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A reverse Ozawa–Rogers estimate

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Abstract. We provide a reverse bilinear estimate for the one-dimensional Klein–Gordon equation that complements a result of Ozawa and Rogers. The proof relies on the cosine formula for frequency vectors adapted to the Klein–Gordon equation.

1. Introduction

Consider the Klein–Gordon equation in space-time $\mathbb{R}^{1+1}$

$$\partial_t^2 u + (1-\Delta)u = 0. \tag{1.1}$$

Recall the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

In El Escorial 2012, Ozawa and Rogers presented the following estimate for (1.1).

Theorem 1.1 (Ozawa–Rogers [9]). *Suppose that* $\operatorname{supp} \widehat{f_1} \cap \operatorname{supp} \widehat{f_2} = \emptyset$. *Then*

$$\begin{aligned} &\big\| e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2 \big\|_{L^2_{t,x}(\mathbb{R}^{1+1})} \\ &\quad \le \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 (1+\xi_1^2)^{3/4} (1+\xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}. \end{aligned} \tag{1.2}$$

These bilinear estimates were first considered by Klainerman and Machedon in [6–8]. Related results for the Schrödinger equation and the wave equation can be found in Ozawa–Tsutsumi [10], Carneiro [4], and Bez–Rogers [3]. For recent developments in this direction, see Jeavons [5], Bez–Jeavons–Ozawa [2], and Beltran–Vega [1]. The proof of (1.2) relies on the estimate

$$\frac{|\xi_2 - \xi_1|}{(1+\xi_1^2)^{3/4}(1+\xi_2^2)^{3/4}} \le \left| \frac{\xi_2}{\sqrt{1+\xi_2^2}} - \frac{\xi_1}{\sqrt{1+\xi_1^2}} \right|, \tag{1.3}$$

which was proved via "an unlikely combination of five trigonometric identities".

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The aim of this note is to provide the following converse to (1.2).

Theorem 1.2. *Suppose that* $\operatorname{supp}\widehat{f_1} \cap \operatorname{supp}\widehat{f_2} = \emptyset$. *Then*

$$\begin{aligned}
&\|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_{L^2_{t,x}(\mathbb{R}^{1+1})} \\
&\quad \geq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 (1+\xi_1^2)^{1/4} (1+\xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}. \qquad (1.4)
\end{aligned}$$

Our proof of (1.4) relies on the following elementary estimate

$$\frac{|\xi_2 - \xi_1|}{(1+\xi_1^2)^{1/4}(1+\xi_2^2)^{1/4}} \geq \left| \frac{\xi_2}{\sqrt{1+\xi_2^2}} - \frac{\xi_1}{\sqrt{1+\xi_1^2}} \right|, \qquad (1.5)$$

which will be proved in the next section by the cosine formula.

Remark 1.3. An anonymous referee generously pointed out the following insightful observation: the sharpness of (1.3) and (1.5) can be verified by taking respectively

$$(\xi_1, \xi_2) = (N+1, N+2) \qquad \text{and} \qquad (\xi_1, \xi_2) = (1/N, 2/N),$$

and letting $N \to \infty$. This also explains, as in [9], the sharpness of $\frac{1}{(2\pi)^2}$ in (1.4).

2. Proof of (1.5)

Consider the following two unit vectors:

$$e_1 = \frac{(\xi_1, 1)}{\sqrt{1+\xi_1^2}} \qquad \text{and} \qquad e_2 = \frac{(\xi_2, 1)}{\sqrt{1+\xi_2^2}}.$$

Thus

$$|e_1 - e_2|^2 \geq \left| \frac{\xi_2}{\sqrt{1+\xi_2^2}} - \frac{\xi_1}{\sqrt{1+\xi_1^2}} \right|^2.$$

Set $\theta = \arccos \langle e_1, e_2 \rangle$. Applying the cosine formula, we have

$$|e_1 - e_2|^2 = 2 - 2\cos\theta$$

and

$$\begin{aligned}
|\xi_2 - \xi_1|^2 &= |(\xi_2, 1) - (\xi_1, 1)|^2 \\
&= 1 + \xi_1^2 + 1 + \xi_2^2 - 2\sqrt{1+\xi_1^2}\sqrt{1+\xi_2^2}\cos\theta.
\end{aligned}$$

Therefore, by the mean value inequality

$$\begin{aligned}|\xi_2-\xi_1|^2 &\geq (2-2\cos\theta)\sqrt{1+\xi_1^2}\sqrt{1+\xi_2^2}\\ &= |e_1-e_2|^2\sqrt{1+\xi_1^2}\sqrt{1+\xi_2^2}\\ &\geq \left|\frac{\xi_2}{\sqrt{1+\xi_2^2}}-\frac{\xi_1}{\sqrt{1+\xi_1^2}}\right|^2\sqrt{1+\xi_1^2}\sqrt{1+\xi_2^2}.\end{aligned}$$

Taking the square root, we obtain (1.5).

3. Proof of Theorem 1.2

The remaining arguments are *entirely the same* as in Ozawa–Rogers [9]. For completeness we reload their elegant proof. Introduce the symmetric functions

$$\begin{aligned}F(\xi_1,\xi_2) &= \frac{1}{2}\big(\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)+\widehat{f}_1(\xi_2)\widehat{f}_2(\xi_1)\big),\\ J(\xi_1,\xi_2) &= \left|\frac{\xi_2}{\sqrt{1+\xi_2^2}}-\frac{\xi_1}{\sqrt{1+\xi_1^2}}\right|.\end{aligned}$$

Thus we have

$$\begin{aligned}&e^{it\sqrt{1-\Delta}}f_1(x)e^{it\sqrt{1-\Delta}}f_2(x)\\ &\quad= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)e^{ix(\xi_1+\xi_2)+it\left(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2}\right)}d\xi_1 d\xi_2\\ &\quad= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}F(\xi_1,\xi_2)e^{ix(\xi_1+\xi_2)+it\left(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2}\right)}d\xi_1 d\xi_2\\ &\quad= \frac{2}{(2\pi)^2}\int_{\xi_2\geq\xi_1}F(\xi_1,\xi_2)e^{ix(\xi_1+\xi_2)+it\left(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2}\right)}d\xi_1 d\xi_2\\ &\quad= \frac{2}{(2\pi)^2}\int_{\xi_2\geq\xi_1}F(\xi_1,\xi_2)e^{ix\eta_1+it\eta_2}\frac{d\eta_1 d\eta_2}{J(\xi_1,\xi_2)}.\end{aligned}$$

In the last step we used the following change of variables:

$$\eta_1=\xi_1+\xi_2 \qquad \text{and} \qquad \eta_2=\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2}.$$

By Plancherel's theorem and reversing the change of variables, and using (1.5),

$$\begin{aligned}&\left\|e^{it\sqrt{1-\Delta}}f_1 e^{it\sqrt{1-\Delta}}f_2\right\|^2_{L^2_{t,x}(\mathbb{R}^{1+1})}\\ &\quad= \frac{4}{(2\pi)^2}\int_{\xi_2\geq\xi_1}|F(\xi_1,\xi_2)|^2\frac{d\eta_1 d\eta_2}{J(\xi_1,\xi_2)^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\xi_1 d\xi_2}{J(\xi_1, \xi_2)} \\
&\geq \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&= \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}.
\end{aligned}
$$

Since $\operatorname{supp} \widehat{f}_1 \cap \operatorname{supp} \widehat{f}_2 = \emptyset$, we have

$$
|F(\xi_1, \xi_2)|^2 = \frac{1}{4} \big(|\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 + |\widehat{f}_1(\xi_2)|^2 |\widehat{f}_2(\xi_1)|^2 \big),
$$

hence

$$
\begin{aligned}
&\frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&\quad = \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} \big(|\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 + |\widehat{f}_1(\xi_2)|^2 |\widehat{f}_2(\xi_1)|^2 \big) \\
&\qquad\qquad \cdot (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&\quad = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}.
\end{aligned}
$$

This proves Theorem 1.2.

References

[1] D. Beltran and L. Vega, Bilinear identities involving the k-plane transform and Fourier extension operators. *Proc. Roy. Soc. Edinburgh Sect. A* **150** (2020), no. 6, 3349–3377 Zbl 1459.42036 MR 4190114

[2] N. Bez, C. Jeavons, and T. Ozawa, Some sharp bilinear space-time estimates for the wave equation. *Mathematika* **62** (2016), no. 3, 719–737 Zbl 1350.35040 MR 3521350

[3] N. Bez and K. M. Rogers, A sharp Strichartz estimate for the wave equation with data in the energy space. *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 3, 805–823 Zbl 1273.35066 MR 3085092

[4] E. Carneiro, A sharp inequality for the Strichartz norm. *Int. Math. Res. Not. IMRN* (2009), no. 16, 3127–3145 Zbl 1178.35090 MR 2533799

[5] C. Jeavons, A sharp bilinear estimate for the Klein–Gordon equation in arbitrary space-time dimensions. *Differential Integral Equations* **27** (2014), no. 1-2, 137–156 Zbl 1313.35036 MR 3161599

[6] S. Klainerman and M. Machedon, Space-time estimates for null forms and the local existence theorem. *Comm. Pure Appl. Math.* **46** (1993), no. 9, 1221–1268 Zbl 0803.35095 MR 1231427

[7] S. Klainerman and M. Machedon, Remark on Strichartz-type inequalities. *Internat. Math. Res. Notices* (1996), no. 5, 201–220 Zbl 0853.35062 MR 1383755

[8] S. Klainerman and M. Machedon, On the regularity properties of a model problem related to wave maps. *Duke Math. J.* **87** (1997), no. 3, 553–589 Zbl 0878.35075 MR 1446618

[9] T. Ozawa and K. M. Rogers, A sharp bilinear estimate for the Klein–Gordon equation in $\mathbb{R}^{1+1}$. *Int. Math. Res. Not. IMRN* (2014), no. 5, 1367–1378 MR 3178601

[10] T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations. *Differential Integral Equations* **11** (1998), no. 2, 201–222 Zbl 1008.35070 MR 1741843

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