Elemente der Mathematik (Online first)
DOI 10.4171/EM/510

Published by EMS Press

Short note

Vandermonde's identity proved by complex analysis

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Abstract. In this short note, we propose a proof of the Vandermonde identity based on elementary complex analysis.

Among the elegant results implied by the binomial theorem, one of the most attractive and widely known identities is Vandermonde's identity. Motivated by an old result of Minsker [2], in this note, we prove a q-analog of the Vandermonde identity using Cauchy's integral formula [1]. The q-analog of the Vandermonde identity is

$$\sum_{k_1+k_2+\cdots+k_q=m} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_q}{k_q} = \binom{n_1+n_2+\cdots+n_q}{m} \tag{1}$$

for any nonnegative integers $n_1, n_2, \ldots, n_q$, m and $0 \le k_i \le n_i$ $(1 \le i \le q)$, and $m \le n_1 + n_2 + \cdots + n_q$.

There are many elementary proofs of identity (1), but here, we prescribe an analytical proof of it.

Lemma 1. *Let N and n be two nonnegative integers with $n \le N$. Then*

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{(1+z)^N}{z^{n+1}}\, dz = \binom{N}{n}.$$

Proof. The proof follows from Cauchy's integral formula by considering the n-th derivative of

$$\frac{(1+z)^N}{n!}.$$

■

Proof of a q-analog of the Vandermonde identity. By Lemma 1, we have

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{(1+z)^{n_1+n_2+\cdots+n_q}}{z^{m+1}} = \binom{n_1+n_2+\cdots+n_q}{m}.$$

But using the binomial expansions of $(1+z)^{n_j}$ and Cauchy's integral formula, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^{n_1+n_2+\cdots+n_q}}{z^{m+1}}\,dz\\
&\quad=\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{z^{m+1}}\left(\sum_{\substack{0\le k_1\le n_1;\\ 0\le k_2\le n_2;\ldots;\\ 0\le k_q\le n_q}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_q}{k_q}z^{k_1+k_2+\cdots+k_q}\right)dz\\
&\quad=\sum_{k_1+k_2+\cdots+k_q=m}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_q}{k_q}.
\end{aligned}$$

This completes the proof. ■

Remark 1. *Thus the following identity is obvious:*

$$\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\cdots+\binom{m}{r}\binom{n}{0}=\binom{m+n}{r}$$

for any nonnegative integers m, n and r, with $0\le r\le m$ and $0\le r\le n$. This identity is named after Alexandre-Théophile Vandermonde, although it was already known in 1303 by the Chinese mathematician Zhu Shijie.

Remark 2. *For two nonnegative integers n and k with $n\ge k$, using Lemma 1, we have*

$$\begin{aligned}
\binom{n}{k}&=\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{k+1}}\,dz\\
&=\frac{1}{2\pi i}\int_{|z|=1}z^{n-k-1}(1+\overline{z})^n\,dz\\
&=\frac{1}{2\pi i}\int_{|z|=1}z^{n-k-1}\sum_{j=0}^{n}\binom{n}{j}\overline{z}^j\,dz\\
&=\frac{1}{2\pi i}\int_{|z|=1}z^{n-k-1}\sum_{j=0}^{n}\frac{\binom{n}{j}}{z^j}\,dz\\
&=\binom{n}{n-k}.
\end{aligned}$$

Remark 3. *If we choose $m=n=r$ in Remark 1, we have the well-known identity (see [2])*

$$\binom{m}{0}^2+\binom{m}{1}^2+\cdots+\binom{m}{m}^2=\binom{2m}{m}.$$

Acknowledgements. The authors are grateful to the anonymous referees for their valuable suggestions which considerably improved the presentation of the paper.

Funding. The research work is supported by the Department of Higher Education, Science and Technology & Biotechnology, Government of West Bengal under the sanction order no. 1303(sanc.)/STBT-11012(26)/17/2021-ST SEC dated 14/03/2022.

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