

A reverse Ozawa–Rogers estimate

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Abstract. We provide a reverse bilinear estimate for the one-dimensional Klein–Gordon equation that complements a result of Ozawa and Rogers. The proof relies on the cosine formula for frequency vectors adapted to the Klein–Gordon equation.

1. Introduction

Consider the Klein–Gordon equation in space-time \mathbb{R}^{1+1}

$$\partial_t^2 u + (1 - \Delta)u = 0. \quad (1.1)$$

Recall the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi} dx.$$

In El Escorial 2012, Ozawa and Rogers presented the following estimate for (1.1).

Theorem 1.1 (Ozawa–Rogers [9]). *Suppose that $\text{supp } \widehat{f}_1 \cap \text{supp } \widehat{f}_2 = \emptyset$. Then*

$$\begin{aligned} & \|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_{L_{t,x}^2(\mathbb{R}^{1+1})} \\ & \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 (1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}. \end{aligned} \quad (1.2)$$

These bilinear estimates were first considered by Klainerman and Machedon in [6–8]. Related results for the Schrödinger equation and the wave equation can be found in Ozawa–Tsutsumi [10], Carneiro [4], and Bez–Rogers [3]. For recent developments in this direction, see Jeavons [5], Bez–Jeavons–Ozawa [2], and Beltran–Vega [1]. The proof of (1.2) relies on the estimate

$$\frac{|\xi_2 - \xi_1|}{(1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4}} \leq \left| \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right|, \quad (1.3)$$

which was proved via “an unlikely combination of five trigonometric identities”.

The aim of this note is to provide the following converse to (1.2).

Theorem 1.2. *Suppose that $\text{supp } \widehat{f}_1 \cap \text{supp } \widehat{f}_2 = \emptyset$. Then*

$$\begin{aligned} & \|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_{L^2_{t,x}(\mathbb{R}^{1+1})} \\ & \geq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}. \end{aligned} \tag{1.4}$$

Our proof of (1.4) relies on the following elementary estimate

$$\frac{|\xi_2 - \xi_1|}{(1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4}} \geq \left| \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right|, \tag{1.5}$$

which will be proved in the next section by the cosine formula.

Remark 1.3. An anonymous referee generously pointed out the following insightful observation: the sharpness of (1.3) and (1.5) can be verified by taking respectively

$$(\xi_1, \xi_2) = (N + 1, N + 2) \quad \text{and} \quad (\xi_1, \xi_2) = (1/N, 2/N),$$

and letting $N \rightarrow \infty$. This also explains, as in [9], the sharpness of $\frac{1}{(2\pi)^2}$ in (1.4).

2. Proof of (1.5)

Consider the following two unit vectors:

$$e_1 = \frac{(\xi_1, 1)}{\sqrt{1 + \xi_1^2}} \quad \text{and} \quad e_2 = \frac{(\xi_2, 1)}{\sqrt{1 + \xi_2^2}}.$$

Thus

$$|e_1 - e_2|^2 \geq \left| \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right|^2.$$

Set $\theta = \arccos \langle e_1, e_2 \rangle$. Applying the cosine formula, we have

$$|e_1 - e_2|^2 = 2 - 2 \cos \theta$$

and

$$\begin{aligned} |\xi_2 - \xi_1|^2 &= |(\xi_2, 1) - (\xi_1, 1)|^2 \\ &= 1 + \xi_1^2 + 1 + \xi_2^2 - 2\sqrt{1 + \xi_1^2} \sqrt{1 + \xi_2^2} \cos \theta. \end{aligned}$$

Therefore, by the mean value inequality

$$\begin{aligned} |\xi_2 - \xi_1|^2 &\geq (2 - 2 \cos \theta) \sqrt{1 + \xi_1^2} \sqrt{1 + \xi_2^2} \\ &= |e_1 - e_2|^2 \sqrt{1 + \xi_1^2} \sqrt{1 + \xi_2^2} \\ &\geq \left| \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right|^2 \sqrt{1 + \xi_1^2} \sqrt{1 + \xi_2^2}. \end{aligned}$$

Taking the square root, we obtain (1.5).

3. Proof of Theorem 1.2

The remaining arguments are *entirely the same* as in Ozawa–Rogers [9]. For completeness we reload their elegant proof. Introduce the symmetric functions

$$\begin{aligned} F(\xi_1, \xi_2) &= \frac{1}{2} (\widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) + \widehat{f}_1(\xi_2) \widehat{f}_2(\xi_1)), \\ J(\xi_1, \xi_2) &= \left| \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right|. \end{aligned}$$

Thus we have

$$\begin{aligned} e^{it\sqrt{1-\Delta}} f_1(x) e^{it\sqrt{1-\Delta}} f_2(x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F(\xi_1, \xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2 \\ &= \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} F(\xi_1, \xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2 \\ &= \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} F(\xi_1, \xi_2) e^{ix\eta_1+it\eta_2} \frac{d\eta_1 d\eta_2}{J(\xi_1, \xi_2)}. \end{aligned}$$

In the last step we used the following change of variables:

$$\eta_1 = \xi_1 + \xi_2 \quad \text{and} \quad \eta_2 = \sqrt{1 + \xi_1^2} + \sqrt{1 + \xi_2^2}.$$

By Plancherel’s theorem and reversing the change of variables, and using (1.5),

$$\begin{aligned} \|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_{L^2_{t,x}(\mathbb{R}^{1+1})}^2 &= \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\eta_1 d\eta_2}{J(\xi_1, \xi_2)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\xi_1 d\xi_2}{J(\xi_1, \xi_2)} \\
&\geq \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&= \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}.
\end{aligned}$$

Since $\text{supp } \widehat{f}_1 \cap \text{supp } \widehat{f}_2 = \emptyset$, we have

$$|F(\xi_1, \xi_2)|^2 = \frac{1}{4} (|\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 + |\widehat{f}_1(\xi_2)|^2 |\widehat{f}_2(\xi_1)|^2),$$

hence

$$\begin{aligned}
&\frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&= \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} (|\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 + |\widehat{f}_1(\xi_2)|^2 |\widehat{f}_2(\xi_1)|^2) \\
&\quad \cdot (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 (1 + \xi_1^2)^{1/4} (1 + \xi_2^2)^{1/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}.
\end{aligned}$$

This proves Theorem 1.2.

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