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Extensions of Cut-and-Choose Fair Division

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In its simplest setting the fair division problem asks that two people "fairly" divide a piece of cake. A widely known solution is: "one cuts the other chooses." The general problem is, like most cakes over which it is posed, very rich and has blossomed since its introduction by Steinhaus in 1948 [9]. In the case of n persons, there are various ways to guarantee all persons at least $1/n$th of the cake by their own assessments [4, 5, 8, 9, 10]. If a piece can be found on which two persons disagree, all can be guaranteed strictly more than $1/n$th of the cake by their assessment [13]. It is also known how to guarantee each of three persons an "envy free" piece (i.e. nobody thinks another has a preferred piece) using at most five cuts [12]. Existence proofs for envy free portions have been given [1, 3, 12] and there are two procedures that generate envy-free pieces for n persons, but neither procedure has a bound (as a function of n) for the number of cuts that may be required [2, 7]. Existence proofs have been given to show there are n pieces

Ob man einen Kuchen teilt, eine Erbschaft oder das Volkseinkommen – wenn die Betroffenen die Teilung nicht als fair empfinden, so sind Konflikte vorprogrammiert. Bei der Teilung eines Kuchens unter zwei Parteien können alle Eltern glücklicherweise von der bekannten und eleganten Lösung profitieren: *Der eine teilt, der andere wählt!* Wen überrascht es, dass sich die Mathematik für diese brilliante Idee weiter interessiert? In der Tat lässt sich das Verfahren auf kompliziertere Situationen verallgemeinern, auf Teilungsprobleme zwischen mehreren Parteien und auf solche mit zusätzlichen, einschränkenden Bedingungen. Wie immer, wenn die Mathematik ein Problem aufgreift, taucht eine Menge neuer und interessanter Fragen auf. Einigen davon gehen Jack Robertson und Bill Webb im vorliegenden Beitrag nach. *ust*

which all n players think are equal [1]. "Moving knife" continuous algorithms have been given for the latter problem for $n = 2$ [7], for the envy free problem for $n = 3$ [11], and the original problem above for any n [10,12]. Players can be guaranteed fair rational unequal portions if that is what they deserve [6], and much attention has been given to trying to minimize the number of cuts used to accomplish the various tasks.

The purpose of this note, after all this activity, is to return to the most basic procedure of two person "cut-and-choose" and explore what can be accomplished with a sequence of such steps. In keeping with the simplicity of the cut-and-choose procedure, we will not give the careful mathematical formulation (found widely in print, e.g. [12]) but rather rely on the intuitive aspects of the procedures. Note that the cut-and-choose procedure assumes only that the cutter can cut one piece into two pieces on which the cutter has no preference, that the chooser can exercise a preference on the two pieces presented, and that the total value of the cake or its pieces is not diminished or enhanced by the cut. Thus, a notable feature of such a simple cut-and-choose procedure is that it requires only a preference ranking and not a numerical evaluation of the pieces.

What divisions can be done utilizing only cut-and-choose procedures? When can fair division be accomplished with a finite number of steps? The following discussion provides the answers to these questions.

Looking first at a specific example, suppose P_1 cuts the cake and P_2 chooses one of the pieces. Now P_2 cuts the unchosen piece and P_1 chooses between the resulting two pieces. Finally P_1 cuts the new unchosen piece, P_2 chooses one piece and P_1 gets what is left. We claim that this procedure quarantees P_1 at least $3/8$ and P_2 at least $5/8$ of the cake. (See Example 1 below.) How do we know this and how do we decide who should cut next? When should we stop? It turns out that the rule is quite easy, works for rational or irrational ratios, and is given in general by:

Procedure I: *A Sequence of Cut-and-Choose Steps that Guarantees Two Players P_1 and P_2 Fair Pieces in the Ratio $\alpha : \beta$ for any Real Numbers $\alpha > 0,\ \beta > 0,\ \alpha + \beta = 1$.*

1. *Write $\alpha = .\alpha_1\ \alpha_2\ \alpha_3 \cdots$ and $\beta = .\beta_1\ \beta_2\ \beta_3 \cdots$ in binary form, using the terminating forms if possible.*
2. *At the ith stage, P_1 cuts if $\alpha_i = 0$; P_2 cuts if $\beta_i = 0$; either cuts if $\alpha_i = \beta_i = 1$. The other chooses and keeps (or banks) the chosen piece.*
3. *The procedure stops in the case $\alpha_i = \beta_i = 1$ with the unchosen piece given to the cutter. Otherwise the procedure continues on the unchosen piece.*

Example 1. Players P_1 and P_2 are to share the cake in the ratio $3 : 5$ or $3/8 : 5/8$.

1. $3/8 = .011$ $\quad$ $5/8 = .101$
2. a. P_1 cuts the cake $X = X_{11} \cup X_{12}$; P_2 chooses (wlog) X_{12}.
   b. P_2 cuts $X_{11} = X_{21} \cup X_{22}$; P_1 chooses (wlog) X_{21}.
   c. Either, say P_1, cuts $X_{22} = X_{31} \cup X_{32}$; P_2 chooses (wlog) X_{32} and X_{31} is given to P_1.
3. The procedure stops since $\alpha_3 = \beta_3 = 1$. Player P_1 has $X_{21} \cup X_{31}$ while P_2 has $X_{12} \cup X_{32}$.

If the binary forms are non-terminating the procedure is countably infinite. Since $\alpha+\beta = 1$ we know $\alpha_i + \beta_i = 1$ unless $\alpha_i = \beta_i = 1$ in the last place of terminating forms.

We must show P_1 and P_2 receive pieces they value at least α and β respectively. Let X_{j1} and X_{j2} be the two pieces produced by the jth cut and let $\mu_1(X_{jk})$ be the fraction of the entire cake X that piece X_{jk} represents according to player P_1; $\mu_2(X_{jk})$ is defined similarly for P_2.

If $X = X_1 \cup X_2$, where X_1 and X_2 are the totality of all pieces assigned to P_1 and P_2 respectively, we must show $\mu_1(X_1) \geq \alpha$ and $\mu_2(X_2) \geq \beta$. This will be established by induction. After k steps of Procedure I, P_1 and P_2 will have received certain portions of the cake. We will denote these two total banked holdings through step k by Y_k and Z_k respectively and the remaining unchosen piece by R_k. Also set $A_k = .\alpha_1\ \alpha_2 \cdots \alpha_k$ and $B_k = .\beta_1\ \beta_2 \cdots \beta_k$. We show through k steps P_1 and P_2 both think they are doing fine so far and the other is not running ahead.

Claim: For $k = 1, 2, 3 \cdots$

a. $\mu_1(Y_k) \geq A_k$, b. $\mu_1(Z_k) \leq B_k$, c. $\mu_2(Y_k) \leq A_k$,
d. $\mu_2(Z_k) \geq B_k$, and e. $\mu_i(R_k) \leq \frac{1}{2^k}$, $i = 1, 2$.

Proof. For $k = 1$, the chooser should think he or she receives at least half of the cake while the cutter should not think more than half has been given away. That is exactly what cut-and-choose accomplishes. Inequality (e) is clear for $k = 1$ also.

Thus assuming the kth case let us examine what happens at the $(k+1)$st step.

Case 1: $\alpha_{k+1} = 0$, $\beta_{k+1} = 1$.

Player P_1 will cut R_k in two and P_2 will choose and bank one of the two pieces. We know $\mu_i(R_k) = 1 - \mu_i(Y_k) - \mu_i(Z_k)$ for $i = 1, 2$. Since $Y_k = Y_{k+1}$, $\mu_1(Y_{k+1}) = \mu_1(Y_k) \geq A_k = A_{k+1}$ from the induction assumption (a), and $\mu_2(Y_{k+1}) = \mu_2(Y_k) \leq A_k = A_{k+1}$ from (c).

Also $\mu_2(Z_{k+1}) \geq \mu_2(Z_k) + \frac{1}{2}(1 - \mu_2(Y_k) - \mu_2(Z_k)) = \frac{1}{2}(1 + \mu_2(Z_k) - \mu_2(Y_k)) \geq \frac{1}{2}(1 + B_k - A_k) = \frac{1}{2} + B_k - \frac{1}{2}(B_k + A_k) = \frac{1}{2} + B_k - \frac{1}{2}(1 - \frac{1}{2^k}) = B_k + \frac{1}{2^{k+1}} = B_{k+1}$. For (b), $\mu_1(Z_{k+1}) = \mu_1(Z_k) + \frac{1}{2}(1 - \mu_1(Y_k) - \mu_1(Z_k)) = \frac{1}{2}(1 + \mu_1(Z_k) - \mu_1(Y_k)) \leq \frac{1}{2}(1 + B_k - A_k) = B_{k+1}$, as was seen above. Finally for (e), $\mu_i(R_{k+1}) \leq \frac{1}{2}\mu_1(R_k) \leq \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}$.

Case 2: $\alpha_{k+1} = 1$, $\beta_{k+1} = 0$

This is established the same way as Case 1.

Case 3: $\alpha_{k+1} = \beta_{k+1} = 1$.

This case is established by observing that R_{k+1} is empty, $A_{k+1} + B_{k+1} = 1$, and by justifying (b) and (d) exactly as in Case 1. Thus, the claim is established and the procedure accomplishes the required division.

In fact the procedure just described gives the *only* sequence of cut-and-choose steps which guarantees pieces in the ratio $\alpha : \beta$. For suppose we have a prescribed sequence of cut-and-choose steps indicating which player is to cut at each step. We will further assume that if the procedure terminates after the ith step that the cutter receives the unchosen piece at that step. (Note that this is equivalent to the ith stage cutter repeatedly choosing on the remainders as the other cuts. We will always opt for the terminating form.)

On the basis of this information about the procedure, let us define

$$a_i = \begin{cases} 0 \text{ if } P_1 \text{ cuts and does not receive the unchosen piece at} \\ \text{step } i, \text{ or the procedure terminates before step } i. \\ 1 \text{ if } P_2 \text{ cuts and } P_1 \text{ chooses, or, } P_1 \text{ cuts and} \\ \text{receives the unchosen piece at step } i. \end{cases}$$

Define b_i similarly for P_2. It is clear that $(.a_1a_2\cdots) + (.b_1b_2\cdots) = 1$ in either the terminating or non-terminating case.

What is the most cake guaranteed P_1 by this sequence of steps? In particular it may be the case that (i) the two players use the same measure (although they may not realize it) and (ii) P_2 always cuts halves when P_2 cuts (maybe unbeknownst to P_1). From the previous discussion above we know this guarantees P_2 a share B with $\mu_2(B) \geq .b_1b_2\cdots$. Hence, since the players are using the same measure, $\mu_1(X - B) \leq 1 - .b_1b_2\cdots = .a_1a_2\cdots$. So the most this sequence of cut-and-choose steps can guarantee P_1 in the general case is $.a_1a_2\cdots$ of the cake. By the same agrument with the players switched, P_2 is guaranteed in general no more than $.b_1b_2\cdots$ by the procedure. It follows that any sequence of steps other than that found in Procedure I will generate two numbers where either $.a_1a_2\cdots < \alpha$ or $.b_1b_2\cdots < \beta$.

Furthermore, we see that in order to guarantee a fair share for a player, that player must cut halves at each stage. For if the ith stage is the first step where the cutter fails to cut halves, the chooser can take the cutter's larger piece. Again they may be using the same measure. Using the notation in the proof above and assuming P_1 cut non-halves in step i we then have $\mu_j(Y_{i-1}) = A_{i-1}, \mu_j(Z_{i-1}) = B_{i-1}, \mu_j(Z_i) > B_i$ and $\mu_j(R_i) < 1/(2^i)$ for $j = 1, 2$. But, arguing as above, (for the share $.\alpha_{i+1}\alpha_{i+2}\cdots$ on R_i) the most P_1 can then receive is $A_{i-1} + 2^i(\alpha - A_{i-1})\mu_1(R_i) < A_{i-1} + (\alpha - A_{i-1}) = \alpha$.

In summary, Procedure I is the unique sequence of cut-and-choose steps which guarantees the fair shares in the given ratio, and the only strategy which guarantees those fair shares is for the cutter to always cut halves.

Example 2: Suppose P_1, P_2, and P_3 are to share in the ratios $1/2 : 1/3 : 1/6$. Using Procedure I, P_1 and P_2 can first divide the cake in the ratio $1/2 : 1/3$ which is $3/5 : 2/5$. Then P_3 can repeat Procedure I in the ratio $1/6 : 5/6$ with each of P_1 and P_2. Then P_1 will have at least $5/6 \cdot 3/5 = 1/2, P_2$ will have at least $5/6 \cdot 2/5 = 1/3$ and P_3 will have at least $1/6$th of everything. In order to avoid a sequence of three infinite procedures back to back, a single countable process can be described using a diagonalization procedure that permits P_3 to start the division of pieces with P_1 and P_2 as soon as they are chosen by P_1 or P_2. Once a piece is banked in either the P_1, P_3 or P_2, P_3 division, it remains banked and uncut.

For $P_1, \cdots, P_n$ to share in the ratios $\alpha_1 : \alpha_2 : \cdots : \alpha_n$ first have $P_1, \cdots, P_{n-1}$ share in the ratios $\frac{\alpha_1}{1-\alpha_n} : \frac{\alpha_2}{1-\alpha_n} : \cdots : \frac{\alpha_{n-1}}{1-\alpha_n}$. Then P_n will share with each of $P_1, \cdots, P_{n-1}$ in the ratio $\alpha_n : 1-\alpha_n$. The procedures can be diagonalized so that a single countable sequence suffices.

When is a finite procedure possible? For $n = 2$ this is an easy question, the procedure is finite if and only if $\alpha/(\alpha+\beta) = a/2^m$ for some positive integers a and m. Hence, the ratio 5:11 is accomplished by 4 steps while the ratio $1/\pi : (\pi - 1)/\pi$ requires an infinite procedure (even on pies).

The case for three or more players is more interesting. For example, suppose the ratios are 12:3:1. If P_1 and P_2 divide first, the ratio is $4/5 : 1/5$ which leads to an infinite process. However, if P_2 and P_3 divide first in the ratio $3/4 : 1/4$, that process is finite and must be followed by P_1 dividing with each of P_2 and P_3 in the ratio $3/4 : 1/4$ each of which is finite. In general the inductive procedure given above for n players is finite if and only for some permutation $a_1 : a_2 : \cdots : a_n$ of the ratios each of the fractions $a_i/(a_1 + a_2 + \cdots + a_i),\ 2 \le i \le n$, can be written as $a/2^m$.

The example above shows that the order of the divisions can be important. Indeed the procedure cannot be finite for all possible permutations of the ratios. Suppose we have three positive numbers α, β, γ with $\alpha + \beta + \gamma = 1$ so that the overall process is finite regardless which pair goes first. This would require $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$ to have the form $a/2^m$ and $b/2^m$ where $a + b = 2^m$. Thus, $\alpha/\beta = a/b$ is rational as is α/γ. It follows that we can assume $\alpha : \beta : \gamma$ are the same ratios as $n_1 : n_2 : n_3$ where all n_i are integers. Since we are assuming the divisions in the ratios $n_1 : n_2,\ n_1 : n_3,\ n_2 : n_3$ and $(n_1 + n_2) : n_3$ are all finite procedures we have:

$$\begin{aligned} n_1 + n_2 &= 2^j, \\ n_1 + n_3 &= 2^k, \\ n_2 + n_3 &= 2^l, \\ n_1 + n_2 + n_3 &= 2^m, \end{aligned}$$

where $j, k, l \ge 1$ and $m \ge 2$.

Thus $2^m = 2^{j-1} + 2^{k-1} + 2^{l-1}$ and this equality requires (wlog) $j - 1 = m - 1,\quad k - 1 = l - 1 = m - 2$. But then $n_1 + n_2 = 2^m$ and $n_3 = 0$ contradicting $\gamma > 0$.

We have seen above that using a sequence of strict cut-and-choose steps, a finite procedure is only possible when the binary forms terminate. We next describe a **finite** procedure "in the spirit of cut and choose" which applies to all **rational ratios** $a : b$. As before, each step will have a cutter with the other player choosing and banking one of the two pieces produced at that step. We will have to use more than a simple preference evaluation on pieces however.

In what follows we will assume a and b are positive integers and that the cake X is to be divided in the ratio $a : b$ between P_1 and P_2 respectively. We will describe a finite, modified cut-and-choose procedure which we will denote $P(a, b)$.

The procedure $P(1,1)$ is the original cut-and-choose where each person gets at least $1/2$ of the cake. Now assume $P(c,d)$ has been defined for all c and d such that $c+d<n$, so as to assure P_1 receives at least $c/(c+d)$ and P_2 receives at least $d/(c+d)$ of the cake in a finite number of steps. With $a+b=n\geq 3$, we will now describe $P(a,b)$. Intuitively, the cutter cuts in the ratio $\alpha:\beta$ where $\alpha+\beta=a+b,\ \alpha$ and β are integers and α/β is as close to 1 as possible. Formally:

Procedure II: *A Finite Sequence of Modified Cut-and-Choose Steps that Guarantees Two Players, P_1 and P_2, Fair Pieces in the Rational Ratio $a:b$.*

(1) For $a>0, P(a,0)$ is P_1 takes the piece; for $b>0, P(0,b)$ is P_2 takes the piece and the procedure stops. For $a=b=1, P(a,b)$ is either player cuts halves, the other chooses. The remaining piece goes to the cutter and the procedure stops.

Assume for (2) and (3) that $ab\neq 0, a+b>2$ and $(a,b)=1$.

(2) If $a+b$ is even and (wlog) $a<b$ then P_1 cuts equal pieces, $X=X_1\cup X_2$. Then P_2 chooses the larger of X_1 or X_2 and procedure $P\left(a,\dfrac{b-a}{2}\right)$ is applied to the other piece.

(3) If $a+b$ is odd and (wlog) $a<b$ then P_1 cuts $X=X_1\cup X_2$ in the ratio $\dfrac{a+b-1}{2}:\dfrac{a+b+1}{2}$ respectively. Player P_2 chooses X_1 if it is considered to have value at least $\dfrac{a+b-1}{2}\cdot\dfrac{1}{a+b}$ and chooses X_2 if it is at least $\dfrac{a+b+1}{2}\cdot\dfrac{1}{a+b}$. If P_2 chooses X_1 apply $P\left(a,\dfrac{b-a+1}{2}\right)$ to X_2. If P_2 chooses X_2 apply $P\left(a,\dfrac{b-a-1}{2}\right)$ to X_1.

Since $a+\frac{1}{2}(b-a+1)=\frac{1}{2}(a+b)+1/2<a+b=n$, all of the specified finite number of additional procedures are inductively defined. Also, since the two values against which X_1 and X_2 are measured in (3) have sum one, one of the two choices must be satisfactory to P_2.

We can now verify that P_1 will get at least $\dfrac{a}{a+b}$ and P_2 will get at least $\dfrac{b}{a+b}$.

For example, suppose $a+b$ is odd and P_2 chooses X_1. Then P_1 is guaranteed at least $a/\left(\dfrac{b+a+1}{2}\right)$ of X_2 by procedure $P\left(a,\dfrac{b-a+1}{2}\right)$ and P_1 views X_2 as worth $\dfrac{a+b+1}{2}\cdot\dfrac{1}{a+b}$. Thus, P_1 gets at least $\dfrac{2a}{b+a+1}\cdot\dfrac{a+b+1}{2(a+b)}=\dfrac{a}{a+b}$.

Similarly, if $a+b$ is odd and P_2 chooses X_2 then P_1 gets at least $\dfrac{2a}{b+a-1}\cdot\dfrac{a+b-1}{2(a+b)}=\dfrac{a}{a+b}$.

To prove that P_2 will get at least $\dfrac{b}{a+b}$ we could give an argument like that for P_1. Rather let us assume that P_2 is not satisfied by either choice presented in Case 3.

Then

$$\mu_2(X_1) + \left(\frac{b-a+1}{a+b+1}\right)\mu_2(X_2) < \frac{b}{a+b} \text{ if } P_2 \text{ chooses } X_1, \text{ and}$$

$$\mu_2(X_2) + \left(\frac{b-a-1}{a+b-1}\right)\mu_2(X_1) < \frac{b}{a+b}, \text{ if } P_2 \text{ chooses } X_2.$$

Adding these inequalities we obtain $2b\mu_2(X_1) + 2b\mu_2(X_2) < \frac{b}{a+b}(2a+2b)$, or $\mu_2(X_1) + \mu_2(X_2) < 1$. This contradicts $\mu_2(X_1) + \mu_2(X_2) = \mu_2(X) = 1$.

The case for $a+b$ even is similar.

Example 3: Suppose the ratio is $5:8$ with player P_1 receiving the smaller portion. The diagram in Fig. 1 summarizes the branching procedure.

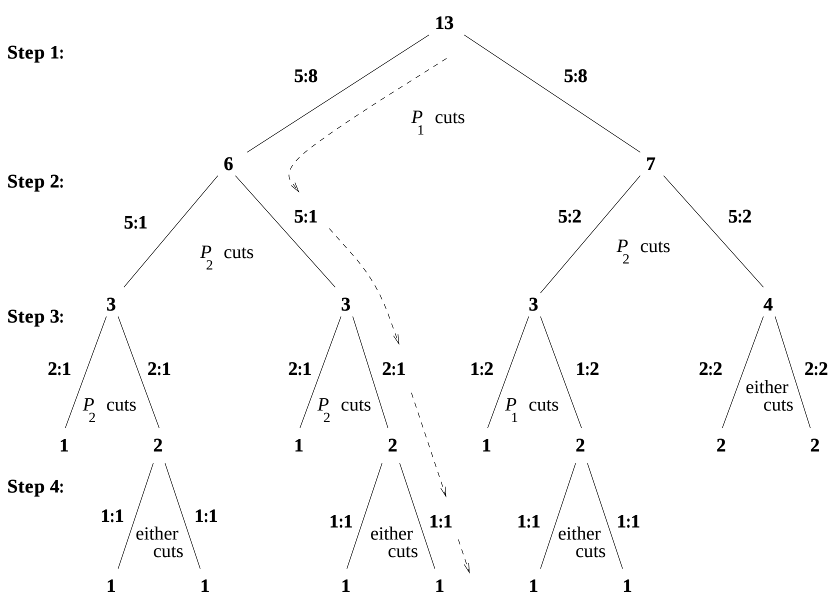

Fig. 1 Branching procedure

For one instance, in the path shown, P_1 cuts in the ratio $6/13 : 7/13$; P_2 chooses and banks the $7/13$ piece; P_2 cuts the other piece in the ratio $3/6; 3/6$; P_1 chooses and banks the left piece; P_2 cuts in the ratio $2/3 : 1/3$; P_1 chooses and banks the $1/3$ piece; either cuts the remaining piece in halves, the other chooses first with the chosen piece given to the last cutter.

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