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Short note

Construction of a quadrilateral from its vertex angles and the diagonal angle

Norbert Hungerbühler and Juan Läuchli

1 Introduction

In a recent article [3], the author challenged the readers by posing the task of constructing a convex quadrilateral from its four vertex angles and the angle between its diagonals: an unsolved problem. Notice that the given set of angles determines the quadrilateral uniquely up to similarity (see [3, Theorem 1]). So, by setting one side to unit length, the quadrilateral is fixed. In the present short note, we describe a construction which solves the task by ruler and compass.

If the quadrilateral is a trapezoid or a parallelogram, the construction is elementary (see below). Therefore, we exclude these trivial cases in the sequel. Before we present the idea of the general construction, it is useful to choose a convenient orientation of the quadrilateral. We fix the side AB and the four vertex angles $\alpha, \beta, \gamma, \delta$, with $\alpha + \beta + \gamma + \delta = 2\pi$ (see Figure 1). We may assume, without loss of generality, that $\alpha + \beta < \pi$ and that $\alpha + \delta < \pi$. If one continues with the mirrored quadrilateral if necessary, one can furthermore assume that the diagonal angle ε over the side AB is less than or equal to $\frac{\pi}{2}$. This way we arrive at the standard situation in Figure 1. Notice that a solution only exists if $\pi - (\alpha + \beta) < \varepsilon < \alpha + \delta$.

The idea of the construction is now explained in Figure 1. The intersection S of the diagonals lies on the blue circular arc over AB which corresponds to the diagonal angle ε. In addition, by moving the side CD parallel and thus preserving all vertex angles, the point S moves on the red locus. If the quadrilateral is a trapezoid or a parallelogram with AB parallel to CD, then the red locus is the straight line joining the midpoints of AB and CD and the construction is indeed elementary. We will see in Section 2 that, in the general case, the red locus is a hyperbola. Hence S is the intersection of the blue arc and the red hyperbola. In general, the intersection of a circle and a conic corresponds to a quartic equation. However, since our two curves have two known intersections (the points A and B), the problem reduces to a quadratic equation. This is, morally speaking, the reason why the solution can be constructed by ruler and compass.

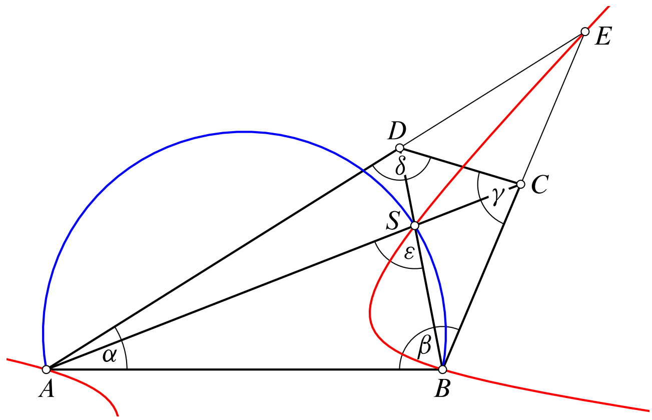

Figure 1. Idea of the construction.

2 Locus of the intersection of the diagonals

Let us denote the angles in the vertices A, B, C, D of the quadrilateral by $\alpha, \beta, \gamma, \delta$ and assume that the length of AB is 1 unit. Let S be the intersection of the diagonals, and $\varepsilon = \sphericalangle ASB$. We are interested in the locus of S when we fix the position of A and B and the size of the angles $\alpha, \beta, \gamma, \delta$ while moving the points C, D. Fortunately, a result from [1] settles this question.

Theorem ([1, Theorem 3.8]). *Suppose we are given a triangle ABE and a point Z. For an arbitrary line passing through Z, let C and D be its intersection points with BE and AE. Then the locus of the intersections of the lines AC and BD is a conic passing through A, B and E and tangent to the lines AZ and BZ.*

Proof. Use a projective transformation that maps ABE to a right isosceles triangle with $AE = BE$ and Z to infinity in the direction perpendicular to AB. It is then easy to see that the locus of the intersections of the lines AC and BD is the circle of Thales over AB. ■

In our setting of the quadrilateral $ABCD$, we apply this theorem with Z being the point at infinity in the direction of CD. We obtain the following.

Corollary. *Let ABE be the triangle with length $AB = 1$, $\sphericalangle BAE = \alpha$ and $\sphericalangle EBA = \beta$. Let the points C and D vary along the sides BE and AE of the triangle ABE in such a way that $\sphericalangle DCB = \gamma$. Then the locus of the intersection S of the diagonals AC and BD is a hyperbola through the points A, B, E with tangents in A and B which are parallel to CD.*

Our next goal is to construct the asymptotes and the axes of the hyperbola. We know a point E and two tangents with points of tangency A and B of this hyperbola. Unfortunately, the construction of a conic given by three points and two tangents which is proposed in [4] does not work in the special case where two of the points are incident with the given tangents. So we propose the following solution.

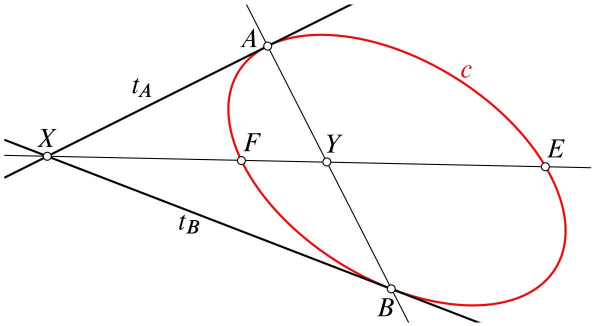

Figure 2. Harmonic points.

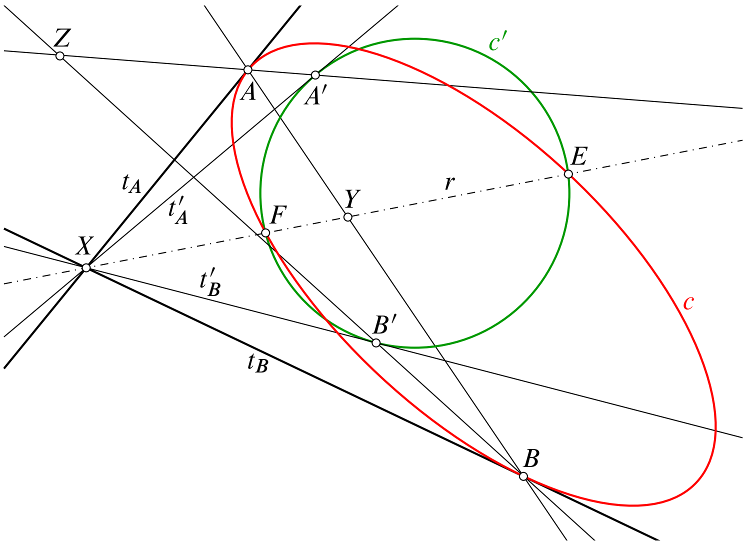

Figure 3. Construction of a perspectivity mapping the circle c' to the conic c.

Construction 1 (Construction of a conic c given by a point E and two tangents t_A, t_B with points of tangency A, B). We consider the situation in Figure 2. Let X denote the intersection of the two tangents and Y the intersection of XE and AB. Observe that X, F, Y, E are harmonic points (see, e.g., [2]). Hence, a further point F of the conic can be constructed.

With the additional information of the position of the point F, we can now find a perspectivity with center Z and axis r, and a circle c' such that c is the image of c' under this perspectivity. The construction is illustrated in Figure 3.

We choose r as the line XE and c' as an arbitrary circle passing through E and F. Then the preimages of t_A and t_B under the perspectivity we want to construct are the tangents t'_A and t'_B from X to c'. Their contact points with c', A' and B', are the preimages of A and B. Hence, the center Z of the perspectivity is the intersection of AA' and BB'. This concludes Construction 1.

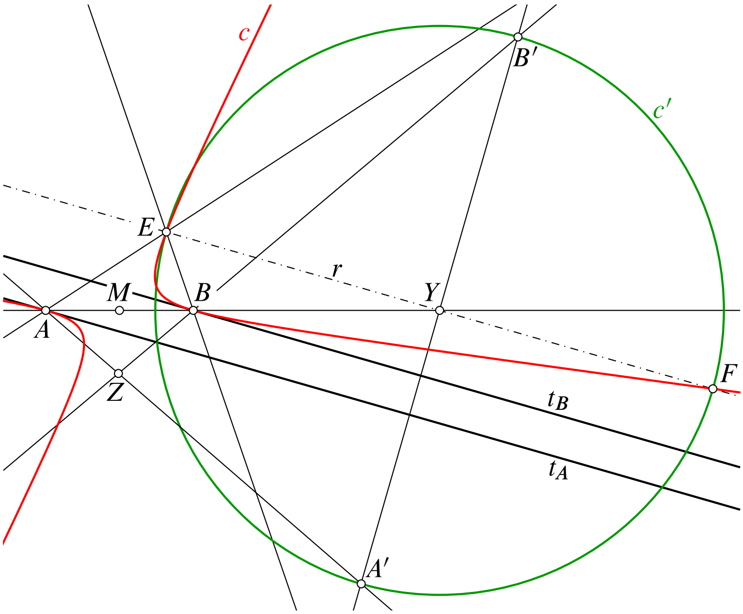

Figure 4. Special case of Construction 1 for parallel tangents t_A and t_B.

We can now apply Construction 1 in our special case where the tangents t_A and t_B are parallel, which means that X is the point at infinity in the direction of these tangents.

Construction 2 (Construction of a conic c given by a point E and two parallel tangents t_A, t_B with points of tangency A, B). The construction is carried out in Figure 4. Choose the axis r as the parallel to t_A, t_B through E. Further, Y is the intersection of r and the line AB. It is convenient to choose c' as the circle with center Y through E since then the fourth harmonic point F is just the second intersection of c' with r. The points A' and B' are obtained as intersections of c' with the perpendicular to r through Y. Again, the center Z of the perspectivity is the intersection of AA' and BB'. Observe that the midpoint M of the line segment AB must be the center of the conic c and that c is a hyperbola in our situation. This concludes Construction 2.

Once we have identified a perspectivity which maps a circle c' to the hyperbola c, it is standard to construct its asymptotes, axes and vertices.

Construction 3 (Construction of asymptotes, axes and vertices). For this step, we only retain the necessary elements from Figure 4. Since we already have the center M of the hyperbola, the construction is particularly simple: see Figure 5. First recall the following. Let l' be an arbitrary line and l its image under the perspectivity. If l'' is the parallel to l through the center Z, then the intersection of l' and l'' is a point on the vanishing line q of the perspectivity: q is parallel to the axis r of the perspectivity and the points on q are mapped to the ideal line. Hence, the intersection I of the line $A'B'$ and the parallel to AB through Z is a point of q, and q itself is the parallel to r through I. So the intersection

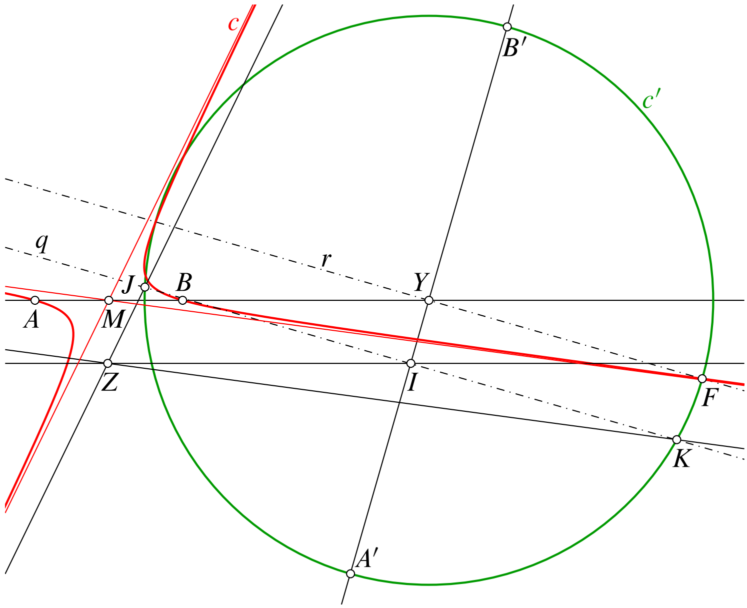

Figure 5. Construction of asymptotes.

points J, K of q and c' are sent to infinity. Therefore, ZJ and ZK are the directions of the asymptotes, and moving these lines parallel through M yields the asymptotes.

The axes of the hyperbola are the angle bisectors of the asymptotes. The vertices V, W are obtained as intersections of the hyperbola and the main axis ℓ. We can construct these intersections easily as follows. Intersect the preimage ℓ' of ℓ with c', giving the intersection points V', W'. Then the images of V', W' are the vertices V, W of c. See [4] for a more general way to construct the vertices which also works if the center of the hyperbola is not known a priori.

3 Intersection of the circular arc and the hyperbola

The idea is now to consider the hyperbola c as the projection of a circle c' in space from a certain center Z as shown in Figure 6. To this end, let m be the tangent of the hyperbola c in the vertex V. The line m intersects the asymptotes in two points G, H of distance $2r$. Now consider the plane Σ' through m which is orthogonal to the plane Σ which contains the hyperbola. We choose c' to be the circle in Σ' which touches m in V and whose center lies at a distance r above V. The center Z of the projection lies at distance r above M.

Now, we consider the situation in Σ' where a remarkable configurations emerges (see Figure 7). We project the points A, B, E in Σ from center Z to Σ'. Notice that this construction can be carried out by ruler and compass on an extra sheet of paper. The vanishing line v is the parallel to m through the center of c', and $A'B'$ is perpendicular to v. The tangents t'_A and t'_B of c' in A' and B' meet in the pole P of $A'B'$ on v. Suppose now

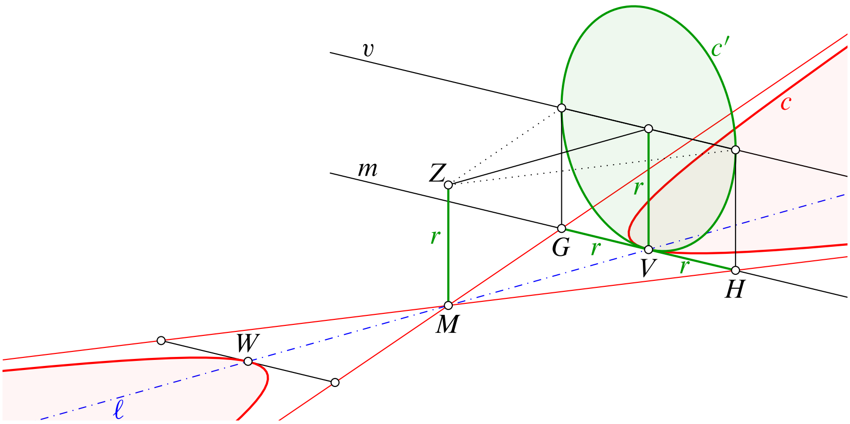

Figure 6. The red hyperbola c as projection with center Z of a green circle c' in space.

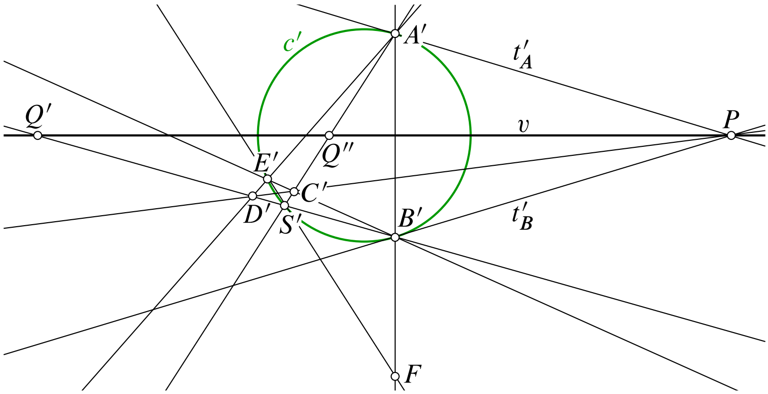

Figure 7. The situation in the plane Σ'.

that the points C and D move parallel on BE and AE, and hence S moves along c. Consider the corresponding projections C', D' and S' in Σ'. In the complete quadrilateral $A'B'S'E'D'F$, we notice that $C'D'$ is the polar line of F (see, e.g., [2]). And since F lies on the polar line of P, the line $C'D'$ passes through P by de La Hire's Theorem. Finally, notice that Q' and Q'' are conjugate points with respect to c'. This implies, in particular, that all circles passing through Q' and Q'' intersect c' orthogonally (see [2]).

It follows from the construction that the lines ZQ' and SB are parallel and that ZQ'' and SA are parallel. To see this, notice that the points B, B', S, S', Q', Z are coplanar and that A, A', S, S', Q'', Z are coplanar. In other words, the lines SA and SB and the lines ZQ'' and ZQ' meet under the same angle ε, and we have $\sphericalangle Q''ZQ' = \sphericalangle ASB$. So the last brick in our construction is the following.

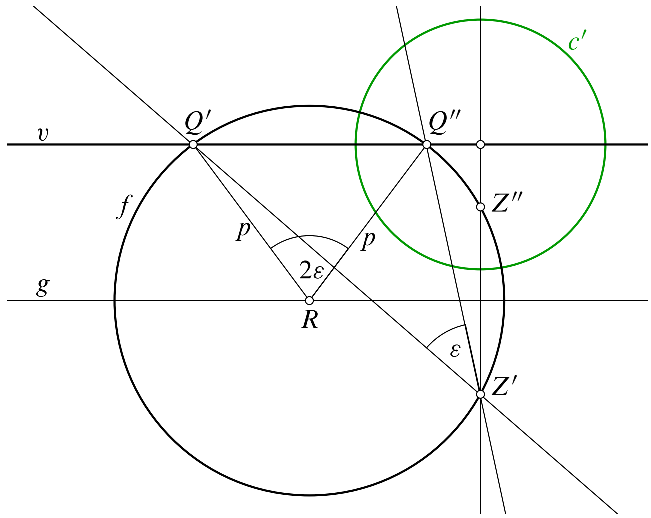

Figure 8. Construction of the conjugate points Q' and Q'' with angle $\sphericalangle Q''Z'Q' = \varepsilon$.

Construction 4 (Construction of conjugate points Q', Q'' seen from Z under a given angle ε). Figure 8 shows again the plane Σ'. The center Z of the projection was flipped around the axis v into the plane Σ' which gives the point Z'. The point Z'' is the conjugate point of Z' with respect to the circle c'. Observe that the points Z', Z'', Q', Q'' are concyclic on a circle f which intersects c' orthogonally. Since $Z'Z''$ is orthogonal to v, the perpendicular bisector g of $Z'Z''$ is parallel to v. The center R of f sees the segment $Q'Q''$ under the angle 2ε. It is therefore an elementary task to construct the radius p of f, and thereupon R and the points Q' and Q''. Notice that Q' is the point at infinity on v, and Q'' the center of c' if $\varepsilon = \frac{\pi}{2}$. This concludes Construction 4.

We can now finally finish the construction of the quadrilateral. Find the points Q', Q'' which correspond to the angle ε by Construction 4. Let S' be the intersection of $Q'B'$ with c'. Project the point S' from Z to the plane Σ, leading to the point S. Complete the quadrilateral by the points C, D such that S is the intersection of the diagonals AC and BD, and the quadrilateral is ready.

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Norbert Hungerbühler
Department of Mathematics, ETH Zentrum
Rämistrasse 101, 8092 Zürich, Switzerland
norbert.hungerbuehler@math.ethz.ch

Juan Läuchli
Fachschaft Mathematik, Kantonsschule Frauenfeld
Ringstrasse 10, 8500 Frauenfeld, Switzerland
juan.laeuchli@kftg.ch