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Regular spatial heptagons based on symmetry

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1 Introduction

Regular spatial heptagons, in the following referred to simply as *heptagons*, are understood to be 7-gons in the Euclidean space E^3, with equal lengths of sides and equal angles α between adjacent sides. The side lengths are normalized to 1, and intersecting sides as well as coinciding vertices are permitted.

For the tetrahedral bond angle $\alpha = \arccos(-\frac{1}{3}) \approx 109.5°$, heptagons have been considered for a long time in stereochemistry, with the aim of examining seven-membered rings of carbon atoms, as they appear, for instance, in cycloheptane. Most articles on this subject are based on a combination of chemical and mathematical approaches. Investigations on heptagons with the tetrahedral angle that refer only to mathematics can be found in [2, 4].

What are the investigations on heptagons with any possible angle α? There is an extensive literature – even dating back to Archimedes – about the special case of the well-known planar heptagons. However, we only know two studies on all nonplanar heptagons – Cox [1] and Kamiyama [3] – both of which are concerned with the configuration

Diese Arbeit befasst sich mit regulären räumlichen Heptagonen, d. h. mit gleichseitigen und gleichwinkligen Siebenecken im euklidischen Raum E^3. Im Vordergrund steht die Frage nach den Zusammenhangskomponenten im Sinne einer stetigen Überführbarkeit innerhalb bestimmter Teilmengen. Dabei nimmt man wesentlich Bezug auf die möglichen Symmetrietypen regulärer Heptagone, welche ausführlich dargelegt werden. Die Menge aller regulären Heptagone mit einem festem Winkel zerfällt je nach Winkelbereich in mehrere Komponenten, zu deren Charakterisierung symmetrische Repräsentanten dienen. Schliesslich zeigt sich, dass die Menge aller regulären räumlichen Heptagone zusammenhängend ist. Animationen zu dieser Arbeit und zusätzliche Informationen zu weiteren Aspekten finden sich in [7].

space. In principle, this involves the following: heptagons are flexible, i.e., they can be continuously transformed while retaining their regularity conditions. The extent to which such transformations are possible depends on the set of considered heptagons and leads to a partition into connected components. In both studies, but with different approaches, the topological structure of the connected components is described for the sets of heptagons with a fixed angle α.

The present article provides an overview of all heptagons, where the focus is put on symmetry. We examine several subsets of heptagons and determine the associated connected components. After presenting some preliminary properties, heptagons of the possible kinds of symmetry are discussed in detail. Next, we consider the sets of heptagons with a fixed angle α, first in the specific case of $\alpha = 60°$ and then for any other α. As a result, we also obtain a characterization of the connected components of these sets. This is essentially based on symmetric heptagons, in contrast to the two studies mentioned above, in which symmetry is not considered at all. Finally, a combination of derived statements reveals that the set of all heptagons is connected.

The results of this article, which are not based on theorems, are obtained from numerical approximations and, thus, are not formally proven. To reproduce computations, it needs a computer algebra system. Animations to outcomes of this paper and some additional properties of heptagons are attached to a website [7] (originally created in connection with [5]).

We use notations for a heptagon with consecutive vertices $v_1, \ldots, v_7$, as shown in Figure 1. The common length of the seven diagonals connecting a vertex with the next but one is denoted by q, and we have

$$q = 2\sin\frac{\alpha}{2}. \tag{1}$$

The other seven diagonals are said to be the *main* diagonals (red), and in general, they differ in length.

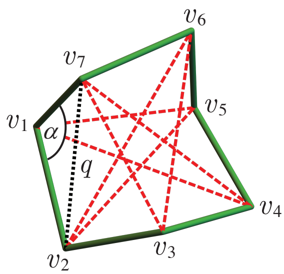
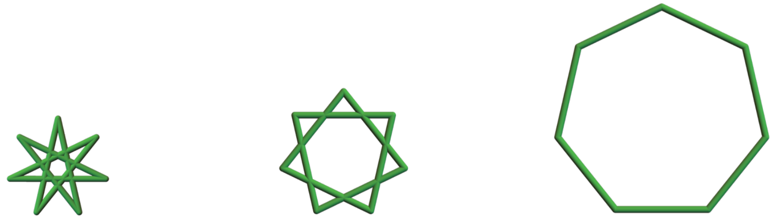


Figure 1. Notations for a heptagon.

According to Figure 2, we denote the three well-known planar heptagons by $star_1$, $star_2$, and $star_3$, although the last is convex and not really star-shaped. The corresponding angles α_i and, by (1), the assigned diagonals q_i are given as follows:

$$\alpha_i = (2i-1)\frac{180°}{7}, \quad q_i = 2\sin\frac{\alpha_i}{2} \quad \text{with } i \in \{1,2,3\}. \tag{2}$$



$star_1$ ($\alpha_1 \approx 25.7°$, $q_1 \approx 0.45$) $star_2$ ($\alpha_2 \approx 77.1°$, $q_2 \approx 1.25$) $star_3$ ($\alpha_3 \approx 128.6°$, $q_3 \approx 1.80$)

Figure 2. The three planar heptagons.

What is the degree of freedom of heptagons? For the coordinates of 7 freely selectable vertices, we have 21 degrees. Congruence invariance reduces this number by 6, normalized side lengths by 7, and equal diagonal lengths q by 6. Thus, 2 degrees remain. Taking into account an additional constraint, such as a symmetry or a fixed angle α, the degree of freedom becomes 1. Thus, one parameter is sufficient to describe the sets of incongruent heptagons in the following sections.

We now give two general properties of heptagons.

Theorem 1. *A heptagon with angle α exists if and only if $\alpha \in [\alpha_1, \alpha_3]$.*

Proof. That the condition $\alpha \in [\alpha_1, \alpha_3]$ is necessary for the existence of a heptagon results from the following property about n-gons in space [6]: for an odd n, the sum of the angles between adjacent sides is at least 180° and at most $(n-2)180°$. That the condition is sufficient follows from Theorem 3 below. ∎

Theorem 2. *A heptagon is asymmetric, plane-symmetric, or line-symmetric.*

Proof. Clearly, a heptagon symmetry is ring-preserving, which means that it must preserve the sequence of the vertices. The symmetry group of the highest order is achieved when all main diagonals are equal. Then it is isomorphic to the dihedral group D_7, and the induced vertex permutations are generated by cycle $\lambda = (v_1 v_2 v_3 v_4 v_5 v_6 v_7)$ and an involution μ. As each λ^k $(1 \leq k \leq 6)$ is a cycle of length seven, it can be induced only by the rotation of a planar heptagon. Thus, we have the symmetry group of each of the three stars, which obviously are both plane- and line-symmetric. The nonplanar heptagons, therefore, are asymmetric, or their symmetry group is isomorphic to a group generated by μ. Since μ is an involution, it can be induced only by a plane, line, or point reflection. The last, however, can be excluded. In fact, an odd number of vertices would coincide with the symmetry center, implying that angle α of at least one of them would be mapped onto itself, and thus $\alpha = 180°$. ∎

Next, we capture already mentioned concepts that in the following we subsume under *connectedness*: a *continuous transformation* of a heptagon is given by continuously varying the lengths of the diagonals (or the underlying vertex coordinates) while retaining the regularity conditions. A set of heptagons is called *connected* if, within the set, for any two heptagons h and h', there is a continuous transformation from h to h'. We write $h \leftrightarrow h'$ for the equivalence relation thus defined. Consequently, a set of heptagons is subdivided into classes of maximal connected subsets, which are said to be the *connected components*.

Lemma. *The set $\tilde{S}$ of all heptagons, which are congruent to those of a connected set S with at least one plane-symmetric heptagon, is also connected.*

Proof. Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two heptagons from $\tilde{S}$. Further, consider heptagons h_1 and h_2 from S congruent to $\tilde{h}_1$ and $\tilde{h}_2$, respectively. We show that there exists a continuous transformation $\tilde{h}_1 \leftrightarrow \tilde{h}_2$ within $\tilde{S}$, composed as follows: $\tilde{h}_1 \leftrightarrow h_1 \leftrightarrow h_2 \leftrightarrow \tilde{h}_2$. The transformation $h_1 \leftrightarrow h_2$ can be realized within S. Therefore, it suffices to indicate $h_1 \leftrightarrow \tilde{h}_1$, as this implies the existence of the reversed $\tilde{h}_1 \leftrightarrow h_1$ and of $h_2 \leftrightarrow \tilde{h}_2$.

If h_1 and $\tilde{h}_1$ are properly congruent, $h_1 \leftrightarrow \tilde{h}_1$ can be implemented with a motion, which is a continuous transformation within $\tilde{S}$. If h_1 and $\tilde{h}_1$ are improperly congruent, we consider first a continuous transformation $h_1 \leftrightarrow pl \leftrightarrow h_1^*$, where pl is a plane-symmetric heptagon from S and h_1^* a mirror image of h_1. By assumption, $h_1 \leftrightarrow pl$ exists within S, and by reflecting each heptagon of this transformation at the symmetry plane of pl, we obtain $pl \leftrightarrow h_1^*$ within $\tilde{S}$. Then, for $h_1^* \leftrightarrow \tilde{h}_1$, a motion can be applied. ∎

2 Plane-symmetric heptagons

Plane-symmetric heptagons allow for an exact representation. Without loss of generality, we can assume that vertex v_1 lies on the symmetry plane, which implies three pairs of equal diagonal lengths: $\overline{v_1v_4} = \overline{v_1v_5}$, $\overline{v_2v_5} = \overline{v_4v_7}$, and $\overline{v_2v_6} = \overline{v_3v_7}$.

Theorem 3. *Define*

$$\mathrm{Q}^+ = [q_1, q_3], \quad \mathrm{Q}^- = [-q_2, -1] \quad \textit{with } q_1,\ q_2, \textit{ and } q_3 \textit{ from } (2),$$

and for given p, let

$$a = -p^3 + p^2 + 2p - 1, \quad b = \sqrt{p^2 - p + 1}, \quad c = \sqrt{-p^2 + p + 3}.$$

For each $p \in \mathrm{Q}^+ \cup \mathrm{Q}^-$ and $w \in \{1, -1\}$, the following vertices form a plane-symmetric heptagon, and (up to congruence) there are no other heptagons that are plane-symmetric:

$$\begin{aligned}
v_1 &= \Big(0, \frac{\sqrt{a(-p^2 + 2p + 1)}}{bc}, \frac{3p^2 + p - 3}{2bc}\Big),\\
v_{2,7} &= \Big(\pm\frac{p}{2}, 0, \frac{b(p + 1)}{2c}\Big),\\
v_{3,6} &= \Big(\pm\frac{p^2 - 1}{2}, 0, \frac{b(p^2 - 2)}{2c}\Big),\\
v_{4,5} &= \Big(\pm\frac{1}{2}, -w\frac{\sqrt{a(p + 1)}}{c}, 0\Big).
\end{aligned}$$

The diagonal length q is $|p|$, and the lengths of the main diagonals are given as follows:

$$\overline{v_1v_4} = \frac{1}{c}\sqrt{\operatorname{sgn}(p)w\frac{2a\sqrt{a+p+2}}{b} + ap + (p+1)^2},$$
$$\overline{v_2v_5} = \sqrt{p^2+p}, \quad \overline{v_2v_6} = \sqrt{p^3-p+1}, \quad \overline{v_3v_6} = |p^2-1|.$$

Proof. The plane-symmetric heptagons are placed in an xyz-coordinate system such that they are symmetric with respect to the yz-plane. Then vertex v_1 lies on this plane, and without loss of generality, we can choose vertices v_2, v_3, v_6, and v_7 (forming an isosceles trapezoid) on the xz-plane, and v_4 and v_5 on the xy-plane. Thus, we use the following ansatz, which already includes $\overline{v_4v_5} = 1$:

$$v_1 = (0, f, g), \quad v_{2,7} = \left(\pm\frac{p}{2}, 0, h\right), \quad v_{3,6} = (\pm k, 0, l), \quad v_{4,5} = \left(\pm\frac{1}{2}, m, 0\right).$$

From $\overline{v_2v_7} = q$, it immediately follows that $p = \pm q$. To obtain a compact representation, it is appropriate to use parameter p instead of q. Then the remaining regularity conditions $\overline{v_1v_2} = \overline{v_2v_3} = \overline{v_3v_4} = 1$ and $\overline{v_1v_3} = \overline{v_2v_4} = \overline{v_3v_5} = |p|$ generate a system of equations with the unknowns f, g, h, k, l, and m. Calculation with a computer algebra system results in eight solutions, consisting of four each, which lead to congruent heptagons related by reflections on the xy-plane, the xz-plane, and the x-axis. Thus, to describe all incongruent heptagons, it suffices to consider two solutions. We take those where, for the arbitrarily chosen reference case $q = 1.2$, the coordinates of v_1 become non-negative. It turns out that these two solutions differ only in the sign of m, which we specify with $w \in \{1, -1\}$. Finally, the auxiliary variables a, b, and c simplify terms.

Considering that q_1, $-q_2$, and q_3 are the zeros of a, it can be shown that $\mathrm{Q}^+ \cup \mathrm{Q}^-$ is the largest range of p such that all occurring roots are real (the root appearing in m is decisive).

The diagonal lengths are obtained from the vertices; in particular, it holds $q = |p|$. ∎

The results of Theorem 3 are illustrated in Figure 3. Taking into account that $q = |p|$, the two closed curves are obtained from the diagonal pairs $(q, \overline{v_1v_4})$ of all plane-symmetric heptagons. For some selected values of q, we show the corresponding heptagons $pl_1, \ldots, pl_{11}$, which are considered from different viewpoints to get the optimal depth effects.

The closed curve containing $star_1$ and $star_3$ (blue) results from heptagons with $p \in Q^+$, and the curve containing $star_2$ (red) from those with $p \in Q^-$; we speak of *large* and *small* heptagons, respectively. Both intersection points of the two curves represent a large and a small heptagon with equal diagonal lengths $\overline{v_1v_4}$ but different $\overline{v_2v_5}$.

Within both closed curves, the solid segments are given by heptagons with $w = 1$, and the dashed ones by those with $w = -1$; they represent what we call *upper* and *lower* heptagons, respectively. The diagonal length $\overline{v_1v_4}$ of an upper heptagon is always larger than or equal to that of a lower heptagon (equal in the case of the stars and pl_4, which are of both types). Note that, for the appropriate conformers of cycloheptane ($q = \frac{2}{3}\sqrt{6}$), an upper heptagon is called a chair and a lower a boat.

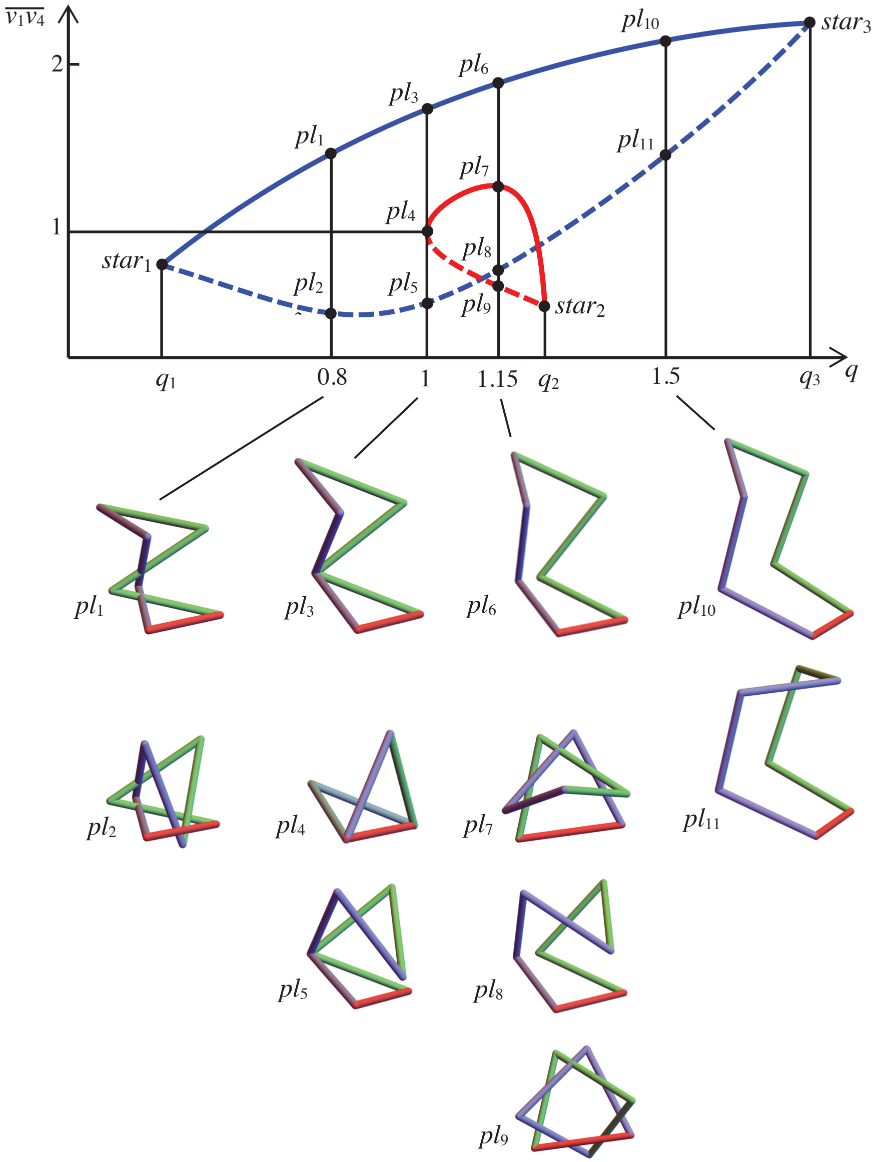


Figure 3. Plane-symmetric heptagons represented by diagonal pairs $(q, \overline{v_1 v_4})$, with examples of some selected values of q.

Remarks. (a) Of course, the described heptagons are incongruent, except in the boundary cases with $p \in \{q_1, -1, -q_2, q_3\}$, where the values $w \in \{1, -1\}$ give the same heptagon.

(b) For a fixed q, we have the following number of incongruent plane-symmetric heptagons:

$$\begin{aligned} &1 \quad \text{for } q = q_1 \text{ or } q = q_3; \\ &2 \quad \text{for } q \in]q_1, 1[\text{ or } q \in]q_2, q_3[; \\ &3 \quad \text{for } q = 1 \text{ or } q = q_2; \\ &4 \quad \text{for } q \in]1, q_2[. \end{aligned} \tag{3}$$

(c) The plane-symmetric heptagons with $q = 1$ have double vertices, namely one in pl_3 and pl_5 and three in pl_4. Furthermore, pl_4 shows an additional plane and line symmetry, but both of which, however, are not ring-preserving.

(d) In a nonplanar plane-symmetric heptagon with $q \neq 1$, there is one intersection point of the sides if the heptagon is small, and two (in one special case even four) if $q < 1$.

The vertex coordinates of the heptagons from Theorem 3 are continuous in p, and for a fixed q, large and small heptagons differ in at least one of the diagonal lengths $\overline{v_1 v_4}$ and $\overline{v_2 v_5}$. From this and the lemma, we obtain the following.

Connectedness 1. *The set of all plane-symmetric heptagons has two connected components, one containing large heptagons and the other small heptagons.*

3 Line-symmetric heptagons

In searching for the vertex coordinates of all line-symmetric heptagons, we obtain a system of equations that we assume no longer allows solutions with radicals. Therefore, we present the results based on numerical approximations; v_1 is presumed to be on the symmetry axis.

The results are shown in Figure 4 analogically to the plane-symmetric case. The pairs $(q, \overline{v_1 v_4})$, each uniquely representing a line-symmetric heptagon, yield a single closed curve. For the same values of q as in Figure 3, the corresponding line-symmetric heptagons $ln_1, \dots, ln_{11}$ are presented. Also analogously, we make the following definition: the segment of the curve between ln_4 and ln_5 with $star_1$ and $star_3$ (blue) stands for *large* heptagons and the remaining segment with $star_2$ (red) for *small* heptagons, the solid segments for *upper* heptagons and the dashed segments for *lower* heptagons.

Remarks. (a) From Figure 4, it follows that, for a fixed q, the number of incongruent line-symmetric heptagons is the same as in the plane-symmetric case (see (3)).

(b) The heptagons ln_3, ln_4, and ln_5 with $q = 1$ have two double vertices, and ln_4 shows two plane symmetries, which are not ring-preserving.

The continuity of the curve in Figure 4, the plane symmetry of the stars, and the lemma imply the following.

Connectedness 2. *The set of all line-symmetric heptagons is connected.*

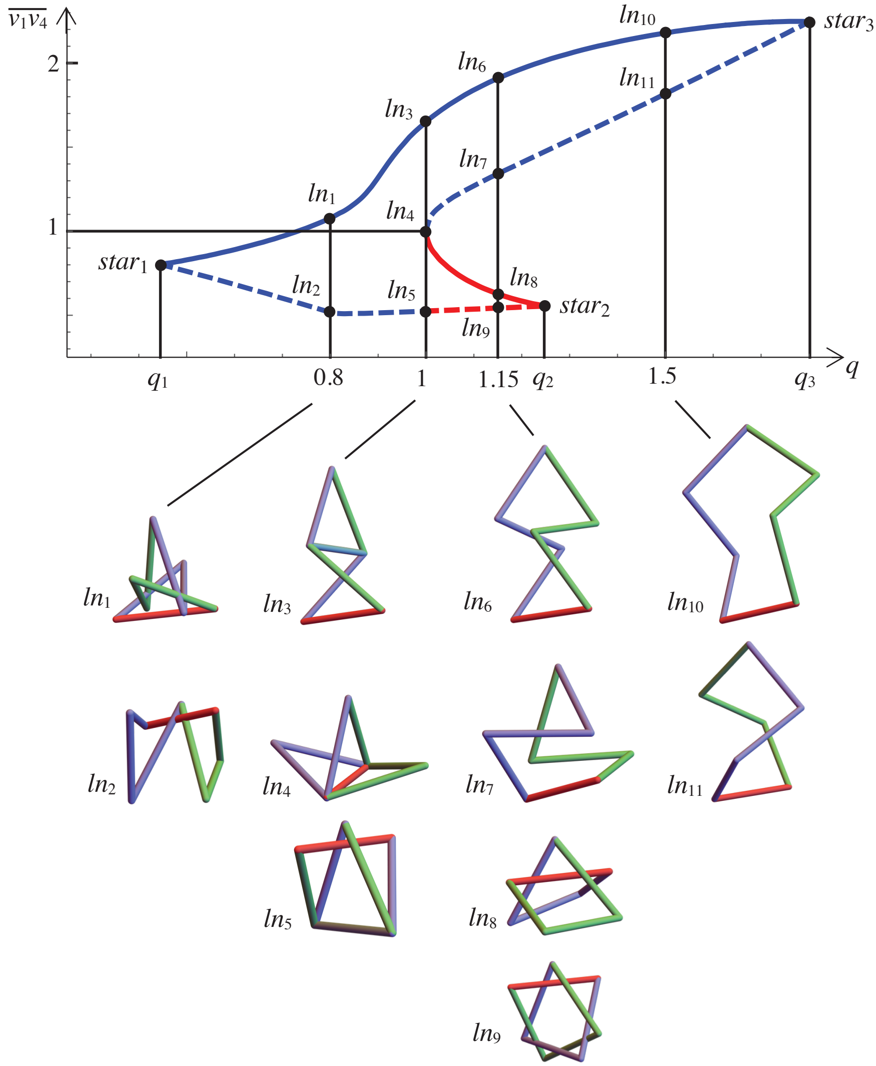


Figure 4. Line-symmetric heptagons represented by diagonal pairs $(q, \overline{v_1 v_4})$, with examples of some selected values of q.

4 Heptagons with $q = 1$

First, consider heptagons with double vertices. As certain diagonals of length q coincide with sides, it follows that $q = 1$. Heptagons with double vertices can be specified with an exact representation.

Theorem 4. *Assume that* $\varphi_0 = \frac{1}{2} \arccos \frac{1}{3}$, *and for given* φ, *let*

$$a = \cos\varphi, \quad b = \sin\varphi.$$

For each $\varphi \in [-2\varphi_0, \pi - \varphi_0]$, *the following vertices form a heptagon with at least one double vertex, and (up to congruence) there are no other heptagons with this property:*

$$v_1 = (0,0,0), \quad v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad v_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right),$$
$$v_4 = (1,0,0), \quad v_5 = v_1, \quad v_6 = \left(\frac{1}{2}, -\frac{\sqrt{3}a}{2}, \frac{\sqrt{3}b}{2}\right),$$
$$v_7' = \left(\frac{3a-1}{5-3a}, \frac{2\sqrt{3}(1-a)}{5-3a}, \frac{2\sqrt{3}b}{5-3a}\right) \quad \text{if } \varphi \in [-2\varphi_0, 2\varphi_0],$$
$$\text{or} \quad v_7'' = v_4 \quad \text{if } \varphi \in [-\varphi_0, \pi - \varphi_0].$$

Proof. Without loss of generality, we can choose as a double vertex $v_5 = v_1$. Because $q = 1$, the vertices of the tetragon $t = v_1v_2v_3v_4$ form a regular tetrahedron, which is placed in an xyz-coordinate system, as indicated. From $\overline{v_6v_1} = \overline{v_6v_4} = 1$, it follows that v_6 lies on a circle parallel to the yz-plane with center $(\frac{1}{2}, 0, 0)$ and radius $\sqrt{3}/2$, and we use the angle parameter φ (by radian) to obtain the coordinates of v_6. Finally, the system of equations, resulting from the remaining regularity conditions $\overline{v_7v_1} = \overline{v_7v_2} = \overline{v_7v_6} = 1$, yields the two solutions v_7' and v_7''.

We show that the indicated intervals for φ are sufficient to describe all incongruent heptagons with double vertices. This is done by verifying that the complementary sets with respect to a full circle interval of length 2π give no further incongruent heptagons.

For heptagons with v_7', consider the symmetry plane P of the tetragon t passing through the double vertex v_1. The boundaries of the interval $[-2\varphi_0, 2\varphi_0]$ yield plane-symmetric heptagons, namely pl_3 for $\varphi = -2\varphi_0$ and pl_5 for $\varphi = 2\varphi_0$ (see Section 2), both with P as the symmetry plane. Since, for each φ, there exists exactly one heptagon with v_7', the extension of φ beyond these interval boundaries must lead to mirrored heptagons with respect to P.

The situation is similar for heptagons with v_7''. Let L be the symmetry axis of the tetragon t passing through the midpoint of the double side v_4v_5. Here, the boundaries of the interval $[-\varphi_0, \pi - \varphi_0]$ result in line-symmetric heptagons, which are ln_3 for $\varphi = -\varphi_0$ and ln_5 for $\varphi = \pi - \varphi_0$ (see Section 3), both with L as the symmetry axis. Since each φ uniquely determines a heptagon with v_7'', the extension to the complementary interval gives mirrored heptagons with respect to L. ∎

For $\varphi = 0$, it holds that $v_7' = v_7''$. Thus, we have a linkage between heptagons with v_7' and v_7''. Figure 5 shows the corresponding (asymmetric) heptagon, which is characterized by the fact that four points (v_1, v_2, v_4, and v_6) form a rhombus. We speak of a *linkage heptagon* and denote it by *lk*.

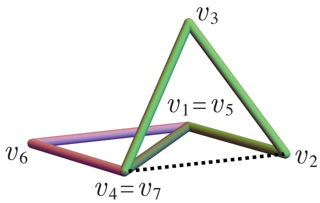


Figure 5. Linkage heptagon *lk*.

Remarks. (a) Apart from *lk*, all other heptagons of Theorem 4 are incongruent. This is because two such heptagons with coinciding diagonals always coincide in v_6 and in v_7.

(b) There exist two further symmetric heptagons with $q = 1$ (see Sections 2 and 3), which are given by Theorem 4 as follows: ln_4 with v_7' for $\varphi = \pi - 4\varphi_0$ and pl_4 with v_7'' for $\varphi = \pi - 2\varphi_0$. Note that ln_4 is the only heptagon with two double vertices but without a double side, whereas pl_4 is the only one with two double sides.

Now, let us turn to all heptagons with $q = 1$. In [1], it is said that these heptagons must have at least one double vertex, a statement that is based on numerical approximations, and we confirmed it with our own investigations; however, a formal proof is still pending. Therefore, there is an interesting unsolved problem that we highlight.

Conjecture. *Heptagons with $q = 1$ always have at least one double vertex.*

Provided that this conjecture is true, Theorem 4 includes (up to congruence) all heptagons with $q = 1$. Then the continuity of the vertex coordinates, the linkage heptagon *lk*, and the lemma imply the following.

Connectedness 3. *The set of all heptagons with $q = 1$ is connected.*

We add that continuous transformations of heptagons with $q = 1$ lead to two other branching possibilities besides *lk*, which are given by ln_4 (possible switch to new double vertex) and pl_4 (possible switch to new double side). Taking into account all successively occurring branches, a complex network – whose structure is presented in the appendix of [1] – emerges.

5 Heptagons with a fixed $q \neq 1$

Again, we are assuming that the solutions of a system of equations for the vertex coordinates of heptagons with a fixed q cannot be expressed in terms with radicals. Once more, it is necessary to resort to numerical approximations, and together with the lemma, we obtain the following.

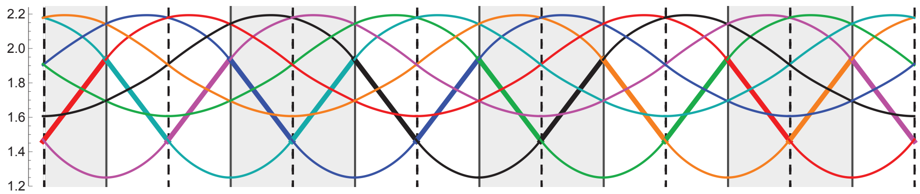
Connectedness 4. *Each connected component of the set of all nonplanar heptagons with a fixed $q \neq 1$ contains (up to congruence) exactly one plane- and one line-symmetric heptagon given as follows:* (i) *both large or both small,* (ii) *one upper and one lower for $q \in \,]q_1, 1[$ and both upper or both lower for $q \in \,]1, q_3[$.*

Remark. Look at the examples in Figures 3 and 4. Each of the following pairs of symmetric heptagons characterizes a connected component:

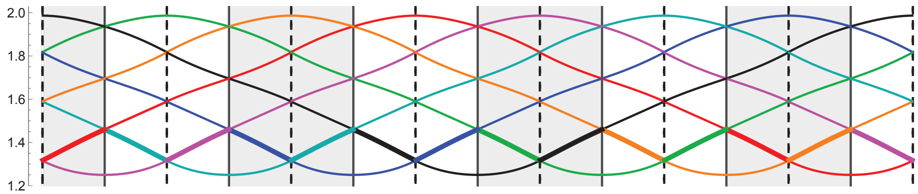
$$(pl_1, ln_2),\ (pl_2, ln_1) \qquad \text{for } q = 0.8;$$
$$(pl_6, ln_6),\ (pl_8, ln_7),\ (pl_7, ln_8),\ (pl_9, ln_9) \qquad \text{for } q = 1.15;$$
$$(pl_{10}, ln_{10}),\ (pl_{11}, ln_{11}) \qquad \text{for } q = 1.5.$$

How can heptagons with a fixed $q \neq 1$ be generated? Basically, a q-preserving continuous transformation is needed between the two characterizing symmetric heptagons of the connected component under consideration. This will now be explained in more detail for the two connected components of $q = 1.5$ with Figure 6.

In both cases, consider first the area on the very left (shaded). The curves restricted in it show the varying lengths of the seven main diagonals during a continuous transformation, where one diagonal (bold red line segment) is the chosen parameter, and thus the variable of the horizontal coordinate axis. The points of the curves on an imagined vertical line give the diagonal lengths of a single heptagon, being line-symmetric at the left and plane-symmetric at the right border of the area (dashed and solid lines, respectively). The heptagons of this first area represent (up to congruence) the connected component.



Component characterized by (pl_{10}, ln_{10}).



Component characterized by (pl_{11}, ln_{11}).

Figure 6. Transformation within the connected components of $q = 1.5$.

This transformation process can be extended as follows: the farther areas are successive mirror images of the previous one. This leads to a continued continuous transformation, where the thought horizontal coordinate axis can be interpreted as the time axis of an associated animation so that the entire diagram shows time-dependent diagonal lengths. The first 14 vertical lines alternatingly represent the line- and plane-symmetric heptagons, where in both cases each of the seven vertices comes to lie once on the symmetry element.

At the very right of the diagram, each of the main diagonals becomes the length from that of the starting heptagon at the very left. By passing a plane-symmetric heptagon, the orientation changes, i.e., from two asymmetric heptagons that are mirror-inverted to each other, one always appears in a shaded area and the other in a white area. This implies that a transformation run from the left to the right changes the orientation, and it therefore needs a second run to get the original orientation. In contrast to the case of $q = 1$, this transformation process allows no branching possibilities.

We conclude by considering all heptagons. Due to Connectedness 4, each asymmetric heptagon with $q \neq 1$ is connected to a plane-symmetric and to a line-symmetric heptagon with the same q. According to Sections 2 and 3, each symmetric heptagon is connected to one with $q = 1$, and together with Connectedness 3, it results in the following.

Connectedness 5. *The set of all heptagons is connected.*

A subset of all heptagons with a fixed q, however, consists of the following number of connected components (cf. (3)):

$$\begin{array}{ll} 1 & \text{for } q = q_1,\ q = 1, \text{ or } q = q_3; \\ 2 & \text{for } q \in \,]q_1, 1[\, \text{ or } q \in \,]q_2, q_3[; \\ 3 & \text{for } q = q_2; \\ 4 & \text{for } q \in \,]1, q_2[.^{1} \end{array}$$

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[1] In [3], this number is given by 8, which we cannot comprehend.