset M$ (of positive Lebesgue measure or at least not too small), we have studied in [8] how L^p norms of the restrictions to E of arbitrary finite linear combinations of the form

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x)$$

can dominate Sobolev norms of ϕ on the whole M. Our result [8, Thm. 2] is the following.

Theorem 3. *Let (M, g) be a Riemannian manifold of class $C^1 \cap W^{2,\infty}$, possibly with boundaries ∂M. There exists $\delta \in (0, 1)$ such that for any $m > 0$, there exist $C, D > 0$ such that for any $\omega \subset M$ with $\mathrm{vol}_g(\omega) \geq m$ and for any $\Lambda > 0$, we have*

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x) \Rightarrow \|\phi\|_{L^2(M)} \leq C e^{D\Lambda} \|\phi 1_\omega\|_{L^2(M)}. \tag{1.12}$$

We shall use the spectral inequality (1.12) in Section 2.3.

1.4. Outline

In Section 2, we gather some facts about the tools used in the proof of our main result: Section 2.1 is devoted to a characterisation of logarithmic decay, Section 2.3 makes the link between the resolvent operators for waves and Schrödinger and the Helmholtz equation, and Section 2.3 is concerned with some estimates for solutions to the Helmholtz equation obtained thanks to the spectral inequalities mentioned before. The proof of Theorem 1 is carried out in Sections 3 and 4 which treat respectively the case of Dirichlet boundary conditions and the case of Neumann boundary conditions. Each of these two sections is also divided into high frequencies and low frequencies. Finally, Section 5 is concerned with the proof of Theorem 2, which is also divided into Section 5.1 (negative frequencies) and Section 5.2 (nonnegative frequencies).

2. Some tools

In this section, we describe first (in Section 2.1) some abstract results relating the time decay of a semi-group with the growth of the resolvent operator at infinity. Next, in Section 2.2 we focus on the resolvent operators related to the wave equation (1.2) and the Schrödinger equation (1.5), which in both cases lead to a Helmholtz equation of the form

$$\Delta u + \lambda u = S,$$

for some parameter $\lambda \in \mathbb{R}$ and a source term S. Finally, in Section 2.3, we use Theorem 3 to get some estimates for solutions to the Helmholtz equation that will be useful in the sequel.

2.1. Sufficient conditions for logarithmic decay

Consider a Hilbert space $\mathcal{H}$ and the functional equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU, \quad t \geq 0, \qquad \text{with} \qquad U(0) = U_0 \in \mathcal{H}, \tag{2.1}$$

for a possibly unbounded operator A with domain $D(A) \subset \mathcal{H}$. As usual, $z \in \mathbb{C}$ belongs to the resolvent set $\rho(A)$ whenever $(A - z)^{-1} \in \mathcal{L}(\mathcal{H})$. The spectrum of A is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

We focus next on the elements of $\rho(A)$ lying on the imaginary axis. For every $\tau \in \mathbb{R}$ we consider the resolvent mapping $R(\tau) = (A - i\tau)^{-1}$ whenever $i\tau \in \rho(A)$.

2.1.1. Growth of the resolvent and decay of the semi-group. Assume that A is the infinitesimal generator of a C^0-continuous semi-group of operators in $\mathcal{H}$ that we denote $(e^{tA})_{t\geq 0}$, so that the solution to (2.1) writes $U(t) = e^{tA}U_0$. Assume further that

$$\sup_{t\geq 0} \|e^{tA}\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Batty and Duyckaerts have introduced in [3] a quantitative approach to characterise the asymptotic behaviour of the semi-group, i.e., the fact that for some $k \in \mathbb{N}^*$,

$$\lim_{t\to+\infty} m_k(t) = 0, \qquad \text{with} \qquad m_k(t) = \|e^{tA}(\mathrm{Id} - A)^{-k}\|_{\mathcal{L}(\mathcal{H})}, \quad t \geq 0, \tag{2.2}$$

in terms of the purely spectral condition

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \tag{2.3}$$

Observe that this condition ensures that the resolvent operators $R(\tau)$ are well defined for any $\tau \in \mathbb{R}$. Moreover, it is possible to describe the decay rate of (2.2) in terms of the growth of the function

$$\mathcal{M}(\mu) = \sup_{|\tau| \le \mu} \|(A - i\tau)^{-1}\|_{\mathcal{L}(\mathcal{H})}, \qquad \mu \in [0, +\infty). \tag{2.4}$$

The following result, obtained first by Lebeau–Robbiano [12] (for the subexponential growth and with a $\log(\log(t))$ loss), then by Burq [4, Thm. 3] (also for the subexponential growth but without loss) for exterior problems, and finally in greater generality by Batty–Duyckaerts, [3, Thm. 1.5], guarantees logarithmic decay of all m_k as long as $\mathcal{M}$ grows at most exponentially at infinity.

Theorem 4. *Assume that* (2.3) *holds and that*

$$\exists C, c > 0 \quad \text{such that} \quad \mathcal{M}(\mu) < Ce^{c|\mu|} \ (\text{resp. } \mathcal{M}(\mu) < Ce^{c\sqrt{|\mu|}}) \quad \forall \mu \in \mathbb{R}.$$

Then, for any $k > 0$ there exists C_k such that

$$\|e^{tA}(\mathrm{Id} - A)^{-k}\|_{\mathcal{L}(\mathcal{H})} \le \frac{C_k}{\log(2+t)^k}, \qquad \forall t \ge 0$$

$$(\text{resp.} \quad \|e^{tA}(\mathrm{Id} - A)^{-k}\|_{\mathcal{L}(\mathcal{H})} \le \frac{C_k}{\log(2+t)^{2k}}, \qquad \forall t \ge 0).$$

We shall use this result in Sections 3 and 5 to get the decay of solutions to (1.2) and (1.5).

2.2. The resolvent operator and the Helmholtz equation

In this section, we make explicit the choice of the functional framework (compatible with Section 2.1) associated to the wave equation and the Schrödinger equations.

2.2.1. The resolvent operator for waves. Following the notation of Section 2.1, let us set

$$AU = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & -a(x) \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.5}$$

in the Hilbert space $\mathcal{H} = H^1(M) \times L^2(M)$ endowed with the natural inner product. As usual, $D(A) = H^2(M) \times H^1(M)$ if $\partial M = \emptyset$ or $D(A) = (H^2(M) \cap H_0^1(M)) \times H^1(M)$ if we impose Dirichlet boundary conditions (the case of Neumann boundary conditions is slightly more involved and we deal with it in Section 4). The solution of (2.1) is given by

$$U(t) = e^{tA} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{2.6}$$

and the solution of (1.2) is given by the first component of $U(t)$. Let $\tau \in \mathbb{R}$ and consider the resolvent operator $R(\tau) = (A - i\tau)^{-1}$. For any $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{H}$, one has

$$\begin{pmatrix} u \\ v \end{pmatrix} = (A - i\tau)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow \begin{cases} v - i\tau u = f, \\ \Delta u - av - i\tau v = g. \end{cases}$$

Using that $v = i\tau u + f$, we find

$$\Delta u - a(i\tau u + f) + \tau^2 u - i\tau f = g, \qquad \text{in } M,$$

and hence u satisfies the Helmholtz equation

$$\Delta u + \tau^2 u = g + (a + i\tau) f + ia\tau u, \qquad \text{in } M. \tag{2.7}$$

2.2.2. The resolvent operator for Schrödinger. In this case we set $\mathcal{H} = L^2(M;\mathbb{C})$ and

$$A = i\Delta - a(x), \qquad D(A) = H^2(M;\mathbb{C}). \tag{2.8}$$

For $\psi_0 \in \mathcal{H}$ given, the solution of (1.5) is then given by the

$$U(t)\psi_0 = e^{tA}\psi_0, \qquad t \geq 0. \tag{2.9}$$

Now, let $\tau \in \mathbb{R}$ and consider the resolvent operator $R(\tau) = (A - i\tau)^{-1}$. If for some $f \in \mathcal{H}$ the function $\psi \in D(A)$ is such that $(A - i\tau)^{-1}\psi = f$, then ψ satisfies the following Helmholtz equation:

$$\Delta\psi - \tau\psi + ia(x)\psi = -if, \qquad \text{in } M. \tag{2.10}$$

2.3. Estimates for the Helmholtz equation

We state for convenience a unique continuation result for the Helmholtz equation that will be useful in Section 3.2. The unique continuation from *small* sets follows from the Remez inequalities obtained in [13, Sect. 1, eq. (6)].

Lemma 2.1. *Let $\omega \subset M$ be a measurable set with $\mathrm{vol}_g(\omega) > 0$. Let $\lambda \in \mathbb{R}$ be fixed and let u be the solution to the Helmholtz equation*

$$\Delta u + \lambda u = 0, \qquad \text{in } M. \tag{2.11}$$

Then, u satisfies the unique continuation principle on ω, i.e.,

$$u|_\omega = 0 \quad \Rightarrow \quad u|_M = 0. \tag{2.12}$$

We state next an estimate for solutions to the Helmholtz equation with source that will be used in Section 3.1 to get a suitable exponential bound in the high frequency regime. The following result follows from the spectral inequality (1.12) in Theorem 3 above.

Proposition 2.2. *Let $\omega \subset M$ be a measurable set with $\mathrm{vol}_g(\omega) > 0$. There exist constants $C = C(\omega) > 0$ and $D = D(\omega) > 0$ such that for every $\mu \in \mathbb{R}$ and $S \in L^2(M)$, the solution to the Helmholtz equation*

$$\Delta u + \mu^2 u = S \quad \text{in } M, \quad u|_{\partial M} = 0 \quad \text{(Dirichlet)} \quad \text{or} \quad \partial_\nu u|_{\partial M} = 0 \quad \text{(Neumann)} \tag{2.13}$$

satisfies

$$\|u\|_{L^2(M)} \leq Ce^{D|\mu|}(\|S\|_{L^2(M)} + \|1_\omega u\|_{L^2(M)}). \tag{2.14}$$

Proof. Let u be given by (2.13). If $S \in L^2(M)$, using the orthonormal basis (e_k) as in Section 1.2, we can write

$$S = \sum_{k \in \mathbb{N}} S_k e_k.$$

Then, we can split u into "hyperbolic" and "elliptic" frequencies as follows:

$$u = 1_{|\Delta+\mu^2|\leq 1} u + 1_{|\Delta+\mu^2|>1} u,$$

where

$$1_{|\Delta+\mu^2|>1} u = \sum_{\substack{k \in \mathbb{N};\\ \mu^2 - k^2 > 1}} \frac{S_k}{\mu^2 - k^2} e_k.$$

Thanks to this explicit expression, we have

$$\|1_{|\Delta+\mu^2|>1} u\|_{L^2(M)} \leq \|S\|_{L^2(M)}. \tag{2.15}$$

Now applying Theorem 3 on the "hyperbolic" frequencies, we get

$$\begin{aligned}
\|1_{|\Delta+\mu^2|\leq 1} u\|_{L^2(M)} &\leq Ce^{D|\mu|}\|1_{|\Delta+\mu^2|\leq 1} u\|_{L^2(\omega)}, \\
&\leq Ce^{D|\mu|}(\|u\|_{L^2(\omega)} + \|1_{|\Delta+\mu^2|>1} u\|_{L^2(\omega)}), \\
&\leq Ce^{D|\mu|}(\|u\|_{L^2(\omega)} + \|S\|_{L^2(M)}),
\end{aligned}$$

where we have used (2.15). ■

3. Proof of Theorem 1 for Dirichlet boundary conditions

Following the notation of Section 2.1, let $\tau \in \mathbb{R}$ and consider the resolvent operator $R(\tau) = (A - i\tau)^{-1}$. Throughout this section we assume that the boundary conditions are of Dirichlet type only and use the notation of Section 2.2.1.

In Section 3.1, we prove the resolvent estimate for the wave equations for $\tau \geq \tau^*$ (with τ^* a constant sufficiently large). Then in Section 3.2 we prove the estimate for $\tau \leq \tau^*$.

3.1. Proof of Theorem 1: High frequencies

Proposition 3.1. *Let $F \subset E \subset M$ and a damping a satisfying* (1.7)*. Then there exist $\tau_* \geq 1$ large enough and constants $C_h, c_h > 0$ independent of τ such that for every $|\tau| \geq \tau_*$ we have*

$$\|U\|_{H^1(M)\times L^2(M)} \leq C_h e^{c_h|\tau|} \|(f,g)\|_{H^1(M)\times L^2(M)}, \tag{3.1}$$

for every $h = (f,g) \in \mathcal{H}$ and every $U = (u,v) = (A - i\tau)^{-1}h \in D(A)$.

Proof. First recall that for any $h = \binom{f}{g} \in \mathcal{H}$, the element $U = \binom{u}{v} = (A - i\tau)^{-1}h$ satisfies the Helmholtz equation (2.7) with the boundary conditions

$$u|_{\partial M} = 0$$

and the identities

$$v - i\tau u = f \qquad \text{and} \qquad \Delta u - av - i\tau v = g, \qquad \text{in } M.$$

The first equation yields

$$\|v\|_{L^2} \leq |\tau|\|u\|_{L^2} + \|f\|_{L^2}$$

and the second one, after multiplying by $\overline{u}$ and integrating, gives, for any $|\tau| \geq 1$,

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &\leq \int_M |((a + i\tau)v + g)\overline{u}| \, \mathrm{d}x \\
&= \int_M |((a + i\tau)(f + i\tau u) + g)\overline{u}| \, \mathrm{d}x \\
&\lesssim (1 + \beta + \tau^2)\|u\|_{L^2}(\|f\|_{L^2} + \|g\|_{L^2} + \|u\|_{L^2}),
\end{aligned}$$

where we have used that $a \leq \beta$. We deduce

$$\|u\|_{H^1} \leq C(1 + \beta + \tau^2)(\|f\|_{L^2} + \|g\|_{L^2} + \|u\|_{L^2}).$$

Hence, it is sufficient to estimate $\|u\|_{L^2}$ to get an estimate on $\|U\|_{H^1(M)\times L^2(M)}$. Let us focus on $\|u\|_{L^2}$. Recalling that u satisfies the Helmholtz equation (2.7), using Proposition 2.2 with

$$\omega = F \qquad \text{and} \qquad S = g + (a + i\tau)f + ia\tau u, \quad \mu = \tau,$$

the estimate (2.14) yields

$$\begin{aligned}\|u\|_{L^2(M)} &\le Ce^{D|\tau|}\big(\|g+(a+i\tau)f+ia\tau u\|_{L^2(M)}+\|1_F u\|_{L^2(M)}\big)\\ &\le Ce^{D|\tau|}\big(\|g\|_{L^2(M)}+(|\tau|+\beta)\|f\|_{L^2(M)}+|\tau|\|au\|_{L^2(M)}\\ &\quad+\|1_F u\|_{L^2(M)}\big)\\ &\le (1+|\tau|+\beta)Ce^{D|\tau|}\|(f,g)\|_{\mathcal{H}}+Ce^{D|\tau|}\big(|\tau|\|au\|_{L^2(M)}\\ &\quad+\|1_F u\|_{L^2(M)}\big).\end{aligned}$$

On the other hand, (1.7) implies

$$\|1_F u\|_{L^2(M)} \le \frac{\sqrt{\beta}}{\alpha}\|\sqrt{a}u\|_{L^2(M)}, \tag{3.2}$$

and

$$\|au\|_{L^2(M)} \le \sqrt{\beta}\|\sqrt{a}u\|_{L^2(M)}. \tag{3.3}$$

As a result, we get

$$\begin{aligned}\|u\|_{L^2(M)} &\le (1+|\tau|+\beta)Ce^{D|\tau|}\|(f,g)\|_{\mathcal{H}}\\ &\quad+\Big(|\tau|+\frac{1}{\alpha}\Big)\sqrt{\beta}Ce^{D|\tau|}\|\sqrt{a}u\|_{L^2(M)}.\end{aligned} \tag{3.4}$$

Next, we need to estimate $\|\sqrt{a}u\|_{L^2(M)}$ in terms of $\|(f,g)\|_{\mathcal{H}}$ and τ. Using (2.7), we obtain

$$\int_M(\Delta u+\tau^2 u)\overline{u}\,\mathrm{d}x=\int_M(g+(i\tau+a)f+ia\tau u)\overline{u}\,\mathrm{d}x$$

and hence,

$$-\int_M|\nabla_x u|^2\,\mathrm{d}x+\tau^2\int_M|u|^2\,\mathrm{d}x=\int_M(g+(i\tau+a)f)\overline{u}\,\mathrm{d}x+i\tau\int_M a|u|^2\,\mathrm{d}x.$$

Taking the imaginary part, we find

$$\begin{aligned}|\tau|\|\sqrt{a}u\|^2_{L^2(M)} &= \Big|\mathrm{Im}\int_M(g+(i\tau+a)f)\overline{u}\,\mathrm{d}x\Big|\\ &\le \frac{1}{2\varepsilon}\|g+(i\tau+a)f\|^2_{L^2(M)}+\frac{\varepsilon}{2}\|u\|^2_{L^2(M)}\\ &\le \frac{1}{2\varepsilon}\big(\|g\|_{L^2(M)}+\|(i\tau+a)f\|_{L^2(M)}\big)^2+\frac{\varepsilon}{2}\|u\|^2_{L^2(M)}\\ &\le \frac{1}{\varepsilon}\|g\|^2_{L^2(M)}+\frac{1}{\varepsilon}(|\tau|^2+\beta^2)\|f\|^2_{L^2(M)}+\frac{\varepsilon}{2}\|u\|^2_{L^2(M)},\end{aligned}$$

for every $\varepsilon>0$. Choosing

$$\varepsilon=\frac{|\tau|e^{-2D|\tau|}}{2(|\tau|+\frac{1}{\alpha})^2\beta C^2}$$

one finds

$$\|\sqrt{a}u\|_{L^2(M)}^2 \le \frac{2}{\tau^2}\Big(|\tau| + \frac{1}{\alpha}\Big)^2 \beta C^2 e^{2D|\tau|}(\|g\|_{L^2(M)}^2 + (|\tau|^2 + \beta^2)\|f\|_{L^2(M)}^2) \\ + \frac{e^{-2D|\tau|}}{4(|\tau| + \frac{1}{\alpha})^2 \beta C^2}\|u\|_{L^2(M)}^2$$

and thus,

$$\|\sqrt{a}u\|_{L^2(M)} \le \sqrt{2}\Big(1 + \frac{1}{\alpha|\tau|}\Big)\sqrt{\beta}C(1 + |\tau| + \beta)e^{D|\tau|}\|(f,g)\|_{\mathcal{H}} \\ + \frac{e^{-D|\tau|}}{2(|\tau| + \frac{1}{\alpha})\sqrt{\beta}C}\|u\|_{L^2(M)}.$$

Now, we get from (3.4)

$$\begin{aligned}\|u\|_{L^2(M)} &\le (1 + |\tau| + \beta)Ce^{D|\tau|}\|(f,g)\|_{\mathcal{H}} + \Big(|\tau| + \frac{1}{\alpha}\Big)\sqrt{\beta}Ce^{D|\tau|}\|\sqrt{a}u\|_{L^2(M)} \\ &\le (1 + |\tau| + \beta)Ce^{D|\tau|}\|(f,g)\|_{\mathcal{H}} \\ &\quad + \frac{\sqrt{2}}{|\tau|}\Big(|\tau| + \frac{1}{\alpha}\Big)^2 \beta C^2 e^{2D|\tau|}(1 + |\tau| + \beta)\|(f,g)\|_{\mathcal{H}} + \frac{1}{2}\|u\|_{L^2(M)}.\end{aligned}$$

Then, if τ_* is large enough so that $\tau^* \ge 1$, we have

$$\begin{aligned}\frac{1}{2}\|u\|_{L^2(M)} &\le C(1 + |\tau| + \beta)\Big(1 + e^{D|\tau|}\sqrt{2}C\beta\Big(|\tau| + \frac{1}{\alpha}\Big)^2\Big)e^{D|\tau|}\|(f,g)\|_{\mathcal{H}} \\ &\le C'e^{3D|\tau|}\|(f,g)\|_{\mathcal{H}},\end{aligned} \tag{3.5}$$

for every $|\tau| \ge \tau_*$, where C' is a positive constant depending only on $\alpha, \beta, C, D, \tau^*$. Hence, estimate (3.1) follows for some constants C_h, c_h large enough, depending only on α, β, C, D, and τ_*. ∎

3.2. Proof of Theorem 1: Low frequencies (Dirichlet boundary conditions)

Proposition 3.2. *Let $F \subset E \subset M$ and a damping a satisfying (1.7). For any $\tau_* > 0$ (we shall choose τ_* given by Proposition 3.1), there exists a constant $C_\ell > 0$ such that for any $|\tau| < \tau_*$, every $h = (f,g) \in \mathcal{H}$ and every U such that $U = (A - i\tau)^{-1}h$,*

$$\|U\|_{H^1(M)\times L^2(M)} \le C_\ell\|(f,g)\|_{H^1(M)\times L^2(M)}. \tag{3.6}$$

Proof. In this case, we proceed by contradiction (we follow [7, Sect. 4]). Assume that the estimate is not true, i.e., that there exist sequences $(U_n) \subset \mathcal{H}$ and $(\tau_n) \subset \mathbb{R}$ with $|\tau_n| \le \tau_*$ such that

$$\|U_n\|_{H^1(M)\times L^2(M)} = 1, \quad \forall n \in \mathbb{N}, \qquad \text{and} \qquad (A - i\tau_n)U_n \to 0 \text{ as } n \to +\infty. \tag{3.7}$$

Writing $U_n = \binom{u_n}{v_n}$, we have

$$v_n - i\tau_n u_n \to 0, \quad \text{in } H^1(M) \quad \text{and} \quad \Delta u_n - i\tau_n a(x)u_n + \tau_n^2 u_n \to 0 \quad \text{in } L^2(M). \tag{3.8}$$

Now multiplying the last limit by $\overline{u_n}$ and integrating by parts we find

$$-\|\nabla u_n\|^2_{L^2(M)} - i\tau_n \int_M a(x)|u_n|^2 \, \mathrm{d}x + \tau_n^2 \|u_n\|^2_{L^2(M)} \to 0.$$

Taking real and imaginary parts yields

$$-\|\nabla u_n\|^2_{L^2(M)} + \tau_n^2 \|u_n\|^2_{L^2(M)} \to 0, \qquad \text{and} \qquad \tau_n \int_M a(x)|u_n|^2 \, \mathrm{d}x \to 0. \tag{3.9}$$

The sequence (τ_n) is bounded (in modulus) by τ_* and consequently we can assume that it converges to some limit τ. We distinguish now two cases.

Case $\tau = 0$. In this case, we would have

$$\|\nabla u_n\|^2_{L^2(M)} \to 0,$$

thanks to (3.9). Hence, by Poincaré's inequality, we would also have

$$\|u_n\|^2_{L^2(M)} \to 0.$$

But then, the first part of (3.8) would also imply

$$\|v_n\|^2_{L^2(M)} \to 0.$$

Henceforth,

$$\|U_n\|^2_{H^1(M)} \to 0,$$

which is a contradiction with (3.7).

Case $\tau \neq 0$. In this case, using (3.9) we may write

$$\lim_{n\to\infty} \|\nabla u_n\|^2_{L^2(M)} = \lim_{n\to\infty} \tau^2 \|u_n\|^2_{L^2(M)},$$

and

$$\lim_{n\to\infty} \int_M a(x)|u_n|^2(x) \, \mathrm{d}x = 0.$$

Using (3.7), we also have

$$\lim_{n\to\infty} \|v_n\|^2_{L^2(M)} = \lim_{n\to\infty} \tau^2 \|u_n\|^2_{L^2(M)}.$$

Then, as

$$1 = \lim_{n\to\infty} \|U_n\|^2_{H^1(M)\times L^2(M)} = \lim_{n\to\infty} (1 + 2\tau^2)\|u_n\|^2_{L^2(M)}, \tag{3.10}$$

which means that the sequence (u_n) is bounded in H^1. Then, Rellich's compactness theorem implies that there exists $u \in H^1(M)$ such that

$$u_n \to u \quad \text{in } L^2(M)\text{-strong}, \qquad \text{and} \qquad \nabla u_n \to \nabla u \quad \text{in } L^2(M)\text{-weak},$$

and a fortiori, upon extracting a subsequence, we also have

$$u_n \to u \qquad \text{a.e. in } M.$$

Now, thanks to Fatou's lemma and the second part of (3.9) we deduce

$$\int_M a(x)|u|^2 \,\mathrm{d}x \le \liminf_{n\to\infty} \int_M a(x)|u_n|^2 \,\mathrm{d}x = 0$$

and hence, using (1.7),

$$u = 0 \qquad \text{a.e. in } F.$$

This is enough to apply Lemma 2.1 which yields

$$u = 0 \qquad \text{a.e. in } M.$$

But this is a contradiction with (3.10). This concludes the proof. ∎

3.3. End of the proof of Theorem 1 (Dirichlet boundary conditions)

Once we have dealt with high and low frequencies in the previous sections, the proof of Theorem 1 is a consequence of Theorem 4.

Proof of Theorem 1. Combining Proposition 3.1 and Proposition 3.2, we find that the estimate

$$\|U\|_{H^1\times L^2} \le Ce^{2c|\tau|}\|h\|_{H^1\times L^2}$$

holds for every $\tau \in \mathbb{R}$, every $h \in H^1(M) \times L^2(M)$, $U = R(\tau)h$ and the constants

$$C = \max(C_\ell, C_h), \qquad c = c_h.$$

As a consequence, the function $\mathcal{M}$ defined by (2.4) satisfies in this case the growth

$$\mathcal{M}(\mu) \le Ce^{2c|\mu|},$$

for every $\mu \in \mathbb{R}$. Hence, Theorem 4 yields that for any $k \in \mathbb{N}$,

$$\|U(t)(\mathrm{Id} - A)^{-k}\|_{\mathcal{L}(H^1\times L^2)} \le \frac{C_k}{\log(2+t)^k}, \qquad \forall t \ge 0, \tag{3.11}$$

where we have used the notation of (2.5) and (2.6). Next, let $k \in \mathbb{N}$ be fixed and let $h_0 = \binom{u_0}{u_1} \in H^{k+1}(M) \times H^k(M)$. Then, $W_0 := (\mathrm{Id} - A)^k h_0 \in L^2(M)$ and inequality (3.11) implies

$$\begin{aligned}
\mathcal{E}_w(t, u_0, u_1) &= \|U(t)h_0\|^2_{H^1 \times L^2} \\
&= \|U(t)(\mathrm{Id} - A)^{-k} W_0\|^2_{H^1 \times L^2} \\
&\leq \frac{C_k}{\log(2+t)^{2k}} \|W_0\|^2_{H^1 \times L^2} \\
&\leq \frac{C_k}{\log(2+t)^{2k}} \|(\mathrm{Id} - A)^k h_0\|^2_{H^1 \times L^2},
\end{aligned}$$

for every $t \geq 0$. The proof is completed by noticing that for $k = 1$ the domain of A is $(H^2(M) \cap H^1_0(M)) \times H^1_0(M)$ with Dirichlet boundary conditions (resp. $H^2(M) \times H^1(M)$ if $\partial M = \emptyset$), and

$$\|(\mathrm{Id} - A)h_0\|^2_{H^1 \times L^2} \leq C\|h_0\|^2_{H^2 \times H^1}. \qquad \blacksquare$$

4. Proof of Theorem 1: The case of Neumann boundary conditions

In the case $\tau = 0$, we cannot use Poincaré's inequality (that we used when dealing with the low frequencies in Proposition 3.2) and we have to change slightly the functional framework. Here we follow the exposition in [5, Appendix]. For the sake of completeness, we recall the argument (which is taken from [5, Appendix]) below and focus on the low frequency regime $|\tau| \leq \tau^*$.

For $s = 1, 2$, we define by $\dot{H}^s = H^s(M)/\mathbb{R}$ the quotient space of $H^s(M)$ by the constant functions, endowed with the norm

$$\|\dot{u}\|_{\dot{H}^1} = \|\nabla u\|_{L^2}, \qquad \|\dot{u}\|_{\dot{H}^2} = \|\Delta u\|_{L^2},$$

(here $\dot{u}$ denotes the equivalence class of a function u in $\dot{H}$). We define the operator

$$\dot{A} = \begin{pmatrix} 0 & \Pi \\ \dot{\Delta} & -a \end{pmatrix}$$

on $\dot{H}^1 \times L^2$ with domain $\dot{H}^2 \times H^1$, where Π is the canonical projection $H^1 \to \dot{H}^1$ and $\dot{\Delta}$ is defined by

$$\dot{\Delta}\dot{u} = (\dot{\Delta u})$$

(independent of the choice of $u \in \dot{u}$). The operator $\dot{A}$ is maximal dissipative and hence defines a semi-group of contractions on $\dot{\mathcal{H}} = \dot{H}^1 \times L^2$. Indeed, for $U = \binom{\dot{u}}{v}$,

$$\mathrm{Re}(\dot{A}U, U)_{\dot{\mathcal{H}}} = \mathrm{Re}(\nabla u, \nabla v)_{L^2} + (\dot{\Delta}\dot{u} - av, v)_{L^2} = -(av, v)_{L^2},$$

and

$$(\dot{A}-\mathrm{Id})\begin{pmatrix}\dot{u}\\ v\end{pmatrix}=\begin{pmatrix}\dot{f}\\ g\end{pmatrix}\Leftrightarrow \Pi v-\dot{u}=\dot{f},\quad \text{and}\quad \dot{\Delta}\dot{u}-(a+1)v=g,$$
$$\Leftrightarrow \Pi v-\dot{u}=\dot{f},\qquad\qquad (4.1)$$
$$\text{and}\quad \Delta v-(1+a)v=g+\Delta f\in H^{-1}(M),$$

and we can solve this equation by variational theory. Notice that this shows that the resolvent $(\dot{A}-\mathrm{Id})^{-1}$ is well defined and continuous from $\dot{H}^1\times L^2$ to $\dot{H}^2\times H^1$. We have further the following.

Lemma 4.1. *The injection $\dot{H}^2\times H^1$ to $\dot{H}^1\times L^2$ is compact.*

This follows from identifying $\dot{H}^s$ with the kernel of the linear form $u\mapsto\int_M u$. We also have the following corollary.

Corollary 4.2. *The operator $(\dot{A}-\mathrm{Id})^{-1}$ is compact on $\dot{\mathcal{H}}$.*

On the other hand, it is very easy to show that for $(u_0,u_1)\in H^1\times L^2$,

$$\begin{pmatrix}\Pi & 0\\ 0 & \mathrm{Id}\end{pmatrix}e^{tA}=e^{t\dot{A}}\begin{pmatrix}\Pi & 0\\ 0 & \mathrm{Id}\end{pmatrix},$$

and hence, the logarithmic decay is equivalent to the logarithmic decay (in norm) of $e^{t\dot{A}}$ (and consequently, according to Theorem 4 equivalent to resolvent estimates for $\dot{A}$). The high frequency resolvent estimates in our new setting are handled with the exact same proof as for Dirichlet boundary conditions (we did not use Poincaré's inequality in this regime) and consequently we omit the proof. Let us focus on the low frequency regime and revisit our proof above in this new functional setting. We prove the following proposition.

Proposition 4.3. *Let $F\subset E\subset M$ and a damping a satisfying* (1.7)*. For any $\tau_*>0$ (we shall choose τ_* given by Proposition* 3.1*), there exist constants $C>0$ such that for any $|\tau|<\tau_*$, and every $h=(f,g)\in\mathcal{H}$ every U such that $U=(\dot{A}-i\tau)^{-1}h$,*

$$\|U\|_{\dot{H}^1(M)\times L^2(M)}\le C\|(f,g)\|_{\dot{H}^1(M)\times L^2(M)}.$$

Proof. We argue by contradiction. Suppose there exist sequences (τ_n), (U_n), (F_n) such that

$$(\dot{A}-i\tau_n)U_n=F_n,\qquad \|U_n\|_{\dot{\mathcal{H}}}>n\|F_n\|_{\dot{\mathcal{H}}}.$$

Since $U_n\neq 0$, we can assume $\|U_n\|_{\dot{\mathcal{H}}}=1$. Extracting subsequences (still indexed by n for simplicity) we can also assume that $\tau_n\to\tau\in\mathbb{R}$ as $n\to\infty$. We write

$$U_n=\begin{pmatrix}\dot{u}_n\\ v_n\end{pmatrix},\qquad F_n=\begin{pmatrix}\dot{f}_n\\ g_n\end{pmatrix},$$

and distinguish according to two cases.

Zero frequency case: $\tau = 0$. In this case, we have

$$\dot{A}U_n = o(1)_{\dot{\mathcal{H}}} \Leftrightarrow \Pi v_n = o(1)_{\dot{H}^1}, \qquad \Delta \dot{u}_n - a v_n = o(1)_{L^2}.$$

We deduce that there exists $(c_n) \subset \mathbb{C}$ such that

$$v_n - c_n = o(1)_{H^1}, \qquad \Delta u_n - a c_n = o(1)_{L^2}.$$

But

$$0 = \int_M \Delta u_n \, dx \Rightarrow c_n \int_M a \, dx = o(1) \Rightarrow c_n = o(1).$$

As a consequence, we get $v_n = o(1)_{L^2}$ and $\Delta u_n = o(1)_{L^2}$, which implies that $\dot{u}_n = o(1)_{\dot{H}^1}$. This contradicts $\|U_n\|_{\dot{\mathcal{H}}} = 1$. As a result, (4.3) follows for $\tau = 0$.

Low (nonzero) frequency case: $\tau \in \mathbb{R}^*$. In this case, we have

$$(\dot{A} - i\tau)U_n = o(1)_{\dot{\mathcal{H}}} \Leftrightarrow \Pi v_n - i\tau \dot{u}_n = o(1)_{\dot{H}^1}, \qquad \Delta \dot{u}_n - (i\tau + a)v_n = o(1)_{L^2}.$$

We deduce

$$\Delta v_n - i\tau(a + i\tau)v_n = o(1)_{L^2} + \Delta(o(1)_{\dot{H}^1}) = o(1)_{H^{-1}}.$$

Since (v_n) is bounded in L^2, from this equation, we deduce that (Δv_n) is bounded in H^{-1} and consequently (v_n) is bounded in H^1. Extracting another subsequence, we can assume that (v_n) converges in L^2 to v which satisfies

$$\Delta v + \tau^2 v - i\tau a v = 0, \qquad \text{in } M.$$

Taking the imaginary part of the scalar product with $\overline{v}$ in L^2 gives (since $\tau \neq 0$) $\int_M a|v|^2 \, dx = 0$, and consequently $av = 0$ which implies that v is an eigenfunction of the Laplace operator and vanishes on F. Let us recall the classical result.

Proposition 4.4. *Let v be an eigenfunction of our Laplace operator*

$$-\Delta v = \lambda v.$$

Assume that v vanishes on a set F of positive Lebesgue measure. Then $v = 0$.

We deduce that $v_n = o(1)_{L^2}$. Now, we have

$$\Delta \dot{u}_n = (i\tau + a)v_n + o(1)_{L^2} = o(1)_{L^2} \Rightarrow \dot{u}_n = o(1)_{\dot{H}^1},$$

but this contradicts $\|U_n\|_{\dot{\mathcal{H}}} = 1$ and (4.3) follows also in this case. This ends the proof. ∎

Remark 4.5. Proposition 4.4 is a straightforward consequence of the resolvent estimate from Theorem 3 applied to this eigenfunction. However, while this quantitative result is in turn consequence of the quite recent deep analysis in [13], the qualitative result claimed in Proposition 4.4 (v vanishing on F implies v vanish on the whole M) has been known for much longer (see [15]), as it relies only on *qualitative* estimates rather than *quantitative* estimates.

Once we have established Proposition 4.3, the proof of Theorem 1 with Neumann boundary conditions follows the same lines of Section 3.3 without significant modifications. This ends the proof of Theorem 1.

5. Proof of Theorem 2: Schrödinger equation

In order to prove Theorem 2 it is enough to prove the following resolvent estimates.

Proposition 5.1. *There exists $C > 0$ such for any $\tau \in \mathbb{R}$, the operator*

$$(\Delta - \tau + ia) : D(A) \to L^2(M)$$

is invertible with bounded inverse

$$\|(\Delta - \tau + ia)^{-1}\|_{\mathcal{L}(L^2(M))} \leq Ce^{c\sqrt{|\tau|}}.$$

5.1. Estimates when $\tau < 0$

In this case, we may use Proposition 2.2 directly. We get the following result.

Proposition 5.2. *Assume that (1.7) holds for some α, β and let $f \in L^2(M; \mathbb{C})$ be given. Then, for any $\tau < 0$, the resolvent $(\Delta - \tau + ia)^{-1}$ satisfies*

$$\|(\Delta - \tau + ia)^{-1} f\|_{L^2(M)} \leq 2(1 + C)^2 \Big(1 + \beta \Big(1 + \frac{1}{\alpha}\Big)^2\Big) e^{2D\sqrt{|\tau|}} \|f\|_{L^2},$$

where C, D are the constants in (2.14).

Proof. Recall that $\psi = -i(\Delta - \tau + ia)^{-1} f$ satisfies the Helmholtz equation (2.10). Then, thanks to (2.14) there exist some constants $C, D > 0$ independent of τ such that

$$\|\psi\|_{L^2(M)} \leq Ce^{D\sqrt{|\tau|}} (\|i(f + a\psi)\|_{L^2(M)} + \|1_F \psi\|_{L^2(M)}).$$

On the other hand, from (3.2) and (3.3), we get

$$\|\psi\|_{L^2(M)} \leq Ce^{D\sqrt{|\tau|}} \Big(\|f\|_{L^2} + \Big(1 + \frac{1}{\alpha}\Big)\sqrt{\beta}\|\sqrt{a}\psi\|_{L^2(M)}\Big). \tag{5.1}$$

Next, using the Helmholtz equation (2.10) we obtain

$$\int_M (\Delta\psi - \tau\psi)\overline{\psi}\,\mathrm{d}x = \int_M (-if - ia\psi)\overline{\psi}\,\mathrm{d}x$$

and hence,

$$\int_M |\nabla_x\psi|^2\,\mathrm{d}x + \tau\int_M |\psi|^2\,\mathrm{d}x = i\int_M f\overline{\psi}\,\mathrm{d}x + i\int_M a|\psi|^2\,\mathrm{d}x.$$

Now, taking the imaginary part and using Cauchy–Schwarz's and Young's inequalities we find

$$\begin{aligned}\|\sqrt{a}\psi\|_{L^2(M)} = \sqrt{-\operatorname{Im}\int_M f\overline{\psi}\,\mathrm{d}x} &\leq \|f\|_{L^2(M)}^{1/2}\|\psi\|_{L^2(M)}^{1/2}\\ &\leq \frac{1}{4\varepsilon}\|f\|_{L^2(M)} + \varepsilon\|\psi\|_{L^2(M)},\end{aligned}$$

for every $\varepsilon > 0$. Injecting this in (5.1) yields

$$\begin{aligned}\|\psi\|_{L^2(M)} \leq Ce^{D\sqrt{|\tau|}}\Big(1 + \sqrt{\beta}\Big(1 + \frac{1}{\alpha}\Big)\frac{1}{2\sqrt{\varepsilon}}\Big)\|f\|_{L^2}\\ + Ce^{D\sqrt{|\tau|}}\sqrt{\beta\varepsilon}\Big(1 + \frac{1}{\alpha}\Big)\|\psi\|_{L^2}.\end{aligned}$$

Next, choosing

$$\varepsilon = \frac{e^{-2D\sqrt{|\tau|}}}{4(1+\frac{1}{\alpha})^2\beta C^2}$$

we get

$$\|\psi\|_{L^2(M)} \leq 2Ce^{D\sqrt{|\tau|}}\Big(1 + \beta\Big(1 + \frac{1}{\alpha}\Big)^2 Ce^{D\sqrt{|\tau|}}\Big)\|f\|_{L^2}.$$

As a consequence, we get the exponential growth estimate

$$\|\psi\|_{L^2(M)} \leq \sqrt{2}(1 + C)^2\Big(1 + \beta\Big(1 + \frac{1}{\alpha}\Big)^2\Big)e^{2D\sqrt{|\tau|}}\|f\|_{L^2},$$

for any $\tau > 0$ given. ■

5.2. Estimates when $0 \leq \tau$

In this section, we shall just rely on the following Poincaré-type inequality.

Proposition 5.3. *Assume that* $a \geq 0$ *and* $\int_M a(x)\,\mathrm{d}x > 0$. *Then there exists* $C_P = C_P(a) > 0$ *such that for all* $u \in H^1(M)$,

$$C_P\int_M (|\nabla_x u|^2(x) + a(x)|u|^2(x))\,\mathrm{d}x \geq \|u\|^2_{H^1(M)}. \tag{5.2}$$

Proof. We follow a standard proof and argue by contradiction. Otherwise, there would exist a sequence $(u_n) \in H^1(M)$ (that we can assume of norm 1 in H^1) such that

$$\int_M (|\nabla_x u_n|^2(x) + a(x)|u_n|^2(x))\,\mathrm{d}x \leq \frac{1}{n}\|u_n\|^2_{H^1(M)}.$$

Since (u_n) is bounded in $H^1(M)$ and M is compact, by Rellich's compactness theorem there exists $u \in H^1$ such that we can extract a subsequence (still denoted by (u_n)) such that

$$\|u_n - u\|_{L^2} \to 0, \qquad \text{as } n \to +\infty.$$

Moreover, as

$$\|\nabla_x u_n\|_{L^2} \to 0, \qquad \|\sqrt{a} u_n\|_{L^2(M)} \to 0,$$

we deduce that $\nabla_x u = 0$ and thus u must be constant in M. But

$$|u|^2 \int_M a(x)\,\mathrm{d}x = \int_M a(x)|u|^2(x)\,\mathrm{d}x = \lim_{n\to+\infty} \int_M a(x)|u_n|^2(x)\,\mathrm{d}x,$$

which implies that $u = 0$. This gives a contradiction with the fact that

$$\|u_n\|_{H^1} = 1, \quad \|\nabla_x u_n\|_{L^2} \to 0, \quad \|u_n - u\|_{L^2} \to 0 \qquad \Rightarrow \qquad \|u\|_{L^2} = 1.$$

Hence, (5.2) follows for some positive constant C_P. ∎

Proposition 5.4. *Assume that* (1.7) *holds for some* α, β *and let* $f \in L^2(M;\mathbb{C})$ *be given. Then, there exists* $C > 0$ *such that for* $\tau \geq 0$, *and any* $f \in L^2(M)$, *we have*

$$\|(\Delta - \tau + ia(x))^{-1} f\|_{L^2(M)} \leq \sqrt{2} C_P \|f\|_{L^2(M)},$$

where C_P *is the Poincaré's constant above.*

Proof. For $f \in L^2(M)$ given, let $\psi = -i(\Delta - \tau + ia)^{-1} f$. Recalling that ψ satisfies the Helmholtz equation (2.10), after multiplying by $\overline{\psi}$ and integrating by parts, we get

$$-\int_M |\nabla\psi|^2\,\mathrm{d}x - \tau \int_M |\psi|^2\,\mathrm{d}x + i \int_M a(x)|\psi|^2\,\mathrm{d}x = \int_M f\overline{\psi}\,\mathrm{d}x. \tag{5.3}$$

The modulus of the left-hand side in (5.3) is larger than

$$\frac{1}{\sqrt{2}} \int_M (|\nabla\psi|^2(x) + a(x)|\psi|^2(x))\,\mathrm{d}x.$$

Using Poincaré's inequality (5.2) on the left and Cauchy–Schwarz on the right, we get

$$\|\psi\|^2_{H^1} \leq \sqrt{2} C_P \|f\|_{L^2} \|\psi\|_{L^2} \Rightarrow \|\psi\|_{H^1} \leq \sqrt{2} C_P \|f\|_{L^2}.$$ ∎

5.3. Conclusion of the proof of Theorem 2

Proof of Theorem 2. Combining Proposition 5.4 and Proposition 5.2, we find that the estimate

$$\|R(\tau)f\|_{L^2(M)} \leq C_0 e^{2D\sqrt{|\tau|}} \|f\|_{L^2}$$

holds for every $\tau \in \mathbb{R}$, every $f \in L^2(M)$ and the constant

$$C_0 := \max\Big\{2(1+C)^2\Big(1+\beta\Big(1+\frac{1}{\alpha}\Big)^2\Big), 2C_P\Big\}.$$

As a consequence, the function $\mathcal{M}$ defined by (2.4) satisfies in this case the growth

$$\mathcal{M}(\mu) \leq C_1 e^{2D\sqrt{|\mu|}},$$

for every $\mu \in \mathbb{R}$. Hence, Theorem 4 yields that for any $k \in \mathbb{N}$,

$$\|U(t)(\mathrm{Id}-A)^{-k}\|_{\mathcal{L}(L^2)} \leq \frac{C_k}{\log(2+t)^{2k}}, \qquad \forall t \geq 0, \tag{5.4}$$

where we have used the notation of (2.8) and (2.9). Next, let $k \in \mathbb{N}$ be fixed and let $\psi_0 \in H^{2k}(M)$ Then, $\Phi_0 := (\mathrm{Id}-A)^k \psi_0 \in L^2(M)$ and inequality (5.4) implies

$$\begin{aligned} \mathcal{E}_S(t,\psi_0) &= \|U(t)\psi_0\|_{L^2}^2 = \|U(t)(\mathrm{Id}-A)^{-k}\Phi_0\|_{L^2}^2 \\ &\leq \frac{C_k}{\log(2+t)^{4k}}\|\Phi_0\|_{L^2}^2 \leq \frac{C_k}{\log(2+t)^{4k}}\|(\mathrm{Id}-A)^k\psi_0\|_{L^2}^2, \end{aligned}$$

for every $t \geq 0$. The proof is completed by noticing that for $k = 1$ the domain of A is $H^2(M) \cap H_0^1(M)$ with Dirichlet boundary conditions (resp. $H^2(M)$ with Neumann boundary conditions or if $\partial M = \emptyset$) and

$$\|(\mathrm{Id}-A)\psi_0\|_{L^2}^2 \leq C\|\psi_0\|_{H^2}^2. \qquad \blacksquare$$

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