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# Cohomology algebra of orbit spaces of free involutions on the product of projective space and 4-sphere

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**Abstract.** Let $X$ be a finitistic space with the mod 2 cohomology of the product space of a projective space and a 4-sphere. Assume that $X$ admits a free involution. In this paper we study the mod 2 cohomology algebra of the quotient of $X$ by the action of the free involution and derive some consequences regarding the existence of $\mathbb{Z}_2$-equivariant maps between such $X$ and an $n$-sphere.

## 1. Introduction

The study of the orbit space of a topological group $G$-action on a topological space $X$ is a classical topic in topology. In particular, the finitistic space plays an important role in the cohomology theory of transformation groups. A paracompact Hausdorff space $X$ is said to be *finitistic* if every open covering of $X$ has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect nontrivially. Finitistic spaces behave nicely under compact Lie group $G$ actions. More precisely, the space $X$ is finitistic if and only if the orbit space $X/G$ is finitistic ([6, 7]).

For a given topological space $X$ with the action of a topological group $G$, it is often difficult to determine the topological type or homotopy type of $X/G$. Orbit spaces of free actions of finite groups on spheres have been studied extensively by Livesay [13], Rice [16], Ritter [17], Rubinstein [19] and many others. Tao [25] determined orbit spaces of free involutions on $S^1 \times S^2$. Later Ritter [18] extended the results to free actions of cyclic groups of order $2^n$. However, there are few known results on compact manifolds other than a sphere. Hence we try to determine the cohomology algebra of the orbit space of some more examples.

To deal with more general spaces, by the notation $X \sim_{\mathbb{Q}} Y$ (resp. $X \sim_p Y$, $p$ a prime), we mean that $X$ and $Y$ have the same rational (resp. mod $p$) cohomology

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algebras, not necessarily induced by a map between $X$ and $Y$. Let us list some related results.

- R. M. Dotzel and others ([11]) have determined the cohomology algebra of orbit spaces of $\mathbb{Z}_p$-action (resp. $S^1$-action) on a finitistic space $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$).
- H. K. Singh and T. B. Singh have determined the mod 2 cohomology algebras of orbit spaces of free $\mathbb{Z}_2$-action on a finitistic space $X \sim_2 \mathbb{R}P^n$ and $X \sim_2 \mathbb{C}P^n$ in [20], and also determined the mod $p$ and rational (resp. mod $p$) cohomology algebras of orbit spaces of free $S^1$-action on a finitistic space $X \sim_F S^1 \times \mathbb{C}P^{m-1}$ with $F = \mathbb{Z}_p$ or $\mathbb{Q}$ (resp. mod $p$ cohomology lens space $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$) in [21].
- M. Singh has determined the cohomology algebras of orbit spaces of free involutions on a finitistic space $X \sim_2 \mathbb{R}P^n \times \mathbb{R}P^m$, $X \sim_2 \mathbb{C}P^n \times \mathbb{C}P^m$ in [22] and $X \sim_2 L^{2m-1}(p; q_1, \ldots, q_m)$ in [23].
- P. Dey and M. Singh have calculated the mod 2 cohomology algebras of orbit spaces of free $\mathbb{Z}_2$ and $S^1$-action on a compact Hausdorff space with mod 2 cohomology algebra of a real or complex Milnor manifold ([9]).
- A. M. M. Morita et al. have calculated the possible $\mathbb{Z}_2$-cohomology rings of orbit spaces of free actions of $\mathbb{Z}_2$ (or fixed point free involutions) on the Dold manifold $P(1, n)$ with $n$ odd ([15]).
- P. Dey has determined the possible mod 2 cohomology algebra of orbit spaces of free involutions on a finite dimensional CW-complex homotopic to the Dold manifold $P(m, n)$ ([8]).
- In [24], S. K. Singh and others have determined the cohomology algebra of orbit spaces of free involutions on a finitistic space $X \sim_2 \mathbb{F}P^m \times S^3$, where $\mathbb{F}P^m$ is a projective space, and $\mathbb{F}$ stands for either the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers or the division ring $\mathbb{H}$ of quaternions.
- As applications of cohomology algebras, the existence of $\mathbb{Z}_2$-equivariant maps $X \to S^n$ or $S^n \to X$ is discussed in [9, 20, 22–24].

This paper deals with the free action of $\mathbb{Z}_2$ on a finitistic space $X$ with mod 2 cohomology of the product of a projective space and 4-sphere, i.e., a space $X \sim_2 \mathbb{F}P^m \times S^4$, along with the cohomology algebra of orbit spaces under free involutions.

The paper is organized as follows: In Section 2, we recall the Leray–Serre spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \to B_G$, and list some known results. Section 3 consists of three main Theorems 3.1, 3.2, 3.3 and two Lemmas 3.6 and 3.7. In Section 4, we prove three main theorems which describe the possible cohomology algebras of orbit spaces. In the last Section 5, as applications of the main theorems, we discuss the existence of $\mathbb{Z}_2$-equivariant maps $X \to S^n$ or $S^n \to X$.

## 2. Preliminaries

We now recall the Borel construction and some results on its spectral sequence. Let $G$ be a compact Lie group acting on a finitistic space $X$. Let $E_G \to B_G$ be the universal principal $G$-bundle. The *Borel construction* on $X$ is defined as the orbit space

$$X_G = (X \times E_G)/G,$$

where $G$ acts diagonally (and freely) on the product $X \times E_G$. The projection $X \times E_G \to E_G$ gives a fibration ([1, Chapter IV]), called the *Borel fibration*,

$$X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G.$$

Throughout, we use the Čech cohomology with $\mathbb{Z}_2$ coefficients, and suppress it from the notation.

We exploit the Leray–Serre spectral sequence $\{E_r^{k,l}, d_r\}$ associated to the Borel fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G$ ([14, Theorem 5.2]), such that

(1) $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$, and

$$E_{r+1}^{k,l} = \frac{\ker d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}}{\operatorname{im} d_r : E_r^{k-r,l+r-1} \to E_r^{k,l}}.$$

(2) The infinity terms $E_\infty^{k,n-k}$ are isomorphic to the successive quotients $F_k^n/F_{k+1}^n$ in a filtration $0 \subset F_n^n \subset \cdots \subset F_1^n \subset F_0^n = H^n(X_G)$ of $H^n(X_G)$.

(3) The $E_2$-term of this spectral sequence is given by

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X)),$$

where $\mathcal{H}^l(X)$ is a locally constant sheaf with stalk $H^l(X)$, and the $E_2$-term converges to $H^*(X_G)$ as an algebra.

If $\pi_1(B_G)$ acts trivially on $H^*(X)$, then the system of local coefficients is simple, that is, the cohomology with local coefficients $H^k(B_G; \mathcal{H}^l(X))$ is just the (ordinary) cohomology $H^k(B_G; H^l(X))$ so that, by the universal coefficient theorem, we have

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).$$

Further, if the system of local coefficients is simple, the restriction of the product structure in the spectral sequence to the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$ coincide with the cup products on $H^*(B_G)$ and $H^*(X)$, respectively. The edge homomorphisms

$$H^k(B_G) \cong E_2^{k,0} \twoheadrightarrow E_3^{k,0} \twoheadrightarrow \cdots \twoheadrightarrow E_k^{k,0} \twoheadrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(X_G)$$

and

$$H^l(X_G) \twoheadrightarrow E_\infty^{0,l} = E_{l+2}^{0,l} \subset E_{l+1}^{0,l} \subset \cdots \subset E_2^{0,l} \cong H^l(X)$$

are the homomorphisms

$$\pi^* : H^k(B_G) \to H^k(X_G),$$
$$i^* : H^l(X_G) \to H^l(X)$$

respectively. The graded commutative algebra $H^*(X_G)$ is isomorphic to $\mathrm{Tot}E_\infty^{*,*}$, the total complex of $E_\infty^{*,*}$, given by

$$(\mathrm{Tot}E_\infty^{*,*})^q = \bigoplus_{k+l=q} E_\infty^{k,l}.$$

Next, we recall some known results.

**Proposition 2.1** ([26, Corollary 9.6]). *If a topological group $G = \mathbb{Z}_2$ acts freely on a topological space $X$ such that $X \to X/G$ is a principal $G$-bundle, then the equivariant cohomology $H_G^*(X) = H^*(X_G)$ is isomorphic to $H^*(X/G)$.*

**Proposition 2.2** ([2, Theorem 1.5, p. 374]). *Let $G = \mathbb{Z}_2$ act on a finitistic space $X$ with $H^i(X) = 0$ for all $i > n$. Then $H^i(X_G)$ is isomorphic to $H^i(X^G)$ for $i > n$, where $X^G$ is the fixed point set of the $G$-action.*

**Proposition 2.3** ([2, Corollary 7.2, p. 406]). *Let $G = \mathbb{Z}_2 = \langle g \rangle$ act on a finitistic space $X$. Then the element $cg^*(c) \in H^{2n}(X)^G = H^0(B_G; H^{2n}(X)) = E_2^{0,2n}$ is a permanent cocycle in the spectral sequence of $X \hookrightarrow X_G \to B_G$, for any $c \in H^n(X)$.*

**Proposition 2.4** ([2, Theorem 7.4, p. 407]). *Let $G = \mathbb{Z}_2 = \langle g \rangle$ act on a finitistic space $X$. Suppose that $H^i(X) = 0$ for all $i > 2n$ and $H^{2n}(X) = \mathbb{Z}_2$. Suppose that $c \in H^n(X)$ is an element such that $cg^*(c) \neq 0$, then the fixed point set is non-empty.*

**Proposition 2.5** ([2, Corollary 7.5, p. 407]). *Let $G = \mathbb{Z}_2 = \langle g \rangle$ act on a finitistic space $X \sim_2 S^n \times S^n$ and suppose that $g^* \neq 1$ on $H^n(X)$. Then the fixed point set is non-empty.*

## 3. Cohomology algebra of orbit space of free $\mathbb{Z}_2$-action on $X \sim_2 \mathbb{F}P^m \times S^4$

Assume that $X$ is a finitistic space equipped with a free involution and has the mod 2 cohomology of $\mathbb{F}P^m \times S^n$, i.e.,

$$H^*(X) = \mathbb{Z}_2[a, b]/\langle a^{m+1}, b^2 \rangle,$$

where, $\deg a = \lambda$, when $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $\lambda = 1, 2$ or $4$, respectively, and $\deg b = n$. Now, we present three main theorems of this paper. More concretely, we determine the cohomology algebras of orbit spaces of free involutions on $X \sim_2 \mathbb{F}P^m \times S^4$.

**Theorem 3.1.** *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times S^4$. If $m = 5$ or $m = 7$, assume further that the action of $G$ on $H^*(X; \mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$. Then $H^*(X/G)$ is isomorphic to one of the following graded commutative algebras:*

$$\mathbb{Z}_2[x, y, z]/I_1, \quad \deg x = 1, \deg y = 2, \deg z = 4;$$
$$\mathbb{Z}_2[x, y]/I_k, \quad \deg x = 1, \deg y = 1, \ k = 2, 3, \ldots, 9;$$

*where the ideal $I_k$ is listed as follows:*

(1) $I_1 = \langle x^2, y^{\frac{m+1}{2}}, z^2 \rangle$, *where $m$ is odd.*

(2) $I_2 = \langle x^5, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^4 y^{m-3} \rangle$, *where $\alpha_i \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. If $m = 1$, then $\alpha_3 = \alpha_4 = 0$. If $m = 2$, then $\alpha_4 = 0$.*

(3) $I_3 = \langle x^{m+5}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^{m+1}, x^4 y \rangle$, *where $\alpha_i \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. If $m = 1$, then $\alpha_3 = \alpha_4 = 0$. If $m = 2$, then $\alpha_4 = 0$.*

(4) $I_4 = \langle x^{m+5}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^3 y^2 + \beta_1 x^4 y + \beta_2 x^5, x^{m+3} y \rangle$, *where $m \geqslant 2$ and $\alpha_i, \beta_1, \beta_2 \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. If $m = 2$, then $\alpha_3 = 0$.*

(5) $I_5 = \langle x^{m+4}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^3 y^2 + \beta_1 x^4 y + \beta_2 x^5 \rangle$, *where $m \geqslant 2$ and $\alpha_i, \beta_1, \beta_2 \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. If $m = 2$, then $\alpha_3 = 0$.*

(6) $I_6 = \langle x^{m+5}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^2 y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5, x^{m+1} y^2 + \gamma_1 x^{m+2} y + \gamma_2 x^{m+3}, x^{m+3} y \rangle$, *where $m \geqslant 3$ and $\alpha_i, \beta_j, \gamma_1, \gamma_2 \in \mathbb{Z}_2$, $i = 1, \ldots, 4$, $j = 1, 2, 3$.*

(7) $I_7 = \langle x^{m+4}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^2 y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5, x^{m+1} y^2 + \gamma_1 x^{m+2} y + \gamma_2 x^{m+3} \rangle$, *where $m \geqslant 3$ and $\alpha_i, \beta_j, \gamma_1, \gamma_2 \in \mathbb{Z}_2$, $i = 1, \ldots, 4$, $j = 1, 2, 3$.*

(8) $I_8 = \langle x^{m+5}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^2 y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5, x^{m+2} y \rangle$, *where $m \geqslant 3$ and $\alpha_i, \beta_j \in \mathbb{Z}_2$, $i = 1, \ldots, 4$, $j = 1, 2, 3$.*

(9) $I_9 = \langle x^{m+3}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, x^2 y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5 \rangle$, *where $m \geqslant 3$ and $\alpha_i, \beta_j \in \mathbb{Z}_2$, $i = 1, \ldots, 4$, $j = 1, 2, 3$.*

**Theorem 3.2.** *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{C}P^m \times S^4$. If $m = 3$, assume further that the action of $G$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{C}P^3 \times S^4$. Then $H^*(X/G)$ is isomorphic to one of the following graded commutative algebras*

$$\begin{aligned} &\mathbb{Z}_2[x, y, z]/I_1, \quad && \deg x = 1, \deg y = 4, \deg z = 4;\\ &\mathbb{Z}_2[x, y]/I_k, && \deg x = 1, \deg y = 2, k = 2, 3; \end{aligned}$$

*where the ideal $I_k$ is listed as follows:*

(1) $I_1 = \langle x^3, y^{\frac{m+1}{2}}, z^2\rangle$, *where $m$ is odd.*

(2) $I_2 = \langle x^5, y^{m+1} + \alpha_1 x^2 y^m + \alpha_2 x^4 y^{m-1}\rangle$, *where $\alpha_1, \alpha_2 \in \mathbb{Z}_2$.*

(3) $I_3 = \langle x^{2m+5}, y^{m+1} + \alpha_1 x^2 y^m + \alpha_2 x^{2m+2}, x^3 y\rangle$, *where $\alpha_1, \alpha_2 \in \mathbb{Z}_2$.*

**Theorem 3.3.** *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{H}P^m \times S^4$. When $m \equiv 3 \pmod 4$, assume further that the action of $G$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{H}P^m \times S^4$. Then $H^*(X/G)$ is isomorphic to one of the following graded commutative algebras:*

$$\begin{aligned} &\mathbb{Z}_2[x, y, z]/I_1, \quad && \deg x = 1, \deg y = 8, \deg z = 4;\\ &\mathbb{Z}_2[x, y]/I_2, && \deg x = 1, \deg y = 4; \end{aligned}$$

*where the ideal $I_1$ and $I_2$ are as follows:*

(1) $I_1 = \langle x^5, y^{\frac{m+1}{2}} + \beta x^4 y^{\frac{m-1}{2}} z, z^2 + \gamma y + \alpha x^4 z\rangle$, *where $\alpha, \beta, \gamma \in \mathbb{Z}_2$ and $m$ is odd. If $m = 1$, then $\beta = \gamma = 0$.*

(2) $I_2 = \langle x^5, y^{m+1}\rangle$.

**Example 3.4.** When $m$ is odd there are standard free involutions of $\mathbb{R}P^m$ and $\mathbb{C}P^m$. The map

$$[x_0, x_1, \ldots, x_{m-1}, x_m] \mapsto [-x_1, x_0, \ldots, -x_m, x_{m-1}]$$

defines a free involution of $\mathbb{R}P^m$ with the orbit space $\mathbb{R}P^m/\mathbb{Z}_2 \sim_2 S^1 \times \mathbb{C}P^{\frac{m-1}{2}}$ ([24, Example 3.3]). Quotienting by the product of the above map with the trivial $\mathbb{Z}_2$-action on $S^4$, the mod 2 cohomology algebra of the orbit space $\mathbb{R}P^m \times S^4/\mathbb{Z}_2$ is that of $S^1 \times \mathbb{C}P^{\frac{m-1}{2}} \times S^4$, which account for case (1) in Theorem 3.1.

Similarly, the map

$$[z_0 : z_1 : \cdots : z_{m-1} : z_m] \mapsto [-\overline{z_1} : \overline{z_0} : \cdots : -\overline{z_m} : \overline{z_{m-1}}]$$

defines a free quaternionic involution of $\mathbb{C}P^m$ with the orbit space $\mathbb{C}P^m/\mathbb{Z}_2 \sim_2 \mathbb{R}P^2 \times \mathbb{H}P^{\frac{m-1}{2}}$ ([24, Example 3.7]). Quotienting by the product of the above map with the trivial $\mathbb{Z}_2$-action on $S^4$, the mod 2 cohomology algebra of the orbit space $\mathbb{C}P^m \times S^4/\mathbb{Z}_2$ is that of $\mathbb{R}P^2 \times \mathbb{H}P^{\frac{m-1}{2}} \times S^4$, which account for case (1) in Theorem 3.2.

The same construction as above does not apply to $\mathbb{H}P^m \times S^4$. For $m = 1$, namely, $X \sim_2 S^4 \times S^4$, by Proposition 2.5 we see that there is no free involution on $X$. For $m > 1$, $\mathbb{H}P^m$ has the fixed point property ([12, Example 4L.4]), where the fixed point property of a topological space means that every continuous map (not necessarily a self-homeomorphism) from the topological space to itself has a fixed point.

**Example 3.5.** Consider the trivial $\mathbb{Z}_2$-action on $\mathbb{F}P^m$ and the antipodal action of $\mathbb{Z}_2$ on $S^4$, then the orbit space of the free involution on $\mathbb{F}P^m \times S^4$ is $\mathbb{F}P^m \times \mathbb{R}P^4$. The cohomology algebra of $\mathbb{F}P^m \times \mathbb{R}P^4$ account for the cases (2) of main Theorems 3.1, 3.2 and 3.3 with all coefficients zero.

When $\alpha_1 = \alpha_2 = 0$, Theorem 3.2 (2) describes the cohomology ring of the Dold manifold $P(4, m)$. The Dold manifold $P(n, m)$ is the orbit space of $S^n \times \mathbb{C}P^m$ by the free involution that acts antipodally on $S^n$ and by complex conjugation on $\mathbb{C}P^m$. Following [10], the ring structure of $H^*(P(n, m))$ is given by

$$H^*(P(n,m)) = \mathbb{Z}_2[x, y]/\langle x^{n+1}, y^{m+1}\rangle,$$

where $\deg x = 1$, $\deg y = 2$.

An open question coming from Theorems 3.1, 3.2 and 3.3 is to search for possible more exotic free involutions and identify the respective cohomology algebras.

The proofs of the above three main theorems are based on spectral sequence arguments. To make the calculation of spectral sequence easier, we firstly prove the following general result, which is an extension of [24, Lemma 3.1].

**Lemma 3.6.** *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{F}P^m \times S^n$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\lambda = 1$, $2$ or $4$, respectively. Then the action of $G$ on $H^*(X; \mathbb{Z}_2)$ is trivial with possibly two exceptions,*

(i) $m \equiv 3 \pmod 4$ *and* $n = \lambda$;

(ii) $\lambda m = n + j$, $j \equiv \lambda \pmod{2\lambda}$, $0 \leqslant j < n$ *and* $\frac{n}{\lambda} \equiv 0 \pmod 2$.

*Proof.* The mod 2 cohomology algebra $H^*(X; \mathbb{Z}_2)$ has two generators $a$ and $b$ satisfying $a^{m+1} = 0$ and $b^2 = 0$. Let $g$ be the generator of $G = \mathbb{Z}_2$. By the naturality of the cup product, we get

$$g^*(a^i b) = g^*(a)^i g^*(b) \quad \text{for all } i \geqslant 0,$$

where $g^*$ is the mod 2 cohomology isomorphism $H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$.

Firstly, we claim that

$$g^*(a) = a, \text{ except the case: } m \equiv 3 \pmod 4 \text{ and } n = \lambda.$$

► If $\deg a = \lambda \neq n = \deg b$, we clearly have $g^*(a) = a$.

- For $m = 1, n = \lambda$, the mod 2 cohomology of $X$ is the same as of

$$S^1 \times S^1,\ S^2 \times S^2 \text{ or } S^4 \times S^4.$$

If $G$ acts nontrivially on $H^*(X;\mathbb{Z}_2)$, by Proposition 2.5, we have that $X^G$ is non-empty, which contradicts the action being free.

- For $m > 1$ and $m \equiv 1 \pmod 4$, $n = \lambda$. Since the orders of $a$ and $b$ are $m+1$ and 2, respectively, it follows that $g^*(a) \neq b$. Let $c = a^{\frac{m+1}{2}} \in H^{\lambda\frac{m+1}{2}}(X;\mathbb{Z}_2)$. If $g^*(a) = a + b$, then

$$cg^*(c) = a^{\frac{m+1}{2}}(a+b)^{\frac{m+1}{2}} = a^{\frac{m+1}{2}}\left(a^{\frac{m+1}{2}} + \tfrac{m+1}{2}a^{\frac{m-1}{2}}b\right) = a^m b \neq 0.$$

By Proposition 2.4, $X^G$ is non-empty, which contradicts the action being free. So $g^*(a) = a$.

- For $m > 1$ even, $n = \lambda$. If $g^*(a) = a + b$, then $a^{m+1} = 0$ gives $0 = g^*(a^{m+1}) = (a+b)^{m+1} = (m+1)a^m b = a^m b$, a contradiction.

Therefore, except for the case when $m \equiv 3 \pmod 4$ and $n = \lambda$, we have $g^*(a) = a$ and Lemma 3.6 is reduced to show that

$$g^* : H^n(X;\mathbb{Z}_2) \to H^n(X;\mathbb{Z}_2)$$

is the identity isomorphism.

If $\lambda \nmid n$ *or* $\lambda m < n$, the cohomology group $H^j(X;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ or zero for any $j \geqslant 0$, Lemma 3.6 is obvious. Thus we need to consider that $\lambda \mid n$ *and* $\lambda m \geqslant n$, $m > 1$.

If $G$ acts nontrivially on $H^*(X;\mathbb{Z}_2)$, then we get $g^*(b) = a^{\frac{n}{\lambda}}$ or $g^*(b) = a^{\frac{n}{\lambda}} + b$. If $g^*(b) = a^{\frac{n}{\lambda}}$, then $g^*(a^m b) = a^{m+\frac{n}{\lambda}} = 0$. Since $g^*$ is an isomorphism, this gives $a^m b = 0$, which is a contradiction. So we must have

$$g^*(b) = a^{\frac{n}{\lambda}} + b. \tag{3.1}$$

From now on to the end of the proof of Lemma 3.6, we show that (3.1) does not hold.

- If $\lambda \mid n$ *and* $\lambda m \geqslant 2n$, we have $0 = g^*(b^2) = (a^{\frac{n}{\lambda}} + b)^2 = a^{\frac{2n}{\lambda}}$, a contradiction. Thus (3.1) cannot happen.
- In the following, we assume that $\lambda \mid n$ *and* $2n > \lambda m \geqslant n$.

(1) For the case $\lambda m = n + j$, $j \equiv 0 \pmod{2\lambda}$ and $0 \leqslant j < n$, set $c = a^{\frac{j}{2\lambda}} b \in H^{\frac{\lambda m+n}{2}}(X;\mathbb{Z}_2)$. We have $cg^*(c) = a^m b \neq 0$, which contradicts Proposition 2.4. Thus (3.1) cannot happen.

(2) Now, let us consider the case $\lambda m = n + j$, $j \equiv \lambda \pmod{2\lambda}$ and $0 \leqslant j < n$. When $l \neq n, n+\lambda, \dots, n+j$, the coefficient sheaf $\mathcal{H}^l(X;\mathbb{Z}_2)$ is constant with

stalk $H^l(X;\mathbb{Z}_2)$ isomorphic to $\mathbb{Z}_2$ or zero. Then $g^* : H^l(X;\mathbb{Z}_2) \to H^l(X;\mathbb{Z}_2)$ is clearly the identity isomorphism, so $\pi_1(B_G) \cong G$ acts trivially on $H^l(X;\mathbb{Z}_2)$, and the $E_2$-term of the Leray–Serre spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \to B_G$ is

$$E_2^{k,l} \cong H^k(B_G;\mathbb{Z}_2) \otimes H^l(X;\mathbb{Z}_2), \quad k \geqslant 0, l \neq n, n+\lambda, n+2\lambda, \ldots, n+j. \tag{3.2}$$

To consider the $G$-action on $H^l(X;\mathbb{Z}_2)$ when $l = n, n+\lambda, \ldots, n+j$, recall that $B_G = \mathbb{R}P^\infty$ is a connected CW-complex with one cell in each dimension,

$$\mathbb{R}P^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots .$$

$E_G = S^\infty$ is the universal covering space of $\mathbb{R}P^\infty$, and the corresponding cell decomposition is

$$S^\infty = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup e^2_+ \cup e^2_- \cup \cdots ,$$

with $e^i_\pm$ being the upper and lower hemispheres of the $i$-sphere. According to [5, §5.2.1], the action of $\pi_1(B_G) \cong \mathbb{Z}_2$ on $S^\infty$ gives $C_*(S^\infty)$ the structure of a $\mathbb{Z}[\mathbb{Z}_2]$-chain complex, where

$$\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z}[g]/\langle g^2 - 1\rangle = \{a_0 + a_1 g \mid a_0, a_1 \in \mathbb{Z}\}$$

denotes the group ring. A basis for the free (rank 1) $\mathbb{Z}[\mathbb{Z}_2]$-module $C_i(S^\infty)$ is $e^i_+$. With the choice of the basis, the $\mathbb{Z}[\mathbb{Z}_2]$-chain complex $C_*(S^\infty)$ is isomorphic to

$$\cdots \to \mathbb{Z}[\mathbb{Z}_2] \to \cdots \xrightarrow{1-g} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+g} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-g} \mathbb{Z}[\mathbb{Z}_2] \to 0.$$

Let

$$\tau = 1 - g^*, \;\; \sigma = 1 + g^*.$$

The cochain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(C_*(S^\infty), H^l(X;\mathbb{Z}_2))$ is isomorphic to

$$\cdots \leftarrow H^l(X;\mathbb{Z}_2) \leftarrow \cdots \xleftarrow{\tau} H^l(X;\mathbb{Z}_2) \xleftarrow{\sigma} H^l(X;\mathbb{Z}_2) \xleftarrow{\tau} H^l(X;\mathbb{Z}_2) \leftarrow 0.$$

So the $E_2$-term of the Leray–Serre spectral sequence associated to the fibration $X \hookrightarrow X_G \to B_G$ is given by

$$\begin{aligned} E_2^{k,l} &= H^k(B_G;\mathcal{H}^l(X;\mathbb{Z}_2)) \cong H^k\big(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(C_*(S^\infty), H^l(X;\mathbb{Z}_2))\big) \\ &\cong \begin{cases} \ker\tau, & k = 0, \\ \ker\tau/\operatorname{im}\sigma, & k > 0 \text{ even}, \\ \ker\sigma/\operatorname{im}\tau, & k > 0 \text{ odd}. \end{cases} \end{aligned}$$

For $l = n, n+\lambda, \ldots, n+j$, $H^l(X;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by a basis $a^{\frac{l}{\lambda}}$, $a^{\frac{l-n}{\lambda}}b$. Note that $\tau = \sigma$ and the matrix representation of $\tau$ with the natural basis is $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. It is easy to see that

$$E_2^{k,l} \cong \begin{cases} 0, & k > 0 \text{ and } l = n, n+\lambda, \ldots, n+j, \\ \mathbb{Z}_2, & k = 0 \text{ and } l = n, n+\lambda, \ldots, n+j. \end{cases} \tag{3.3}$$

If $X \sim_2 \mathbb{C}P^m \times S^n$, $\deg a = \lambda = 2$, $\deg b = n$, $n$ being even implies that $E_2^{k,l} = 0$ for $l$ odd. This gives $d_2 = 0 : E_2^{k,l} \to E_2^{k+2,l-1}$ and hence $E_2^{*,*} = E_3^{*,*}$. If $X \sim_2 \mathbb{H}P^m \times S^n$, $\lambda = 4$, $\lambda$ dividing $n$ implies that $E_r^{k,l} = 0$ for $4 \nmid l$. This gives $d_r = 0 : E_r^{k,l} \to E_r^{k+r,l-r+1}$ for $2 \leqslant r \leqslant 4$ and hence $E_2^{*,*} = E_5^{*,*}$. That is to say, for $X \sim_2 \mathbb{F}P^m \times S^n$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we have

$$E_2^{*,*} = E_{\lambda+1}^{*,*}. \tag{3.4}$$

If $\frac{n}{\lambda} \equiv 1 \pmod 2$, by (3.2), (3.4) and the derivation property of the differential,

$$d_{\lambda+1}\left(1 \otimes a^{\frac{n}{\lambda}-1}\right) = \left(\frac{n}{\lambda} - 1\right)\left(1 \otimes a^{\frac{n}{\lambda}-2}\right) d_{\lambda+1}(1 \otimes a) = 0.$$

Note that $d_{\lambda+1} : E_{\lambda+1}^{k,n+j+\lambda} \to E_{\lambda+1}^{k+\lambda+1,n+j}$ is trivial as $E_{\lambda+1}^{k+\lambda+1,n+j} = 0$ (by (3.3)) for all $k$, particularly, $d_{\lambda+1}(t^k \otimes a^{\frac{j}{\lambda}+1} b) = 0$. By (3.4) and (3.2),

$$E_{\lambda+1}^{k,2n+j} = E_2^{k,2n+j} \cong H^k(B_G;\mathbb{Z}_2) \otimes H^{2n+j}(X;\mathbb{Z}_2)$$

is generated by the unique element $t^k \otimes a^{\frac{n+j}{\lambda}} b$. Furthermore, by the multiplicative structure of the spectral sequence, we have

$$d_{\lambda+1}\left(t^k \otimes a^{\frac{n+j}{\lambda}} b\right) = d_{\lambda+1}\left(\left(t^k \otimes a^{\frac{j}{\lambda}+1} b\right)\left(1 \otimes a^{\frac{n}{\lambda}-1}\right)\right) = 0.$$

Consequently,

$$d_{\lambda+1} : E_{\lambda+1}^{k,2n+j} \to E_{\lambda+1}^{k+\lambda+1,2n+j-\lambda} \text{ is trivial for all } k.$$

Set $c = a^{\frac{j-\lambda}{2\lambda}} b$. Then, by Proposition 2.3,

$$1 \otimes c g^*(c) = 1 \otimes a^{\frac{n+j-\lambda}{\lambda}} b \in E_2^{0,2n+j-\lambda}$$

is a permanent cocycle. By degree reasons, $t \otimes 1 \in E_2^{1,0}$ is a permanent cocycle, therefore $t^k \otimes a^{\frac{n+j-\lambda}{\lambda}} b \in E_2^{k,2n+j-\lambda}$ is also a permanent cocycle for all $k$.

By (3.4), when $\lambda = 2$ or 4, $d_r = 0 : E_r^{k,l} \to E_r^{k+r,l-r+1}$ for $2 \leqslant r < \lambda + 1$. Moreover,

$$d_{\lambda+1} : E_{\lambda+1}^{k,2n+j} \to E_{\lambda+1}^{k+\lambda+1,2n+j-\lambda}$$

is trivial for all $k$ and $\lambda = 1, 2$ or 4, hence $t^k \otimes a^{\frac{n+j-\lambda}{\lambda}} b \in E_r^{k,2n+j-\lambda}$ is not hit by any $d_r$-coboundaries, $2 \leqslant r \leqslant \lambda + 1$. Since $X$ has the mod 2 cohomology of

$\mathbb{F}P^m \times S^n$, for $\lambda m = n + j$, we have $H^l(X;\mathbb{Z}_2) = 0$ for $l > 2n + j$. As a result, $d_r : E_r^{k-r,2n+j+r-\lambda-1} \to E_r^{k,2n+j-\lambda}$ is trivial for $\lambda + 1 < r$ as $E_2^{k-r,2n+j+r-\lambda-1} = 0$. So $t^k \otimes a^{\frac{n+j-\lambda}{\lambda}} b \in E_r^{k,2n+j-\lambda}$ is not hit by any $d_r$-coboundaries, $r \geqslant 2$. Then, $t^k \otimes a^{\frac{n+j-\lambda}{\lambda}} b$ survives to a nontrivial element in $E_\infty$. However, this contradicts Proposition 2.2. Thus (3.1) does not happen. Therefore, the action of $G$ on $H^*(X;\mathbb{Z}_2)$ is trivial. ∎

As stated in Lemma 3.6, there are two possible exceptional cases in which we cannot prove that the action of $G$ on $H^*(X;\mathbb{Z}_2)$ is trivial. Alternatively, we prove this when $X$ is additionally assumed to have the integral cohomology of $\mathbb{F}P^m \times S^n$ for $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. The following proof is inspired by the discussions in [8, Theorem 4.5 and Lemma 5.1].

**Lemma 3.7.** *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_{\mathbb{Z}} \mathbb{F}P^m \times S^n$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Then the action of $G$ on $H^*(X;\mathbb{Z}_2)$ is trivial.*

*Proof.* The integral cohomology generators of $H^*(X;\mathbb{Z}) \cong H^*(\mathbb{F}P^m \times S^n;\mathbb{Z})$ are also denoted as $a$ and $b$. Let $g_{\mathbb{Z}}^*$ be the induced integral cohomology homomorphism $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z})$. Then $g_{\mathbb{Z}}^*$ is an automorphism and preserves degrees as well as cup-length, where for a cohomology class $x$, the cup-length of $x$ is the greatest integer $k$ such that $x^k \neq 0$. Note that the cup-length of a sum of the integral generators is the sum of the cup-lengths of the individual generators. For $m = 1, n = \lambda$, the mod 2 cohomology of $X$ is the same as of

$$S^2 \times S^2 \quad \text{or} \quad S^4 \times S^4.$$

If $G$ acts nontrivially on $H^*(X;\mathbb{Z}_2)$, by Proposition 2.5, we have $X^G$ is non-empty, which contradicts the action being free. Therefore, except for the case when $m = 1$, $n = \lambda$, we clearly have

$$g_{\mathbb{Z}}^*(a) = \pm a,\ g_{\mathbb{Z}}^*(b) = \pm b.$$

The integral cohomology group of $X \sim_{\mathbb{Z}} \mathbb{F}P^m \times S^n$ is torsion free for any dimension $l$, so

$$H^l(X;\mathbb{Z}_2) \cong H^l(X;\mathbb{Z}) \otimes \mathbb{Z}_2.$$

Considering the mod 2 reduction $\phi : H^l(X;\mathbb{Z}) \to H^l(X;\mathbb{Z}_2)$, we have the following commutative diagram:

$$\begin{array}{ccc} H^l(X;\mathbb{Z}) & \xrightarrow{g_{\mathbb{Z}}^*} & H^l(X;\mathbb{Z}) \\ \phi \downarrow & & \downarrow \phi \\ H^l(X;\mathbb{Z}_2) & \xrightarrow[g^*]{} & H^l(X;\mathbb{Z}_2). \end{array}$$

Thus the mod 2 cohomology homomorphism $g^*$ is the identity homomorphism. ∎

**Remark 3.8.** For the exceptional cases $m = 5, 7$ in Theorem 3.1, if $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$, the trivial action of $G$ on $H^*(X; \mathbb{Z}_2)$ is easily seen as follows:

It is known that

$$H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}[a_1, a_2]/\langle 2a_1, a_1^{k+1}, a_2^2, a_1a_2\rangle,$$
$$\deg a_1 = 2, \deg a_2 = 2k + 1.$$

Let $g^*_{\mathbb{Z}} : H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ be the induced automorphism. The action of the involution on the generator $b \in H^4(X; \mathbb{Z})$ coming from $H^4(S^4; \mathbb{Z})$ must reduce mod 2 to the identity action. Otherwise we would have that $g^*_{\mathbb{Z}}(b) = \pm b + a_1^2$ and this cannot happen as the class $b \pm a_1^2$ does not square to zero in $H^*(X; \mathbb{Z})$.

Lemmas 3.6, 3.7 and Remark 3.8 are sufficient for the proof of the main Theorems 3.1, 3.2 and 3.3. The parts of the main theorems that are affected by the exceptions of Lemma 3.6 occur only in cases of $\mathbb{R}P^5 \times S^4$, $\mathbb{R}P^7 \times S^4$, $\mathbb{C}P^3 \times S^4$ and $\mathbb{H}P^m \times S^4$, $m \equiv 3 \pmod 4$. But with an additional assumption about the trivial action of $G$ on $H^*(X; \mathbb{Z}_2)$ or the integral cohomology of $\mathbb{F}P^m \times S^4$ mentioned in Lemma 3.7 and Remark 3.8, it is possible to show the above cases of the main theorems.

## 4. Proofs of the main theorems

Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{F}P^m \times S^4$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. By Lemmas 3.6, 3.7 and Remark 3.8, $\pi_1(B_G) \cong \mathbb{Z}_2$ acts trivially on $H^*(X)$, hence, the $E_2$-term of the Leray–Serre spectral sequence associated to the fibration $X \hookrightarrow X_G \to B_G$ has the form

$$E_2^{k,l} = H^k(B_G) \otimes H^l(X).$$

Recall that,

$$H^*(B_G) = \mathbb{Z}_2[t], \quad \text{where } \deg t = 1.$$

### 4.1. Proof of Theorem 3.1

Let $G = \mathbb{Z}_2$ act freely on $X \sim_2 \mathbb{R}P^m \times S^4$. Using the Künneth formula, we observe that,

$$H^l(X) = \begin{cases} \mathbb{Z}_2, & 0 \leqslant l \leqslant \min\{3, m\} \text{ or } \max\{4, m+1\} \leqslant l \leqslant m+4, \\ (\mathbb{Z}_2)^2, & 4 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a \in H^1(X)$ and $b \in H^4(X)$ be the generators of the cohomology algebra of $H^*(X)$, satisfying $a^{m+1} = 0$ and $b^2 = 0$. By degree reasons, $t \otimes 1 \in E_2^{1,0}$ is a permanent cocycle and survives to a nontrivial element $x \in E_\infty^{1,0}$, i.e., by the edge homomorphism,

$$x = \pi^*(t) \in E_\infty^{1,0} \subset H^1(X_G). \tag{4.1}$$

Since $\mathbb{Z}_2$ acts freely on $X$, by Proposition 2.2, the spectral sequence does not collapse. Otherwise, we get $H^i(X/G) \neq 0$ for infinitely many values of $i > m+4$. It implies that some differential $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ must be nontrivial. Note that $E_2^{*,*}$ is generated by $t \otimes 1 \in E_2^{1,0}$, $1 \otimes a \in E_2^{0,1}$ and $1 \otimes b \in E_2^{0,4}$. There can only be nontrivial differentials $d_r$ on these generators when $2 \leqslant r \leqslant 5$. It follows immediately that there are five possibilities for nontrivial differentials on generators,

(i) $d_2(1 \otimes a) \neq 0$;

(ii) $d_2(1 \otimes a) = 0$, $d_r(1 \otimes b) = 0$, $r = 2, 3, 4$ and $d_5(1 \otimes b) \neq 0$;

(iii) $d_2(1 \otimes a) = 0$, $d_r(1 \otimes b) = 0$, $r = 2, 3$ and $d_4(1 \otimes b) \neq 0$;

(iv) $d_2(1 \otimes a) = 0$, $d_2(1 \otimes b) = 0$ and $d_3(1 \otimes b) \neq 0$;

(v) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) \neq 0$.

In the following, we discuss each case separately.

**Case (i).** $d_2(1 \otimes a) = t^2 \otimes 1 \neq 0$.

If $m$ is even, then $a^{m+1} = 0$ gives $0 = d_2((1 \otimes a^m)(1 \otimes a)) = t^2 \otimes a^m$, a contradiction. Hence $m$ must be odd. There are two possible subcases: either $d_2(1 \otimes b) = t^2 \otimes a^3 \neq 0$ or $d_2(1 \otimes b) = 0$.

If $d_2(1 \otimes b) = t^2 \otimes a^3 \neq 0$ (in this subcase, $m \geqslant 3$), by the derivation property of the differential, we have

$$\begin{cases} d_2(1 \otimes a^j) = j(t^2 \otimes a^{j-1}), & 1 \leqslant j \leqslant m, \\ d_2(1 \otimes a^j b) = t^2 \otimes a^{j+3} + j(t^2 \otimes a^{j-1} b), & 0 \leqslant j \leqslant m-3, \\ d_2(1 \otimes a^j b) = j(t^2 \otimes a^{j-1} b), & m-2 \leqslant j \leqslant m. \end{cases}$$

Note that

$$d_2(1 \otimes ab) = \begin{cases} t^2 \otimes b + t^2 \otimes a^4, & m \geqslant 5, \\ t^2 \otimes b, & m = 3, \end{cases}$$

$$\begin{aligned} d_2 d_2(1 \otimes ab) &= \begin{cases} d_2(t^2 \otimes b + t^2 \otimes a^4), & m \geqslant 5, \\ d_2(t^2 \otimes b), & m = 3, \end{cases} \\ &= t^4 \otimes a^3 \neq 0. \end{aligned}$$

This contradicts $d_2 d_2 = 0$, thus $d_2(1 \otimes b) = 0$. By the derivation property of the differential, we have

$$\begin{cases} d_2(1 \otimes a^j) = j(t^2 \otimes a^{j-1}), & 1 \leqslant j \leqslant m, \\ d_2(1 \otimes a^j b) = j(t^2 \otimes a^{j-1} b), & 0 \leqslant j \leqslant m. \end{cases}$$

The $E_2$-term and $d_2$-differentials look like Figure 1. In all Figures of this paper, we write $t^k a^l, t^k a^{l-4} b$ for $t^k \otimes a^l, t^k \otimes a^{l-4} b \in E_2^{k,l}$ respectively. Each black dot represents a $\mathbb{Z}_2$ summand and the two types of lines (colored by red, cyan) represent multiplication by $a$ and $b$, and the arrowed line (colored by blue) represents a non-trivial differential. In columns $k-2$ and $k$, if there is no arrowed line starting from a black dot, then $d_2$ vanishes on this class.

Since

$$E_3^{k,l} = \frac{\ker\{d_2 : E_2^{k,l} \to E_2^{k+2,l-1}\}}{\operatorname{im}\{d_2 : E_2^{k-2,l+1} \to E_2^{k,l}\}},$$

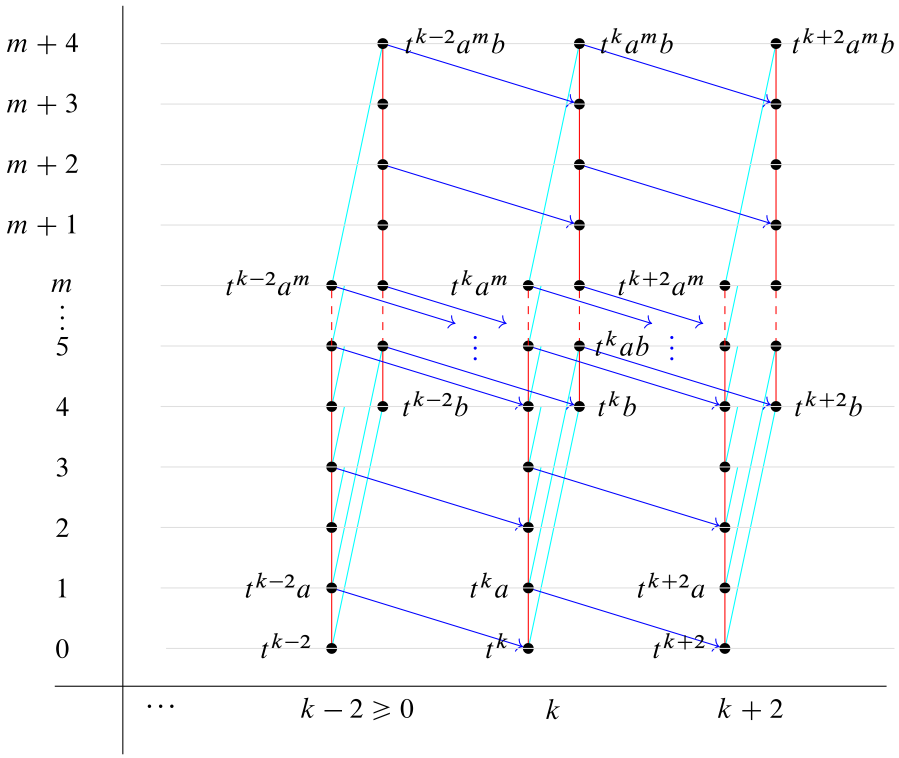


**Figure 1.** $E_2$-term and $d_2$-differentials in Case (i)

it is clear from Figure 1 that

$$E_3^{k,l} = \begin{cases} \mathbb{Z}_2, & k = 0, 1;\ l = 0, 2, m+1, m+3, \\ (\mathbb{Z}_2)^2, & k = 0, 1;\ 4 \leqslant l \leqslant m \text{ and } l \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant 3$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_3^{*,*} = E_\infty^{*,*}.$$

Since $H^*(X_G) \cong \mathrm{Tot}E_\infty^{*,*}$, the additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & 0 \leqslant j \leqslant 3 \text{ or } m+1 \leqslant j \leqslant m+4, \\ (\mathbb{Z}_2)^2, & 4 \leqslant j \leqslant m, \\ 0, & j > m+4. \end{cases}$$

As $E_\infty^{2,0} = 0$, by (4.1), we have $x^2 = 0$. Notice that the elements $1 \otimes a^2 \in E_2^{0,2}$ and $1 \otimes b \in E_2^{0,4}$ are permanent cocycles and are not hit by any $d_r$-coboundaries. Hence, they determine nontrivial elements $u \in E_\infty^{0,2}$ and $v \in E_\infty^{0,4}$, respectively. We have $u^{\frac{m+1}{2}} = 0$ as $a^{m+1} = 0$, and $v^2 = 0$ as $b^2 = 0$. Thus

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x, u, v]/\langle x^2, u^{\frac{m+1}{2}}, v^2\rangle,$$

where $\deg x = 1$, $\deg u = 2$, $\deg v = 4$.

By the identification of the edge homomorphism there exist $y \in H^2(X_G)$ and $z \in H^4(X_G)$ such that $i^*(y) = a^2$ and $i^*(z) = b$, respectively. Notice that $H^j(X_G) = E_\infty^{k,j-k}$, where $k = 0, 1$ and $j - k$ even. Consequently, $y^{\frac{m+1}{2}} \in H^{m+1}(X_G) = E_\infty^{0,m+1}$ is represented by $a^{m+1} \in E_2^{0,m+1}$ and $z^2 \in H^8(X_G) = E_\infty^{0,8}$ is represented by $b^2 \in E_2^{0,8}$. So we have the following relations:

$$y^{\frac{m+1}{2}} = 0, \quad z^2 = 0.$$

Therefore, $H^*(X_G)$ is the graded commutative algebra

$$\mathbb{Z}_2[x, y, z]/\langle x^2, y^{\frac{m+1}{2}}, z^2\rangle,$$

where $\deg y = 2$, $\deg z = 4$ and $m$ is odd. This gives possibility (1) of Theorem 3.1.

**Case (ii).** $d_2(1 \otimes a) = 0$, $d_r(1 \otimes b) = 0$, $2 \leqslant r \leqslant 4$ and $d_5(1 \otimes b) = t^5 \otimes 1 \neq 0$. In this case we have $d_r = 0$, $2 \leqslant r \leqslant 4$, $E_2^{*,*} = E_5^{*,*}$, and

$$\begin{cases} d_5(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_5(1 \otimes a^j b) = t^5 \otimes a^j, & 0 \leqslant j \leqslant m. \end{cases}$$

Furthermore, we have

$$\begin{gathered}E_5^{k-5,l+4} \xrightarrow{d_5} E_5^{k,l} \xrightarrow{d_5} E_5^{k+5,l-4},\\ t^{k-5}\otimes a^l b \xmapsto{d_5} t^k \otimes a^l \xmapsto{d_5} 0,\\ t^{k-5}\otimes a^{l+4} \xmapsto{d_5} 0, \quad t^k\otimes a^{l-4}b \xmapsto{d_5} t^{k+5}\otimes a^{l-4}.\end{gathered}$$

So

$$E_6^{k,l} = \begin{cases}\mathbb{Z}_2, & 0\leqslant k\leqslant 4;\ 0\leqslant l\leqslant m,\\ 0, & \text{otherwise.}\end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant 6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_6^{*,*} = E_\infty^{*,*}.$$

The additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases}\mathbb{Z}_2, & j = 0, m+4,\\ (\mathbb{Z}_2)^2, & j = 1, m+3,\\ (\mathbb{Z}_2)^3, & j = 2, m+2,\\ (\mathbb{Z}_2)^4, & j = 3, m+1,\\ (\mathbb{Z}_2)^5, & 4\leqslant j\leqslant m \text{ (for } m\geqslant 4),\\ 0, & \text{otherwise.}\end{cases} \tag{4.2}$$

Notice that the element $1\otimes a \in E_2^{0,1}$ is a permanent cocycle and is not a $d_r$-coboundary. Hence, it determines a nontrivial element $u \in E_\infty^{0,1}$. As we have remarked, $a^{m+1} = 0$, so

$$u^{m+1} = 0. \tag{4.3}$$

As $E_\infty^{5,0} = 0$, by (4.1), we have $x^5 = 0$. Thus

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^5, u^{m+1}\rangle,$$

where $\deg u = 1$.

Further, choose $y \in H^1(X_G)$ such that $i^*(y) = a$. By considering the filtration on $H^{m+1}(X_G)$,

$$\begin{aligned}0 = F_{m+1}^{m+1} = \cdots = \underbrace{F_5^{m+1} \subset F_4^{m+1}}_{E_\infty^{4,m-3}} &\underbrace{\subset F_3^{m+1}}_{E_\infty^{3,m-2}} \underbrace{\subset F_2^{m+1}}_{E_\infty^{2,m-1}} \underbrace{\subset F_1^{m+1}}_{E_\infty^{1,m}}\\ = F_0^{m+1} = H^{m+1}(X_G),&\end{aligned} \tag{4.4}$$

we get the following relation:

$$y^{m+1} = \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^4 y^{m-3},$$

where $\alpha_i \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y]/\langle x^5, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^4 y^{m-3}\rangle,$$

where $\deg y = 1$. If $m = 1$, then $\alpha_3 = \alpha_4 = 0$. If $m = 2$, then $\alpha_4 = 0$. This gives possibility (2) of Theorem 3.1.

*In the remaining cases, Cases (iii)–(v), there will be classes $u \in E_\infty^{0,1}$, $y \in H^1(X_G)$ defined as above and relation (4.3) will be satisfied.*

**Case (iii).** $d_2(1 \otimes a) = 0$, $d_r(1 \otimes b) = 0$, $r = 2, 3$ and $d_4(1 \otimes b) = t^4 \otimes a \neq 0$. This case implies that $d_r = 0$, $r = 2, 3$, $E_4^{*,*} = E_2^{*,*}$. So we have

$$\begin{cases} d_4(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_4(1 \otimes a^j b) = t^4 \otimes a^{j+1}, & 0 \leqslant j \leqslant m-1, \\ d_4(1 \otimes a^m b) = 0. \end{cases}$$

The $E_4$-term and $d_4$-differentials look like Figure 2. Then

$$E_5^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant 4;\ l = 0,\ m+4, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 3;\ 0 \leqslant l \leqslant m,\ l = m+4, \\ 0, & \text{otherwise.} \end{cases}$$

Since $1 \otimes a$ is a permanent cocycle, by the derivation property of the differential, $d_5(1 \otimes a^j) = 0$, $1 \leqslant j \leqslant m$, and all $d_5 : E_5^{k,l} \to E_5^{k+5,l-4}$ is zero by degree reasons. Similarly, $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $6 \leqslant r \leqslant m+4$. Thus

$$E_{m+5}^{*,*} = E_5^{*,*}.$$

Now, if $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is trivial, then by the multiplicative properties of the spectral sequence, we have $E_{m+5}^{*,*} = E_\infty^{*,*}$. Therefore the bottom line ($l = 0$) and the top line ($l = m+4$) of the spectral sequence survive to $E_\infty$, which reduces to $H^i(X/G) \neq 0$ for all $i > m+4$. That contradicts Proposition 2.2. Thus, $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ must be nontrivial. It follows immediately that $d_{m+5} : E_{m+5}^{k,m+4} \to E_{m+5}^{k+m+5,0}$ is an isomorphism for all $k$. So

$$E_{m+6}^{k,l} = \begin{cases} \mathbb{Z}_2, & 4 \leqslant k \leqslant m+4;\ l = 0, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 3;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_\infty^{*,*} = E_{m+6}^{*,*}.$$

It follows that the cohomology groups $H^j(X_G)$ are the same as (4.2).

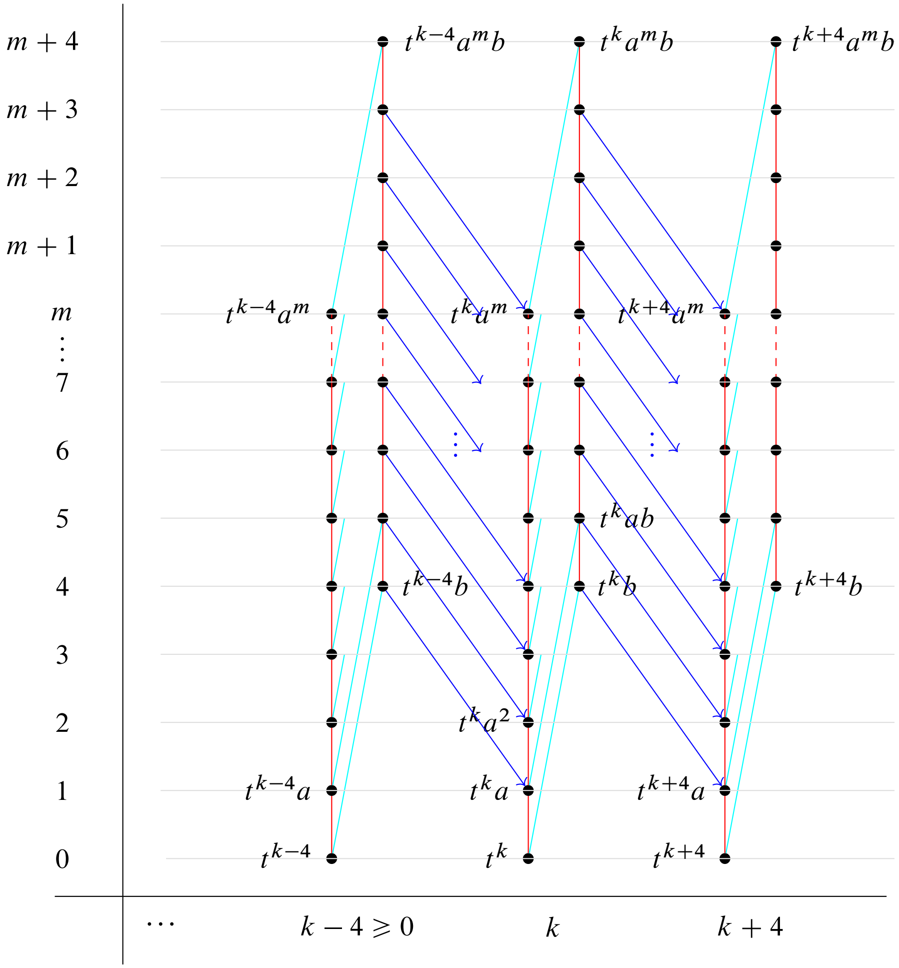


**Figure 2.** $E_4$-term and $d_4$-differentials in Case (iii)

As $E_\infty^{m+5,0} = 0$, by (4.1), we have $x^{m+5} = 0$. Clearly, $x^4u = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{m+5}, u^{m+1}, x^4u\rangle.$$

Now, choose $y' \in H^1(X_G)$ such that $i^*(y') = a$. By considering the filtration on $H^{m+1}(X_G)$,

$$\begin{aligned}\underbrace{0 \subset F_{m+1}^{m+1}}_{E_\infty^{m+1,0}} = \cdots = F_4^{m+1} &\underbrace{\subset}_{E_\infty^{3,m-2}} F_3^{m+1} \underbrace{\subset}_{E_\infty^{2,m-1}} F_2^{m+1} \underbrace{\subset}_{E_\infty^{1,m}} F_1^{m+1} \\ &= F_0^{m+1} = H^{m+1}(X_G),\end{aligned} \tag{4.5}$$

we get the following relation:

$$(y')^{m+1} = \alpha'_1 x(y')^m + \alpha'_2 x^2 (y')^{m-1} + \alpha'_3 x^3 (y')^{m-2} + \alpha'_4 x^{m+1},$$

where $\alpha'_i \in \mathbb{Z}_2$, $i = 1, \ldots, 4$. By considering the filtration on $H^5(X_G)$,

$$\underbrace{0 \subset F_5^5}_{E_\infty^{5,0}} = \underbrace{F_4^5 \subset F_3^5}_{E_\infty^{3,2}} \underbrace{\subset F_2^5}_{E_\infty^{2,3}} \underbrace{\subset F_1^5}_{E_\infty^{1,4}} \underbrace{\subset F_0^5}_{E_\infty^{0,5}} = H^5(X_G),$$

we can write $x^4 y'$ as

$$x^4 y' = \beta x^5, \beta \in \mathbb{Z}_2.$$

By choosing a particular

$$y = y' + \beta x, \tag{4.6}$$

the above relations can be simplified as

$$\begin{aligned} y^{m+1} &= \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^{m+1}, \\ x^4 y &= 0, \end{aligned}$$

where $\alpha_i$ $(i = 1, \ldots, 4) \in \mathbb{Z}_2$. Thus, $H^*(X_G)$ is the graded commutative algebra $\mathbb{Z}_2[x, y]/I$, where $I$ is the ideal given by

$$I = \langle x^{m+5}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^3 y^{m-2} + \alpha_4 x^{m+1}, x^4 y \rangle.$$

If $m = 1$, then $\alpha_3 = \alpha_4 = 0$. If $m = 2$, then $\alpha_4 = 0$. This gives possibility (3) of Theorem 3.1.

**Case (iv).** $d_2(1 \otimes a) = 0$, $d_2(1 \otimes b) = 0$ and $d_3(1 \otimes b) = t^3 \otimes a^2 \neq 0$.

Obviously, $m \geqslant 2$, $d_2 = 0$ and $E_3^{*,*} = E_2^{*,*}$. Since

$$\begin{cases} d_3(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_3(1 \otimes a^j b) = t^3 \otimes a^{j+2}, & 0 \leqslant j \leqslant m-2, \\ d_3(1 \otimes a^j b) = 0, & j = m-1, m, \end{cases}$$

the $E_3$-term and $d_3$-differentials look like Figure 3. Then

$$E_4^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant 3;\ l = 0, 1, m+3, m+4, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ 0 \leqslant l \leqslant m,\ l = m+3, m+4, \\ 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

Consider the bidegrees of $E_4^{*,*}$, $d_r : E_r^{k,l} \to E_r^{k+r, l-r+1}$ is zero for all $4 \leqslant r \leqslant m+2$. So

$$E_{m+3}^{k,l} = E_4^{k,l}, \text{ for all } k, l. \tag{4.8}$$

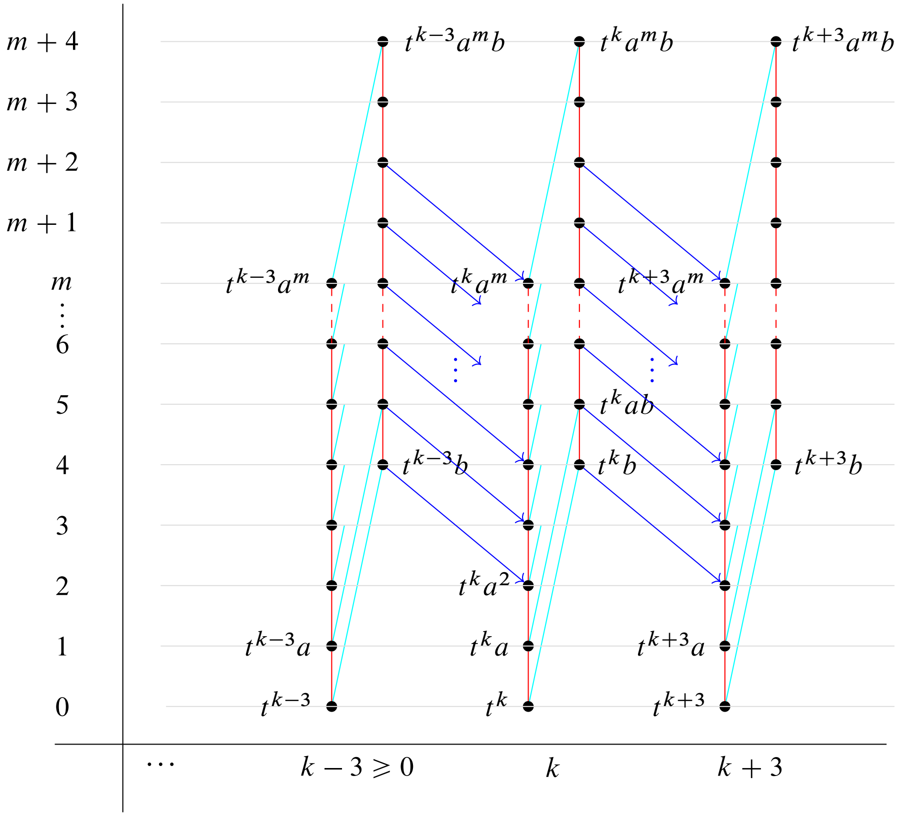


**Figure 3.** $E_3$-term and $d_3$-differentials in Case (iv)

The differential $d_r : E_r^{0,m+3} \to E_r^{r,m+4-r}$ ($r \geqslant m+3$) can only be nontrivial when $r = m+3$ or $m+4$. If $d_r : E_r^{0,m+3} \to E_r^{r,m+4-r}$ is trivial for $r = m+3$ and $r = m+4$, then $d_r = 0 : E_r^{k,m+3} \to E_r^{k+r,m+4-r}$ for any $k$, $r = m+3$ and $r = m+4$. Thus $E_{m+3}^{*,*} = E_\infty^{*,*}$, at least two lines of the spectral sequence survive to $E_\infty$, which contradicts Proposition 2.2. Thus, we get two possibilities:

(iv.1) $d_{m+3} : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is nontrivial.

(iv.2) $d_{m+3} : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is trivial and $d_{m+4} : E_{m+4}^{0,m+3} \to E_{m+4}^{m+4,0}$ is nontrivial.

**Subcase (iv.1).** If $d_{m+3} : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is nontrivial, then $d_{m+3}(1 \otimes a^{m-1}b) = t^{m+3} \otimes a$, and

$$d_{m+3} : E_{m+3}^{k,l} \to E_{m+3}^{k+m+3,l-m-2}$$

is an isomorphism for all $k$ and $l = m+3$ and a trivial homomorphism otherwise.

Consequently,

$$E_{m+4}^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant m+3;\ l = 0, m+4, \\ \mathbb{Z}_2, & 3 \leqslant k \leqslant m+2;\ l = 0, 1, m+4, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ 0 \leqslant l \leqslant m,\ l = m+4, \\ 0, & \text{otherwise.} \end{cases}$$

The differential $d_r : E_r^{0,m+4} \to E_r^{r,m+5-r}$ $(r \geqslant m+4)$ can only be nontrivial when $r = m+5$. If $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is trivial, then $d_{m+5} = 0 : E_{m+5}^{k,m+4} \to E_{m+5}^{k+m+5,0}$ for any $k$. Thus $E_{m+4}^{*,*} = E_\infty^{*,*}$. Therefore the bottom line $(l = 0)$ and the top line $(l = m+4)$ of the spectral sequence survive to $E_\infty$, which contradicts Proposition 2.2. Therefore, the differential $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is nontrivial. Then $d_{m+5} : E_{m+5}^{k,l} \to E_{m+5}^{k+m+5,l-m-4}$ is an isomorphism for all $k$ and $l = m+4$ and a trivial homomorphism otherwise. Consequently,

$$E_{m+6}^{k,l} = \begin{cases} \mathbb{Z}_2, & k = m+3, m+4;\ l = 0, \\ \mathbb{Z}_2, & 3 \leqslant k \leqslant m+2;\ l = 0, 1, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+6}^{*,*} = E_\infty^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_\infty^{m+5,0} = 0$, by (4.1), we have $x^{m+5} = 0$. Clearly, $x^3u^2 = 0$, $x^{m+3}u = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{m+5}, u^{m+1}, x^3u^2, x^{m+3}u\rangle.$$

Similar to the discussion of the filtration (4.4) or (4.5) and the particular choice of $y$ in (4.6), consider (4.9), we get the following relations:

$$\begin{aligned} y^{m+1} &= \alpha_1 xy^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ x^3y^2 &= \beta_1 x^4 y + \beta_2 x^5, \\ x^{m+3}y &= 0, \end{aligned}$$

where $\alpha_i$ $(i = 1, \ldots, 4)$, $\beta_1, \beta_2 \in \mathbb{Z}_2$. So the graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x,y]/I$, where $I$ is the ideal given by

$$\begin{aligned} I = \langle & x^{m+5}, y^{m+1} + \alpha_1 xy^m + \alpha_2 x^2 y^{m-1} + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ & x^3y^2 + \beta_1 x^4 y + \beta_2 x^5, x^{m+3}y\rangle, \end{aligned}$$

where $m \geqslant 2$. If $m = 2$, then $\alpha_3 = 0$. This gives possibility (4) of Theorem 3.1.

**Subcase (iv.2).** If $d_{m+3}: E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is trivial and $d_{m+4}: E_{m+4}^{0,m+3} \to E_{m+4}^{m+4,0}$ is nontrivial, then

$$\begin{aligned} d_{m+3} = 0 : E_{m+3}^{k,l} &\to E_{m+3}^{k+m+3,l-m-2}, \quad \text{for any } k, l, \\ d_{m+4}(1 \otimes a^{m-1}b) &= t^{m+4} \otimes 1, \\ d_{m+4}(1 \otimes a^{m}b) &= t^{m+4} \otimes a. \end{aligned} \tag{4.10}$$

Furthermore, we obtain that

$$d_{m+4} : E_{m+4}^{k,l} \to E_{m+4}^{k+m+4,l-m-3} \tag{4.11}$$

is an isomorphism for all $k, l = m+3, m+4$ and a trivial homomorphism otherwise. Consequently, by (4.8), (4.10) and (4.11), we have

$$E_{m+5}^{k,l} = \begin{cases} \mathbb{Z}_2, & 3 \leqslant k \leqslant m+3;\ l = 0, 1, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+5$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+5}^{*,*} = E_\infty^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_\infty^{m+4,0} = 0$, by (4.1), we have $x^{m+4} = 0$. Clearly, $x^3u^2 = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x, u]/\langle x^{m+4}, u^{m+1}, x^3u^2\rangle.$$

The graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x, y]/I$, where $I$ is the ideal given by

$$\begin{aligned} I = \langle & x^{m+4}, y^{m+1} + \alpha_1 xy^m + \alpha_2 x^2y^{m-1} + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ & x^3y^2 + \beta_1 x^4 y + \beta_2 x^5\rangle, \end{aligned}$$

where $m \geqslant 2$ and $\alpha_i$ $(i = 1, \ldots, 4)$, $\beta_1, \beta_2 \in \mathbb{Z}_2$. If $m = 2$, then $\alpha_3 = 0$. This gives possibility (5) of Theorem 3.1.

**Case (v).** $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t^2 \otimes a^3 \neq 0$.

Obviously, $m \geqslant 3$. We have

$$\begin{cases} d_2(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_2(1 \otimes a^j b) = t^2 \otimes a^{j+3}, & 0 \leqslant j \leqslant m-3, \\ d_2(1 \otimes a^j b) = 0, & m-2 \leqslant j \leqslant m. \end{cases}$$

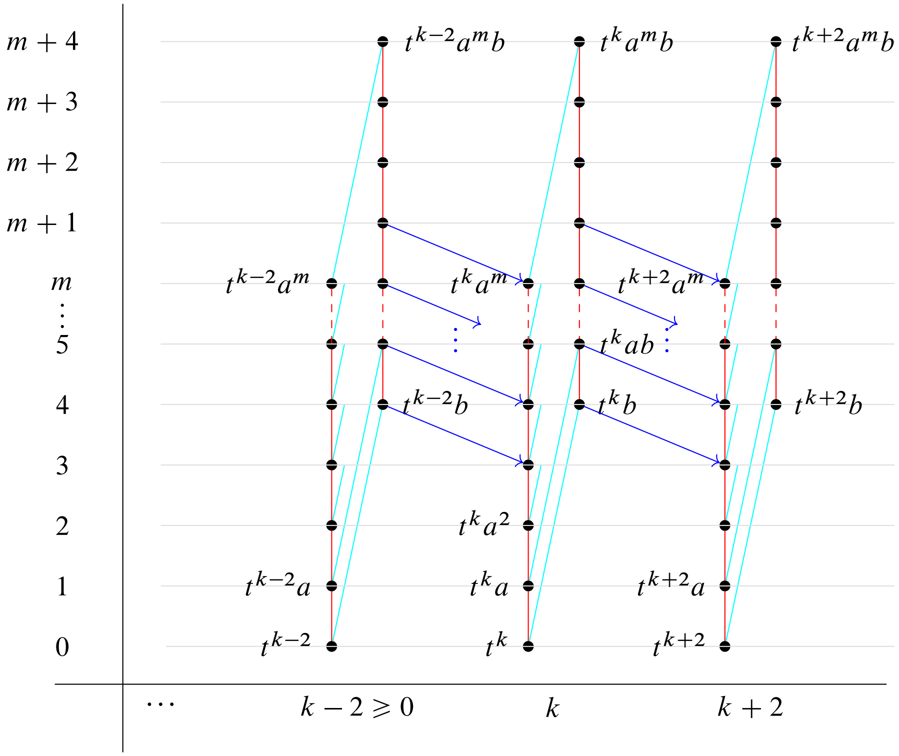


**Figure 4.** $E_2$-term and $d_2$-differentials in Case (v)

The $E_2$-term and $d_2$-differentials look like Figure 4. Then

$$E_3^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant 2;\ l = 0,1,2,m+2,m+3,m+4, \\ \mathbb{Z}_2, & k = 0,1;\ 0 \leqslant l \leqslant m,\ l = m+2,m+3,m+4, \\ 0, & \text{otherwise}. \end{cases} \tag{4.12}$$

Clearly, $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $3 \leqslant r \leqslant m$. So

$$E_3^{k,l} = E_{m+1}^{k,l}, \quad \text{for all } k, l. \tag{4.13}$$

The differential $d_r : E_r^{0,m+2} \to E_r^{r,m+3-r}$ ($r \geqslant m+1$) can only be nontrivial when $r = m+1, m+2, m+3$. If $d_r : E_r^{0,m+2} \to E_r^{r,m+3-r}$ is trivial for $r = m+1$, $m+2, m+3$, then $d_r = 0 : E_r^{k,m+2} \to E_r^{k+r,m+3-r}$ for any $k$, $r = m+1, m+2$ and $m+3$. Thus $E_{m+1}^{*,*} = E_\infty^{*,*}$, at least two lines of the spectral sequence survive to $E_\infty$, which contradicts Proposition 2.2. Therefore, we get the following subcases:

(v.1) $d_{m+1} : E_{m+1}^{0,m+2} \to E_{m+1}^{m+1,2}$ is nontrivial.

(v.2) $d_{m+1} = 0 : E_{m+1}^{0,m+2} \to E_{m+1}^{m+1,2}$ and $d_{m+2} : E_{m+2}^{0,m+2} \to E_{m+2}^{m+2,1}$ is nontrivial.

(v.3) $d_r = 0 : E_r^{0,m+2} \to E_r^{r,m+3-r}$, $r = m+1, m+2$ and $d_{m+3} : E_{m+3}^{0,m+2} \to E_{m+3}^{m+3,0}$ is nontrivial.

**Subcase (v.1).** If $d_{m+1} : E_{m+1}^{0,m+2} \to E_{m+1}^{m+1,2}$ is nontrivial, then $d_{m+1}(1 \otimes a^{m-2}b) = t^{m+1} \otimes a^2$, and $d_{m+1} : E_{m+1}^{k,l} \to E_{m+1}^{k+m+1,l-m}$ is an isomorphism for all $k$ and $l = m+2$ and a trivial homomorphism otherwise. Consequently,

$$E_{m+2}^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant m+1;\ l = 0, 1, m+3, m+4, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m;\ l = 0, 1, 2, m+3, m+4, \\ \mathbb{Z}_2, & k = 0, 1;\ 0 \leqslant l \leqslant m,\ l = m+3, m+4, \\ 0, & \text{otherwise}. \end{cases} \tag{4.14}$$

Clearly, $d_{m+2} : E_{m+2}^{k,l} \to E_{m+2}^{k+m+2,l-m-1}$ is zero by degree reasons. So

$$E_{m+2}^{k,l} = E_{m+3}^{k,l}, \qquad \text{for all } k, l. \tag{4.15}$$

The differential $d_r : E_r^{0,m+3} \to E_r^{r,m+4-r}$ $(r \geqslant m+3)$ can only be nontrivial when $r = m+3, m+4$. If $d_r : E_r^{0,m+3} \to E_r^{r,m+4-r}$ is trivial for $r = m+3, m+4$, then $d_r = 0 : E_r^{k,m+3} \to E_r^{k+r,m+4-r}$ for any $k$, $r = m+3$ and $m+4$. Thus $E_{m+3}^{*,*} = E_\infty^{*,*}$, at least two lines of the spectral sequence survive to infinity, which contradicts Proposition 2.2. Thus, we get two possibilities:

(v.1.1) $d_{m+3} : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is nontrivial.

(v.1.2) $d_{m+3} = 0 : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ and $d_{m+4} : E_{m+4}^{0,m+3} \to E_{m+4}^{m+4,0}$ is nontrivial.

**Subcase (v.1.1).** If $d_{m+3} : E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is nontrivial, then $d_{m+3} : E_{m+3}^{k,l} \to E_{m+3}^{k+m+3,l-m-2}$ is an isomorphism for all $k$ and $l = m+3$ and a trivial homomorphism otherwise. Consequently,

$$E_{m+4}^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant m+3;\ l = 0, m+4, \\ \mathbb{Z}_2, & k = m+1, m+2;\ l = 0, 1, m+4, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m;\ l = 0, 1, 2, m+4, \\ \mathbb{Z}_2, & k = 0, 1;\ 0 \leqslant l \leqslant m,\ l = m+4, \\ 0, & \text{otherwise}. \end{cases}$$

The differential $d_r : E_r^{0,m+4} \to E_r^{r,m+5-r}$ $(r \geqslant m+4)$ can only be nontrivial when $r = m+5$. If $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is trivial, then $d_{m+5} = 0 : E_{m+5}^{k,m+4} \to E_{m+5}^{k+m+5,0}$ for any $k$. Thus $E_{m+4}^{*,*} = E_\infty^{*,*}$, the bottom line $(l = 0)$ and the top line

($l = m + 4$) of the spectral sequence survive to $E_\infty$, which contradicts Proposition 2.2. Therefore, $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ must be nontrivial. Then $d_{m+5} : E_{m+5}^{k,l} \to E_{m+5}^{k+m+5,l-m-4}$ is an isomorphism for all $k$ and $l = m + 4$ and a trivial homomorphism otherwise. Consequently,

$$E_{m+6}^{k,l} = \begin{cases} \mathbb{Z}_2, & k = m+3, m+4;\ l = 0, \\ \mathbb{Z}_2, & k = m+1, m+2;\ l = 0, 1, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m;\ l = 0, 1, 2, \\ \mathbb{Z}_2, & k = 0, 1;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise}. \end{cases} \tag{4.16}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m + 6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+6}^{*,*} = E_\infty^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_\infty^{m+5,0} = 0$, by (4.1), we have $x^{m+5} = 0$. Clearly, $x^2u^3 = 0$, $x^{m+1}u^2 = 0$, $x^{m+3}u = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x, u]/\langle x^{m+5}, u^{m+1}, x^2u^3, x^{m+1}u^2, x^{m+3}u\rangle.$$

Analyzing the filtration of $H^*(X_G)$ as in (4.4) and (4.5) and choosing the particular $y$ as in (4.6), consider (4.16), we get the following relations:

$$\begin{aligned} y^{m+1} &= \alpha_1 xy^m + \alpha_2 x^{m-1}y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ x^2y^3 &= \beta_1 x^3y^2 + \beta_2 x^4 y + \beta_3 x^5, \\ x^{m+1}y^2 &= \gamma_1 x^{m+2}y + \gamma_2 x^{m+3}, \\ x^{m+3}y &= 0 \end{aligned}$$

for some $\alpha_i$ ($i = 1, \ldots, 4$), $\beta_j$ ($j = 1, 2, 3$), $\gamma_1, \gamma_2 \in \mathbb{Z}_2$. So the graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x, y]/I$, where $I$ is the ideal given by

$$\begin{aligned} I = \langle & x^{m+5}, y^{m+1} + \alpha_1 xy^m + \alpha_2 x^{m-1}y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ & x^2y^3 + \beta_1 x^3y^2 + \beta_2 x^4 y + \beta_3 x^5, \\ & x^{m+1}y^2 + \gamma_1 x^{m+2}y + \gamma_2 x^{m+3}, x^{m+3}y\rangle, \end{aligned}$$

where $m \geqslant 3$. This gives possibility (6) of Theorem 3.1.

**Subcase (v.1.2).** If $d_{m+3}: E_{m+3}^{0,m+3} \to E_{m+3}^{m+3,1}$ is trivial and $d_{m+4}: E_{m+4}^{0,m+3} \to E_{m+4}^{m+4,0}$ is nontrivial, then

$$\begin{aligned} d_{m+3} = 0 : E_{m+3}^{k,l} &\to E_{m+3}^{k+m+3,l-m-2}, \quad \text{for any } k, l, \\ d_{m+4}(1 \otimes a^{m-1}b) &= t^{m+4} \otimes 1, \\ d_{m+4}(1 \otimes a^{m}b) &= t^{m+4} \otimes a. \end{aligned} \tag{4.17}$$

Furthermore, we obtain that

$$d_{m+4} : E_{m+4}^{k,l} \to E_{m+4}^{k+m+4,l-m-3} \tag{4.18}$$

is an isomorphism for all $k$ and $l = m+3, m+4$ and a trivial homomorphism otherwise. Consequently, by (4.15), (4.17) and (4.18), we have

$$E_{m+5}^{k,l} = \begin{cases} \mathbb{Z}_2, & m+1 \leqslant k \leqslant m+3;\ l = 0, 1, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m;\ l = 0, 1, 2, \\ \mathbb{Z}_2, & k = 0, 1;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+5$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+5}^{*,*} = E_{\infty}^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_{\infty}^{m+4,0} = 0$, by (4.1), we have $x^{m+4} = 0$. Clearly, $x^2u^3 = 0$, $x^{m+1}u^2 = 0$. Combining with (4.3), then

$$\mathrm{Tot} E_{\infty}^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{m+4}, u^{m+1}, x^2u^3, x^{m+1}u^2 \rangle.$$

The graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x,y]/I$, where $I$ is the ideal given by

$$\begin{aligned} I = \langle & x^{m+4}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ & x^2y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5, x^{m+1}y^2 + \gamma_1 x^{m+2} y + \gamma_2 x^{m+3} \rangle, \end{aligned}$$

where $m \geqslant 3$ and $\alpha_i$ $(i = 1, \ldots, 4)$, $\beta_j$ $(j = 1, 2, 3)$, $\gamma_1, \gamma_2 \in \mathbb{Z}_2$. This gives possibility (7) of Theorem 3.1.

**Subcase (v.2).** If $d_{m+1}: E_{m+1}^{0,m+2} \to E_{m+1}^{m+1,2}$ is trivial and $d_{m+2}: E_{m+2}^{0,m+2} \to E_{m+2}^{m+2,1}$ is nontrivial, then

$$\begin{aligned} d_{m+1} = 0 : E_{m+1}^{k,l} &\to E_{m+1}^{k+m+1,l-m}, \quad \text{for any } k, l, \\ d_{m+2}(1 \otimes a^{m-2}b) &= t^{m+2} \otimes a, \\ d_{m+2}(1 \otimes a^{m-1}b) &= t^{m+2} \otimes a^2. \end{aligned} \tag{4.19}$$

Furthermore, we obtain that

$$d_{m+2} : E_{m+2}^{k,l} \to E_{m+2}^{k+m+2,l-m-1} \tag{4.20}$$

is an isomorphism for all $k$ and $l = m+2, m+3$ and a trivial homomorphism otherwise. Consequently, by (4.13), (4.19) and (4.20), we have

$$E_{m+3}^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant m+2;\ l = 0, m+4, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m+1;\ l = 0,1,2,m+4, \\ \mathbb{Z}_2, & k = 0,1;\ 0 \leqslant l \leqslant m,\ l = m+4, \\ 0, & \text{otherwise}. \end{cases}$$

The differential $d_r : E_r^{0,m+4} \to E_r^{r,m+5-r}$ $(r \geqslant m+3)$ can only be nontrivial when $r = m+5$. If $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is trivial, then $d_{m+5} = 0 : E_{m+5}^{k,m+4} \to E_{m+5}^{k+m+5,0}$ for any $k$. Thus $E_{m+3}^{*,*} = E_\infty^{*,*}$, the bottom line $(l = 0)$ and the top line $(l = m+4)$ of the spectral sequence survive to $E_\infty$, which contradicts Proposition 2.2. Therefore, $d_{m+5} : E_{m+5}^{0,m+4} \to E_{m+5}^{m+5,0}$ is nontrivial. Then $d_{m+5} : E_{m+5}^{k,l} \to E_{m+5}^{k+m+5,l-m-4}$ is an isomorphism for all $k$ and $l = m+4$ and a trivial homomorphism otherwise. Consequently,

$$E_{m+6}^{k,l} = \begin{cases} \mathbb{Z}_2, & m+2 \leqslant k \leqslant m+4;\ l = 0, \\ \mathbb{Z}_2, & 2 \leqslant k \leqslant m+1;\ l = 0,1,2, \\ \mathbb{Z}_2, & k = 0,1;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+6}^{*,*} = E_\infty^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_\infty^{m+5,0} = 0$, by (4.1), we have $x^{m+5} = 0$. Clearly, $x^2u^3 = 0$, $x^{m+2}u = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{m+5}, u^{m+1}, x^2u^3, x^{m+2}u\rangle.$$

The graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x,y]/I$, where $I$ is the ideal given by

$$I = \langle x^{m+5}, y^{m+1} + \alpha_1 xy^m + \alpha_2 x^{m-1}y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1},$$
$$x^2y^3 + \beta_1 x^3y^2 + \beta_2 x^4 y + \beta_3 x^5, x^{m+2}y\rangle,$$

where $m \geqslant 3$ and $\alpha_i$ $(i = 1,\ldots,4)$, $\beta_j$ $(j = 1,2,3) \in \mathbb{Z}_2$. This gives possibility (8) of Theorem 3.1.

**Subcase (v.3).** If $d_r : E_r^{0,m+2} \to E_r^{r,m+3-r}$ is trivial for $r = m+1, m+2$ and $d_{m+3}$ : $E_{m+3}^{0,m+2} \to E_{m+3}^{m+3,0}$ is nontrivial, then

$$\begin{aligned} d_r &= 0 : E_r^{k,l} \to E_r^{k+r,l+1-r}, \quad \text{for any } k, l \text{ and } r = m+1, m+2, \\ d_{m+3}(1 \otimes a^j b) &= t^{m+3} \otimes a^{j-m+2}, j = m-2, m-1, m. \end{aligned} \tag{4.21}$$

Furthermore, we obtain that

$$d_{m+3} : E_{m+3}^{k,l} \to E_{m+3}^{k+m+3,l-m-2} \tag{4.22}$$

is an isomorphism for all $k$ and $l = m+2, m+3, m+4$ and a trivial homomorphism otherwise. Consequently, by (4.13), (4.21) and (4.22), we have

$$E_{m+4}^{k,l} = \begin{cases} \mathbb{Z}_2, & 2 \leqslant k \leqslant m+2;\ l = 0, 1, 2, \\ \mathbb{Z}_2, & k = 0, 1;\ 0 \leqslant l \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant m+4$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{m+4}^{*,*} = E_\infty^{*,*}.$$

We observe that the cohomology groups $H^j(X_G)$ are the same as (4.2).

As $E_\infty^{m+3,0} = 0$, by (4.1), we have $x^{m+3} = 0$. Clearly, $x^2u^3 = 0$. Combining with (4.3), then

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{m+3}, u^{m+1}, x^2u^3\rangle.$$

The graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x,y]/I$, where $I$ is the ideal given by

$$\begin{aligned} I = \langle & x^{m+3}, y^{m+1} + \alpha_1 x y^m + \alpha_2 x^{m-1} y^2 + \alpha_3 x^m y + \alpha_4 x^{m+1}, \\ & x^2y^3 + \beta_1 x^3 y^2 + \beta_2 x^4 y + \beta_3 x^5 \rangle, \end{aligned}$$

where $m \geqslant 3$ and $\alpha_i$ $(i = 1, \ldots, 4)$, $\beta_j$ $(j = 1, 2, 3) \in \mathbb{Z}_2$. This gives possibility (9) of Theorem 3.1. ∎

## 4.2. Proof of Theorem 3.2

Let $G = \mathbb{Z}_2$ act freely on $X \sim_2 \mathbb{C}P^m \times S^4$. For $m \geqslant 2$, we have

$$H^l(X) = \begin{cases} \mathbb{Z}_2, & l = 0, 2, 2m+2, 2m+4, \\ (\mathbb{Z}_2)^2, & l = 4, 6, \ldots, 2m, \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 1$, we have

$$H^l(X) = \begin{cases} \mathbb{Z}_2, & l = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $E_2^{k,l} = H^k(B_G) \otimes H^l(X) = 0$ for $l$ odd. This gives $d_r = 0$ for $r$ even. Let $a \in H^2(X)$ and $b \in H^4(X)$ be generators of the cohomology algebra of $H^*(X)$, satisfying $a^{m+1} = 0$ and $b^2 = 0$. As in the proof of Theorem 3.1, it is clear that $t \otimes 1 \in E_2^{1,0}$ is a permanent cocycle and survives to a nontrivial element $x \in E_\infty^{1,0}$, i.e.,

$$x = \pi^*(t) \in E_\infty^{1,0} \subset H^1(X_G). \tag{4.23}$$

Since $\mathbb{Z}_2$ acts freely on $X$, by Proposition 2.2, the spectral sequence does not collapse. Otherwise, we get $H^i(X/G) \neq 0$ for infinitely many values of $i > 2m + 4$. This implies that some differential $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ must be nontrivial. Note that $E_2^{*,*}$ is generated by $t \otimes 1 \in E_2^{1,0}$, $1 \otimes a \in E_2^{0,2}$ and $1 \otimes b \in E_2^{0,4}$. There can only be nontrivial differentials $d_r$ on the generators when $r = 3, 5$. It follows immediately that there are three possibilities for nontrivial differentials:

(i) $d_3(1 \otimes a) \neq 0$.

(ii) $d_3(1 \otimes a) = 0$, $d_3(1 \otimes b) = 0$ and $d_5(1 \otimes b) \neq 0$.

(iii) $d_3(1 \otimes a) = 0$ and $d_3(1 \otimes b) \neq 0$.

**Case (i).** $d_3(1 \otimes a) = t^3 \otimes 1 \neq 0$.

If $m$ is even, then $a^{m+1} = 0$ gives $0 = d_3((1 \otimes a^m)(1 \otimes a)) = t^3 \otimes a^m$, a contradiction. Hence $m$ must be odd. There are two possible subcases: either $d_3(1 \otimes b) = t^3 \otimes a \neq 0$ or $d_3(1 \otimes b) = 0$.

Firstly, let us consider $d_3(1 \otimes b) = t^3 \otimes a \neq 0$. Note that by the derivation property of the differential we have

$$\begin{cases} d_3(1 \otimes a^j) = j(t^3 \otimes a^{j-1}), & 1 \leqslant j \leqslant m, \\ d_3(1 \otimes a^j b) = j(t^3 \otimes a^{j-1} b) + t^3 \otimes a^{j+1}, & 0 \leqslant j \leqslant m-1, \\ d_3(1 \otimes a^m b) = t^3 \otimes a^{m-1} b. \end{cases}$$

Note that

$$d_3(1 \otimes ab) = \begin{cases} t^3 \otimes b + t^3 \otimes a^2, & m > 1, \\ t^3 \otimes b, & m = 1. \end{cases}$$

$$\begin{aligned} d_3 d_3(1 \otimes ab) &= \begin{cases} d_3(t^3 \otimes b + t^3 \otimes a^2), & m > 1, \\ d_3(t^3 \otimes b), & m = 1. \end{cases} \\ &= t^6 \otimes a \neq 0 \end{aligned}$$

This contradicts $d_3 d_3 = 0$, thus $d_3(1 \otimes b) = 0$.

By the derivation property of the differential we have

$$\begin{cases} d_3(1\otimes a^j) = j(t^3\otimes a^{j-1}), & 1\leqslant j\leqslant m,\\ d_3(1\otimes a^j b) = j(t^3\otimes a^{j-1}b), & 0\leqslant j\leqslant m. \end{cases}$$

The $E_3$-term and $d_3$-differentials look like Figure 5. Then

$$E_4^{k,l} = \begin{cases} \mathbb{Z}_2, & 0\leqslant k\leqslant 2;\ l=0,2m+2,\\ (\mathbb{Z}_2)^2, & 0\leqslant k\leqslant 2;\ l=4,8,12,\dots,2m-2,\\ 0, & \text{otherwise}. \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r\geqslant 4$ as $E_r^{k+r,l-r+1}=0$, so

$$E_4^{*,*} = E_\infty^{*,*}.$$

Since $H^*(X_G)\cong \mathrm{Tot}E_\infty^{*,*}$, the additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & 0\leqslant j\leqslant 2 \text{ or } 2m+2\leqslant j\leqslant 2m+4,\\ (\mathbb{Z}_2)^2, & 4\leqslant j\leqslant 2m \text{ and } j\neq 7,11,15,\dots,2m-3,\\ 0, & \text{otherwise}. \end{cases}$$

As $E_\infty^{3,0}=0$, by (4.23), we have $x^3=0$. Notice that the elements $1\otimes a^2\in E_2^{0,4}$ and $1\otimes b\in E_2^{0,4}$ are permanent cocycles and are not hit by any $d_r$-coboundaries.

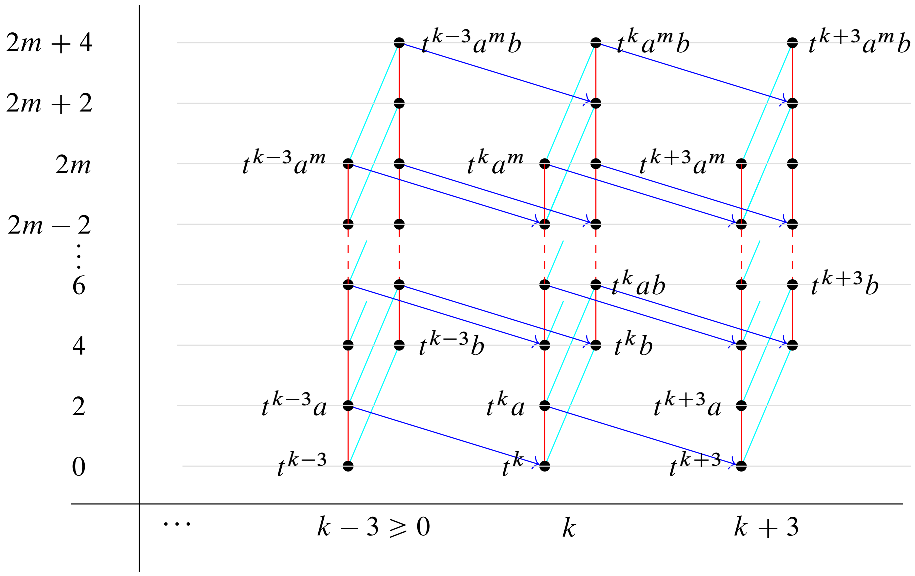


**Figure 5.** $E_3$-term and $d_3$-differentials in Case (i)

Hence, they determine nontrivial elements $u \in E_\infty^{0,4}$ and $v \in E_\infty^{0,4}$, respectively. We have $u^{\frac{m+1}{2}} = 0$ as $a^{m+1} = 0$, and $v^2 = 0$ as $b^2 = 0$. Thus

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x, u, v]/\langle x^2, u^{\frac{m+1}{2}}, v^2\rangle,$$

where $\deg x = 1$, $\deg u = 4$, $\deg v = 4$.

By the edge homomorphism, let $y \in H^4(X_G)$ and $z \in H^4(X_G)$ be such that $i^*(y) = a^2$ and $i^*(z) = b$, respectively. Notice that $y^{\frac{m+1}{2}} \in H^{2m+2}(X_G) = E_\infty^{0,2m+2}$ is represented by $a^{m+1} \in E_2^{0,2m+2}$ and $z^2 \in H^8(X_G) = E_\infty^{0,8}$ is represented by $b^2 \in E_2^{0,8}$. Since the edge homomorphism is an isomorphism in degrees 8 and $2m+2$, we have the following relations:

$$y^{\frac{m+1}{2}} = 0, \qquad z^2 = 0.$$

Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y, z]/\langle x^3, y^{\frac{m+1}{2}}, z^2\rangle,$$

where $\deg x = 1$, $\deg y = 4$, $\deg z = 4$ and $m$ is odd. This gives possibility (1) of Theorem 3.2.

**Case (ii).** $d_3(1 \otimes a) = 0$, $d_3(1 \otimes b) = 0$ and $d_5(1 \otimes b) = t^5 \otimes 1 \neq 0$.

This case implies that $d_3 = 0$. We have

$$\begin{cases} d_5(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_5(1 \otimes a^j b) = t^5 \otimes a^j, & 0 \leqslant j \leqslant m, \end{cases}$$

and

$$\begin{gathered} E_5^{k-5,l+4} \xrightarrow{d_5} E_5^{k,l} \xrightarrow{d_5} E_5^{k+5,l-4}, \\ t^{k-5} \otimes a^{\frac{l}{2}} b \xmapsto{d_5} t^k \otimes a^{\frac{l}{2}} \xmapsto{d_5} 0, \\ t^{k-5} \otimes a^{\frac{l}{2}+4} \xmapsto{d_5} 0, \quad t^k \otimes a^{\frac{l}{2}-4} b \xmapsto{d_5} t^{k+5} \otimes a^{\frac{l}{2}-4}. \end{gathered}$$

So

$$E_6^{k,l} = \begin{cases} \mathbb{Z}_2, & 0 \leqslant k \leqslant 4;\ l = 0, 2, \ldots, 2m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant 6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_6^{*,*} = E_\infty^{*,*}.$$

The additive structure of $H^*(X_G)$ is given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & j = 0, 1, 2m+3, 2m+4, \\ (\mathbb{Z}_2)^2, & j = 2, 2m+2 \text{ or } j = 3, 5, \ldots, 2m+1, \\ (\mathbb{Z}_2)^3, & j = 4, 6, \ldots, 2m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.24}$$

Notice that the element $1 \otimes a \in E_2^{0,2}$ is a permanent cocycle and is not a $d_r$-coboundary. Hence, it determines a nontrivial element $u \in E_\infty^{0,2}$. As we have remarked, $a^{m+1} = 0$, so

$$u^{m+1} = 0. \tag{4.25}$$

As $E_\infty^{5,0} = 0$, by (4.23), we have $x^5 = 0$. Thus

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x, u]/\langle x^5, u^{m+1} \rangle,$$

where $\deg x = 1$, $\deg u = 2$.

Now, choose $y \in H^2(X_G)$ such that $i^*(y) = a$. By considering the filtration on $H^{2m+2}(X_G)$,

$$\begin{aligned} 0 = F_{2m+2}^{2m+2} = \cdots = \underbrace{F_5^{2m+2} \subset F_4^{2m+2}}_{E_\infty^{4,2m-2}} = \underbrace{F_3^{2m+2} \subset F_2^{2m+2}}_{E_\infty^{2,2m}} \\ = F_1^{2m+2} = F_0^{2m+2} = H^{2m+2}(X_G), \end{aligned} \tag{4.26}$$

we get the following relation:

$$y^{m+1} = \alpha_1 x^2 y^m + \alpha_2 x^4 y^{m-1},$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}_2$. Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y]/\langle x^5, y^{m+1} + \alpha_1 x^2 y^m + \alpha_2 x^4 y^{m-1} \rangle,$$

where $\deg x = 1$, $\deg y = 2$. This gives possibility (2) of Theorem 3.2.

*In the remaining Case (iii) there will be classes $u \in E_\infty^{0,2}$, $y \in H^2(X_G)$ defined as above and the relation (4.25) will be satisfied.*

**Case (iii).** $d_3(1 \otimes a) = 0$ and $d_3(1 \otimes b) \neq 0$.

Clearly, $d_3(1 \otimes b) = t^3 \otimes a$. So we have

$$\begin{cases} d_3(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_3(1 \otimes a^j b) = t^3 \otimes a^{j+1}, & 0 \leqslant j \leqslant m-1, \\ d_3(1 \otimes a^m b) = 0. \end{cases}$$

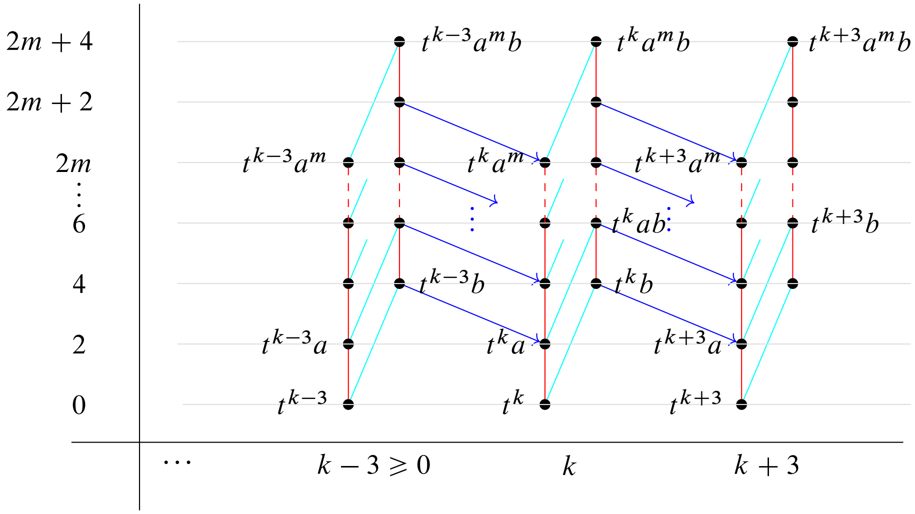


**Figure 6.** $E_3$-term and $d_3$-differentials in Case (iii)

The $E_3$-term and $d_3$-differentials look like Figure 6. Then

$$E_5^{k,l} = E_4^{k,l} = \begin{cases} \mathbb{Z}_2, & k \geqslant 3;\ l = 0, 2m+4, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ l = 0, 2, \ldots, 2m \text{ or } l = 2m+4, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $5 \leqslant r \leqslant 2m+4$. Now, if $d_{2m+5} : E_{2m+5}^{0,2m+4} \to E_{2m+5}^{2m+5,0}$ is trivial, then by the multiplicative properties of the spectral sequence, we have $E_{2m+5}^{*,*} = E_\infty^{*,*}$. Therefore the bottom line ($l = 0$) and the top line ($l = 2m+4$) of the spectral sequence survive to $E_\infty$, which reduces to $H^i(X/G) \neq 0$ for all $i > 2m+4$. This contradicts to Proposition 2.2. Thus, $d_{2m+5} : E_{2m+5}^{0,2m+4} \to E_{2m+5}^{2m+5,0}$ must be nontrivial. It follows immediately that $d_{2m+5} : E_{2m+5}^{k,2m+4} \to E_{2m+5}^{k+2m+5,0}$ is an isomorphism for all $k$. So

$$E_{2m+6}^{k,l} = \begin{cases} \mathbb{Z}_2, & 3 \leqslant k \leqslant 2m+4;\ l = 0, \\ \mathbb{Z}_2, & 0 \leqslant k \leqslant 2;\ l = 0, 2, \ldots, 2m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant 2m+6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_{2m+6}^{*,*} = E_\infty^{*,*}.$$

It follows that the cohomology groups $H^j(X_G)$ are the same (4.24) as in Case (ii).

As $E_\infty^{2m+5,0} = 0$, by (4.23), we have $x^{2m+5} = 0$. Clearly, $x^3u = 0$. Combining with (4.25), then

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x,u]/\langle x^{2m+5}, u^{m+1}, x^3u\rangle,$$

Choose $y' \in H^2(X_G)$ such that $i^*(y') = a$ and let $y = y' + \beta x^2 \in H^2(X_G)$, $\beta \in \mathbb{Z}_2$. As before, we conclude that the graded commutative algebra $H^*(X_G)$ is $\mathbb{Z}_2[x,y]/I$, where $I$ is the ideal given by

$$I = \langle x^{2m+5}, y^{m+1} + \alpha_1 x^2 y^m + \alpha_2 x^{2m+2}, x^3 y\rangle,$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}_2$. This gives possibility (3) of Theorem 3.2. ∎

## 4.3. Proof of Theorem 3.3

Let $G = \mathbb{Z}_2$ act freely on $X \sim_2 \mathbb{H}P^m \times S^4$. We observe that $m \geqslant 1$,

$$H^l(X) = \begin{cases} \mathbb{Z}_2, & l = 0, 4m+4, \\ (\mathbb{Z}_2)^2, & l = 4, 8, \ldots, 4m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $E_2^{k,l} = H^k(B_G) \otimes H^l(X) = 0$ for $l \not\equiv 0 \pmod 4$. This gives $d_r = 0$ for $2 \leqslant r \leqslant 4$ and hence $E_2^{*,*} = E_5^{*,*}$. Let $a \in H^4(X)$ and $b \in H^4(X)$ be generators of the cohomology algebra of $H^*(X)$, satisfying $a^{m+1} = 0$ and $b^2 = 0$. The element $t \otimes 1 \in E_2^{1,0}$ is a permanent cocycle and survives to a nontrivial element $x \in E_\infty^{1,0}$, i.e.,

$$x = \pi^*(t) \in E_\infty^{1,0} \subset H^1(X_G). \tag{4.27}$$

Since $\mathbb{Z}_2$ acts freely on $X$, by Proposition 2.2, the spectral sequence does not collapse. It implies that some differential $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ must be nontrivial. Note that $E_2^{*,*}$ is generated by $t \otimes 1 \in E_2^{1,0}$, $1 \otimes a \in E_2^{0,4}$ and $1 \otimes b \in E_2^{0,4}$. The first nontrivial differential $d_r$ occurs possibly only when $r = 5$. It follows immediately that there are three possibilities for the nontrivial differential:

(i) $d_5(1 \otimes a) \neq 0$ and $d_5(1 \otimes b) \neq 0$.

(ii) $d_5(1 \otimes a) \neq 0$ and $d_5(1 \otimes b) = 0$.

(iii) $d_5(1 \otimes a) = 0$ and $d_5(1 \otimes b) \neq 0$.

**Case (i).** $d_5(1 \otimes a) = t^5 \otimes 1 \neq 0$ and $d_5(1 \otimes b) = t^5 \otimes 1 \neq 0$.

Note that by the derivation property of the differential we have

$$\begin{cases} d_5(1 \otimes a^j) = j(t^5 \otimes a^{j-1}), & 1 \leqslant j \leqslant m, \\ d_5(1 \otimes a^j b) = t^5 \otimes a^j + j(t^5 \otimes a^{j-1} b), & 0 \leqslant j \leqslant m. \end{cases}$$

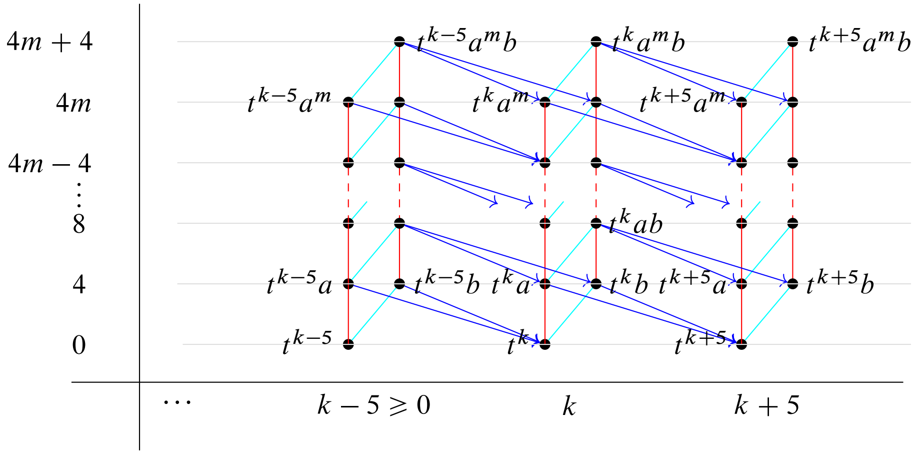


**Figure 7.** $E_5$-term and $d_5$-differentials in Case (i)

If $m$ is even, then $a^{m+1} = 0$ gives $0 = d_5((1 \otimes a^m)(1 \otimes a)) = t^5 \otimes a^m$, a contradiction. Hence $m$ must be odd. The $E_5$-term and $d_5$-differentials look like Figure 7. Then

$$E_6^{k,l} = \begin{cases} \mathbb{Z}_2, & 0 \leqslant k \leqslant 4;\ l = 0, 4, 8, \ldots, 4m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is zero for all $r \geqslant 6$ as $E_r^{k+r,l-r+1} = 0$, so

$$E_6^{*,*} = E_\infty^{*,*}.$$

Since $H^*(X_G) \cong \mathrm{Tot}E_\infty^{*,*}$, we have

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & 0 \leqslant j \leqslant 4m+4 \text{ and } j \neq 4, 8, \ldots, 4m, \\ (\mathbb{Z}_2)^2, & j = 4, 8, \ldots, 4m, \\ 0, & \text{otherwise.} \end{cases}$$

As $E_\infty^{5,0} = 0$, by (4.27), we have $x^5 = 0$. Notice that the elements $1 \otimes a^2 \in E_2^{0,8}$ and $1 \otimes (a+b) \in E_2^{0,4}$ are permanent cocycles and are not hit by any $d_r$-coboundaries. Hence, they determine nontrivial elements $u \in E_\infty^{0,8}$ and $v \in E_\infty^{0,4}$, respectively. We have $u^{\frac{m+1}{2}} = 0$ as $a^{m+1} = 0$, and $v^2 + u = 0$ as $b^2 = 0$. Thus

$$\mathrm{Tot}E_\infty^{*,*} \cong \mathbb{Z}_2[x, u, v]/\langle x^5, u^{\frac{m+1}{2}}, v^2 + u \rangle,$$

where $\deg x = 1$, $\deg u = 8$, $\deg v = 4$.

Let $y \in H^8(X_G)$ and $z \in H^4(X_G)$ be such that $i^*(y) = a^2$ and $i^*(z) = a + b$, respectively. By considering the filtrations of $H^{4m+4}(X_G)$ and $H^8(X_G)$, we have the short exact sequence

$$0 \to E_\infty^{4,j-4} \to H^j(X_G) \to E_\infty^{0,j} \to 0, \quad j = 4m + 4 \text{ or } 8. \tag{4.28}$$

By (4.28), we get the following relations:

$$\begin{aligned} y^{\frac{m+1}{2}} &= \beta x^4 y^{\frac{m-1}{2}} z, && \beta \in \mathbb{Z}_2, \\ z^2 + y &= \alpha x^4 z, && \alpha \in \mathbb{Z}_2. \end{aligned}$$

Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y, z]/\langle x^5, y^{\frac{m+1}{2}} + \beta x^4 y^{\frac{m-1}{2}} z, z^2 + \gamma y + \alpha x^4 z\rangle,$$

where $\deg x = 1$, $\deg y = 8$, $\deg z = 4$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$ and $m$ is odd. Also, $\gamma = 1$ except when $m = 1$.

**Case (ii).** $d_5(1 \otimes a) = t^5 \otimes 1 \neq 0$ and $d_5(1 \otimes b) = 0$.

If $m$ is even, then $0 = d_5(1 \otimes a^{m+1}) = t^5 \otimes a^m$, a contradiction. So $m$ must be odd. Note that by the derivation property of the differential we have

$$\begin{cases} d_5(1 \otimes a^j) = j(t^5 \otimes a^{j-1}), & 1 \leqslant j \leqslant m, \\ d_5(1 \otimes a^j b) = j(t^5 \otimes a^{j-1} b), & 0 \leqslant j \leqslant m. \end{cases}$$

The $E_5$-term and $d_5$-differentials look like Figure 8. Then $E_6^{k,l}$ is the same as in Case (i),

$$E_6^{k,l} = \begin{cases} \mathbb{Z}_2, & 0 \leqslant k \leqslant 4;\ l = 0, 4, 8, \ldots, 4m, \\ 0, & \text{otherwise}. \end{cases}$$

Thus the cohomology groups $H^j(X_G)$ are also the same as in Case (i),

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & 0 \leqslant j \leqslant 4m + 4 \text{ and } j \neq 4, 8, \ldots, 4m, \\ (\mathbb{Z}_2)^2, & j = 4, 8, \ldots, 4m, \\ 0, & \text{otherwise}. \end{cases}$$

As $E_\infty^{5,0} = 0$, by (4.27), we have $x^5 = 0$. Notice that the elements $1 \otimes a^2 \in E_2^{0,8}$ and $1 \otimes b \in E_2^{0,4}$ are permanent cocycles and are not hit by any $d_r$-coboundaries. Hence, they determine nontrivial elements $u \in E_\infty^{0,8}$ and $v \in E_\infty^{0,4}$, respectively. We have $u^{\frac{m+1}{2}} = 0$ as $a^{m+1} = 0$, and $v^2 = 0$ as $b^2 = 0$. Thus

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x, u, v]/\langle x^5, u^{\frac{m+1}{2}}, v^2\rangle,$$

where $\deg x = 1$, $\deg u = 8$, $\deg v = 4$.

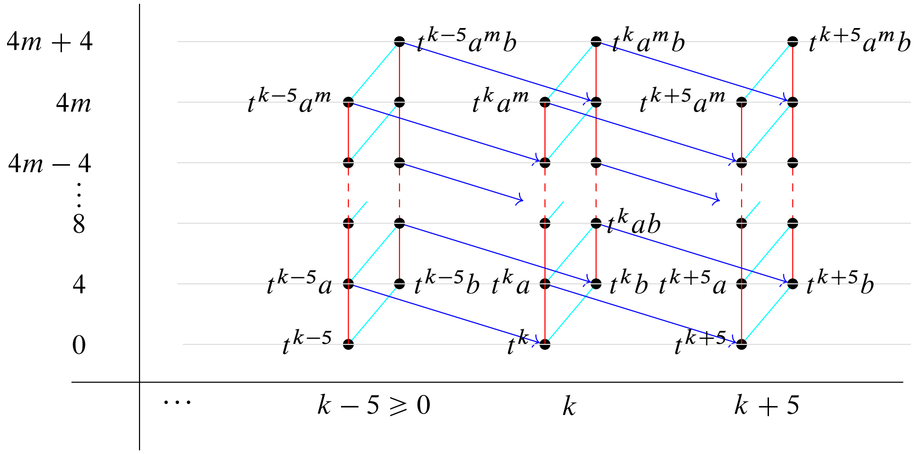


**Figure 8.** $E_5$-term and $d_5$-differentials in Case (ii)

Let $y \in H^8(X_G)$ and $z \in H^4(X_G)$ be such that $i^*(y) = a^2$ and $i^*(z) = b$, respectively. Similar to Case (i), by (4.28), we get the following relations:

$$\begin{aligned} y^{\frac{m+1}{2}} &= \beta x^4 y^{\frac{m-1}{2}} z, && \beta \in \mathbb{Z}_2, \\ z^2 &= \alpha x^4 z, && \alpha \in \mathbb{Z}_2. \end{aligned}$$

Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y, z]/\langle x^5, y^{\frac{m+1}{2}} + \beta x^4 y^{\frac{m-1}{2}} z, z^2 + \alpha x^4 z\rangle,$$

where $\deg x = 1$, $\deg y = 8$, $\deg z = 4$, $\alpha, \beta \in \mathbb{Z}_2$ and $m$ is odd. If $m = 1$, then $\beta = 0$.

By combining results in Case (i) and (ii), we can rewrite the result as follows:

$$H^*(X_G) = \mathbb{Z}_2[x, y, z]/\langle x^5, y^{\frac{m+1}{2}} + \beta x^4 y^{\frac{m-1}{2}} z, z^2 + \gamma y + \alpha x^4 z\rangle,$$

where $\deg x = 1$, $\deg y = 8$, $\deg z = 4$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$ and $m$ is odd. If $m = 1$, then $\beta = 0, \gamma = 0$. This gives possibility (1) of Theorem 3.3.

**Case (iii).** $d_5(1 \otimes a) = 0$ and $d_5(1 \otimes b) \neq 0$.

Immediately, $d_5(1 \otimes b) = t^5 \otimes 1$, so we have

$$\begin{cases} d_5(1 \otimes a^j) = 0, & 1 \leqslant j \leqslant m, \\ d_5(1 \otimes a^j b) = t^5 \otimes a^j, & 0 \leqslant j \leqslant m. \end{cases}$$

and

$$E_5^{k-5,l+4} \xrightarrow{d_5} E_5^{k,l} \xrightarrow{d_5} E_5^{k+5,l-4},$$

$$t^{k-5} \otimes a^{\frac{l}{4}} b \overset{d_5}{\longmapsto} t^k \otimes a^{\frac{l}{4}} \overset{d_5}{\longmapsto} 0,$$

$$t^{k-5} \otimes a^{\frac{l}{4}+1} \overset{d_5}{\longmapsto} 0, \quad t^k \otimes a^{\frac{l}{4}-4} b \overset{d_5}{\longmapsto} t^{k+5} \otimes a^{\frac{l}{4}-4}.$$

Then $E_6^{k,l}$ is the same as in Case (i),

$$E_6^{k,l} = \begin{cases} \mathbb{Z}_2, & 0 \leqslant k \leqslant 4;\ l = 0, 4, \ldots, 4m, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the cohomology groups $H^j(X_G)$ are also the same as in Case (i),

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2, & 0 \leqslant j \leqslant 4m+4 \text{ and } j \neq 4, 8, \ldots, 4m, \\ (\mathbb{Z}_2)^2, & j = 4, 8, \ldots, 4m, \\ 0, & \text{otherwise.} \end{cases}$$

As $E_\infty^{5,0} = 0$, by (4.27), we have $x^5 = 0$. Notice that the element $1 \otimes a \in E_2^{0,4}$ is a permanent cocycle and is not a $d_r$-coboundary. Hence, it determines a nontrivial element $u \in E_\infty^{0,4}$. As we have remarked, $a^{m+1} = 0$, so $u^{m+1} = 0$. Thus

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x, u]/\langle x^5, u^{m+1} \rangle,$$

where $\deg x = 1$, $\deg u = 4$.

Choose $y' \in H^4(X_G)$ such that $i^*(y') = a$ and let $y = y' + \alpha x^4 \in H^4(X_G)$, $\alpha \in \mathbb{Z}_2$. we get the following relation:

$$y^{m+1} = 0.$$

Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x, y]/\langle x^5, y^{m+1} \rangle,$$

where $\deg x = 1$, $\deg y = 4$. This gives possibility (2) of Theorem 3.3. ∎

## 5. Applications to $\mathbb{Z}_2$-equivariant maps

We will now use the above results to study the existence of equivariant maps to and from $X$. This is an application that we find highly motivating. Let $X$ be a compact Hausdorff space with a free involution and the unit $n$-sphere $S^n$ carries the antipodal involution. Let us recall some numerical indices.

**Definition 5.1** ([4]). The index of the involution on $X$ is

$$\mathrm{ind}(X) = \max\{n \mid \text{there exists a } \mathbb{Z}_2\text{-equivariant map } S^n \to X\}.$$

**Definition 5.2** ([4]). The mod 2 cohomology index of the involution on $X$ is

$$\text{co-ind}_2(X) = \max\{n \mid \omega^n \neq 0\},$$

where $\omega \in H^1(X/\mathbb{Z}_2;\mathbb{Z}_2)$ is the Whitney class of the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$.

The above index and co-index are both defined by Conner and Floyd. Further, they gave the relationship between these indices.

**Proposition 5.3** ([4]). *The following holds:* $\mathrm{ind}(X) \leqslant \text{co-ind}_2(X)$.

Given a $G$-space $X$, Volovikov defined a numerical index $i(X)$ as the following:

**Definition 5.4** ([27]). The index $i(X)$ is the smallest $r$ such that for some $k$, $d_r : E_r^{k-r,r-1} \to E_r^{k,0}$ in the cohomology Leray–Serre spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G$ is nontrivial.

Let $\beta_k(X)$ be the $k$-th *Betti number* of the space $X$. Using Volovikov index, Coelho, Mattos and Santos proved the following results.

**Proposition 5.5** ([3, Theorem 1.1]). *Let $G$ be a compact Lie group and $X, Y$ be Hausdorff, path-connected and paracompact free $G$-spaces. With a PID as the coefficient for the cohomology, suppose that $i(X) \geqslant l+1$ for some natural $l \geqslant 1$ and $H^{k+1}(Y/G) = 0$ for some $1 \leqslant k \leqslant l$.*

(i) *If $k = l$ and $\beta_l(X) < \beta_{l+1}(B_G)$, then there is no $G$-equivariant map $f : X \to Y$.*

(ii) *If $1 \leqslant k < l$ and $0 < \beta_{k+1}(B_G)$, then there is no $G$-equivariant map $f : X \to Y$.*

Using these Conner and Floyd indices, we get the following results.

**Proposition 5.6.** *Let $X \sim_2 \mathbb{R}P^m \times S^4$ be a finitistic space with a free involution and consider the antipodal involution on $S^n$. If $m = 5$ or $m = 7$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$. Then the mod 2 co-index of $X$ can only take the values $C = 1, 4, m+2, m+3$ and $m+4$ and there are no $\mathbb{Z}_2$-equivariant maps $S^n \to X$ for $n \geqslant C+1$.*

*Proof.* For the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$, we can take a classifying map

$$f : X/\mathbb{Z}_2 \to B_{\mathbb{Z}_2}.$$

It would uniquely determine a homotopy class of $[X/\mathbb{Z}_2, B_{\mathbb{Z}_2}]$. Let $\eta : X/\mathbb{Z}_2 \to X_{\mathbb{Z}_2}$ be a homotopy inverse of the homotopy equivalence $h : X_{\mathbb{Z}_2} \to X/\mathbb{Z}_2$, then $\pi\eta : X/\mathbb{Z}_2 \to B_{\mathbb{Z}_2}$ also classifies the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$. Therefore, we find the following homotopy equivalence $f \simeq \pi\eta$. Consider the map

$$\pi^* : H^1(B_{\mathbb{Z}_2}) \to H^1(X_{\mathbb{Z}_2}).$$

The characteristic class $t \in H^1(B_{\mathbb{Z}_2})$ of the universal bundle $\mathbb{Z}_2 \hookrightarrow E_{\mathbb{Z}_2} \xrightarrow{\pi} B_{\mathbb{Z}_2}$ is mapped to $\pi^*(t) \in H^1(X_{\mathbb{Z}_2}) \cong H^1(X/\mathbb{Z}_2)$, which is the Whitney class of the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$.

For $X \sim_2 \mathbb{R}P^m \times S^4$, by possibility (1) of Theorem 3.1, we see that $x \neq 0$ and $x^2 = 0$. Thus, co-ind$_2(X) = 1$. By Proposition 5.3, ind$(X) \leqslant 1$, this means that there is no $\mathbb{Z}_2$-equivariant map $S^n \to X$ for $n \geqslant 2$.

In possibility (2) of Theorem 3.1, $x^4 \neq 0$ and $x^5 = 0$. Accordingly, co-ind$_2(X) = 4$, ind$(X) \leqslant 4$ and there is no $\mathbb{Z}_2$-equivariant map $S^n \to X$ for $n \geqslant 5$.

In possibilities (3), (4), (6) and (8) of Theorem 3.1, $x^{m+4} \neq 0$ and $x^{m+5} = 0$. Accordingly, co-ind$_2(X) = m + 4$, ind$(X) \leqslant m + 4$ and there is no $\mathbb{Z}_2$-equivariant map $S^n \to X$ for $n \geqslant m + 5$.

In possibilities (5) and (7) of Theorem 3.1, $x^{m+3} \neq 0$ and $x^{m+4} = 0$. Therefore, co-ind$_2(X) = m + 3$, ind$(X) \leqslant m + 3$ and there is no $\mathbb{Z}_2$-equivariant map $S^n \to X$ for $n \geqslant m + 4$.

Finally, in possibility (9) of Theorem 3.1, $x^{m+2} \neq 0$ and $x^{m+3} = 0$. Thus, we have co-ind$_2(X) = m + 2$, ind$(X) \leqslant m + 2$ and there is no $\mathbb{Z}_2$-equivariant map $S^n \to X$ for $n \geqslant m + 3$. ■

By a similar proof, we get the following results for the $\mathbb{Z}_2$-equivariant maps from $S^n$ to $X \sim_2 \mathbb{C}P^m \times S^4$ or $X \sim_2 \mathbb{H}P^m \times S^4$.

**Proposition 5.7.** *Let $X \sim_2 \mathbb{C}P^m \times S^4$ be a finitistic space with a free involution and consider the antipodal involution on $S^n$. If $m = 3$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{C}P^3 \times S^4$. Then the mod 2 co-index of $X$ can only take the values $C = 2, 4$ and $2m + 4$ and there are no $\mathbb{Z}_2$-equivariant maps $S^n \to X$ for $n \geqslant C + 1$.*

**Proposition 5.8.** *Let $X \sim_2 \mathbb{H}P^m \times S^4$ be a finitistic space with a free involution and consider the antipodal involution on $S^n$. When $m \equiv 3 \pmod 4$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{H}P^m \times S^4$. Then the mod 2 co-index of $X$ can only take the value 4 and there are no $\mathbb{Z}_2$-equivariant maps $S^n \to X$ for $n \geqslant 5$.*

Note that the index of $X \sim_2 \mathbb{F}P^m \times S^4$ (Definition 5.1) can be no more than $m + 4$, $2m + 4$ and 4, when $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ respectively.

We get the following immediate consequences by the proof of Theorem 3.1, Theorem 3.2 and Theorem 3.3.

**Proposition 5.9.** *Let $\mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times S^4$. If $m = 5$ or $m = 7$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$. Then $i(X)$ has one of the following values: $2,\ 5,\ m+3,\ m+4$ or $m+5$.*

**Proposition 5.10.** *Let $\mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{C}P^m \times S^4$. If $m = 3$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{C}P^3 \times S^4$. Then $i(X)$ has one of the following values: $3,\ 5$ or $2m+5$.*

**Proposition 5.11.** *Let $\mathbb{Z}_2$ act freely on a finitistic space $X \sim_2 \mathbb{H}P^m \times S^4$. When $m \equiv 3 \pmod 4$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{H}P^m \times S^4$. Then $i(X) = 5$.*

By Proposition 5.5 and Proposition 5.9, we obtain:

**Proposition 5.12.** *Suppose that $\mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times S^4$ and a path-connected, paracompact Hausdorff space $Y$. If $m = 5$ or $m = 7$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$. Then there is no $\mathbb{Z}_2$-equivariant map $X \to Y$,*

(a) *if $i(X) = 5$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < 5$;*

(b) *if $i(X) = m+3$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < m+3$;*

(c) *if $i(X) = m+4$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < m+4$;*

(d) *if $i(X) = m+5$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < m+5$.*

*Proof.* We observe that $\beta_l(B_{\mathbb{Z}_2};\mathbb{Z}_2) = 1$ for all $l$. By Proposition 5.9, $i(X)$ is one of 2, 5, $m+3$, $m+4$ or $m+5$. We can apply these results to Proposition 5.5. If $i(X) = 5, m+3, m+4$ or $m+5$, then we get the possibilities (a), (b), (c) or (d), respectively. ∎

For the same reason, we obtain the following propositions directly.

**Proposition 5.13.** *Suppose that $\mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 \mathbb{C}P^m \times S^4$ and a path-connected, paracompact Hausdorff space $Y$. If $m = 3$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{C}P^3 \times S^4$. Then there is no $\mathbb{Z}_2$-equivariant map $X \to Y$,*

(a) *if $i(X) = 3$ and $H^k(Y/\mathbb{Z}_2) = 0$ for $k = 2$;*

(b) *if $i(X) = 5$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < 5$;*

(c) *if $i(X) = 2m+5$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < 2m+5$.*

**Proposition 5.14.** *Suppose that $\mathbb{Z}_2$ acts freely on a finitistic space $X \sim_2 \mathbb{H}P^m \times S^4$ and a path-connected, paracompact Hausdorff space $Y$. When $m \equiv 3 \pmod 4$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{H}P^m \times S^4$. If $i(X) = 5$ and $H^k(Y/\mathbb{Z}_2) = 0$ for some $2 \leqslant k < 5$, then there is no $\mathbb{Z}_2$-equivariant map $X \to Y$.*

Replacing $Y$ in the above by $S^n$, we obtain the following results.

**Corollary 5.15.** *Let $X \sim_2 \mathbb{R}P^m \times S^4$ be a finitistic space and the unit $n$-sphere $S^n$ be equipped with a free involution. If $m = 5$ or $m = 7$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{R}P^m \times S^4$. Then, there is no $\mathbb{Z}_2$-equivariant map $X \to S^n$,*

(a) *if $i(X) = 5$ and $n < 4$;*

(b) *if $i(X) = m + 3$ and $n < m + 2$;*

(c) *if $i(X) = m + 4$ and $n < m + 3$;*

(d) *if $i(X) = m + 5$ and $n < m + 4$.*

**Corollary 5.16.** *Let $X \sim_2 \mathbb{C}P^m \times S^4$ be a finitistic space and the unit $n$-sphere $S^n$ be equipped with a free involution. If $m = 3$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{C}P^3 \times S^4$. Then, there is no $\mathbb{Z}_2$-equivariant map $X \to S^n$,*

(a) *if $i(X) = 3$ and $n < 2$;*

(b) *if $i(X) = 5$ and $n < 4$;*

(c) *if $i(X) = 2m + 5$ and $n < 2m + 4$.*

**Corollary 5.17.** *Let $X \sim_2 \mathbb{H}P^m \times S^4$ be a finitistic space and the unit $n$-sphere $S^n$ be equipped with a free involution. When $m \equiv 3 \pmod 4$, assume further that the action of $\mathbb{Z}_2$ on $H^*(X;\mathbb{Z}_2)$ is trivial or $X \sim_{\mathbb{Z}} \mathbb{H}P^m \times S^4$. If $i(X) = 5$ and $n < 4$, then there is no $\mathbb{Z}_2$-equivariant map $X \to S^n$.*

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