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An interesting theorem on circumscribed hexagon

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1 Introduction

In [3], an interesting theorem of Dao Thanh Oai about six circumcenters associated with a cyclic hexagon was introduced.

Theorem 1 (Dao). *Let A_i, $i = 1, 2, \dots, 6$, be six points on a circle. Taking subscripts modulo 6, we denote, for $i = 1, 2, \dots, 6$, the intersection of the lines $A_i A_{i+1}$ and $A_{i+2} A_{i+3}$ by B_{i+3}, and the circumcenter of the triangle $A_i A_{i+1} B_{i+2}$ by C_{i+3}. The lines C_1C_4, C_2C_5, and C_3C_6 are concurrent. (See Figure 1.)*

With a dual way of thinking, when we replace a cyclic hexagon with a circumscribed hexagon and the circumscribed circle centers with the inscribed circle centers, we get

Gelegentlich gelingt es, durch analoges Denken aus bekannten Theoremen neue Sätze zu gewinnen. Diese Methode ist besonders fruchtbar in der ebenen Euklidischen Geometrie. In der vorliegenden Arbeit gehen die Autoren von einem erst vor wenigen Jahren gefundenen Satz von Dao aus. Indem sie in diesem Satz statt eines einbeschriebenen ein umbeschriebenes Sechseck betrachten und Umkreise durch Inkreise ersetzen, gelangen sie zu einem Theorem, in dem statt kopunktalen Geraden kollineare Punkte auftreten. Der Beweis erfolgt durch Rechnung in komplexen Koordinaten.

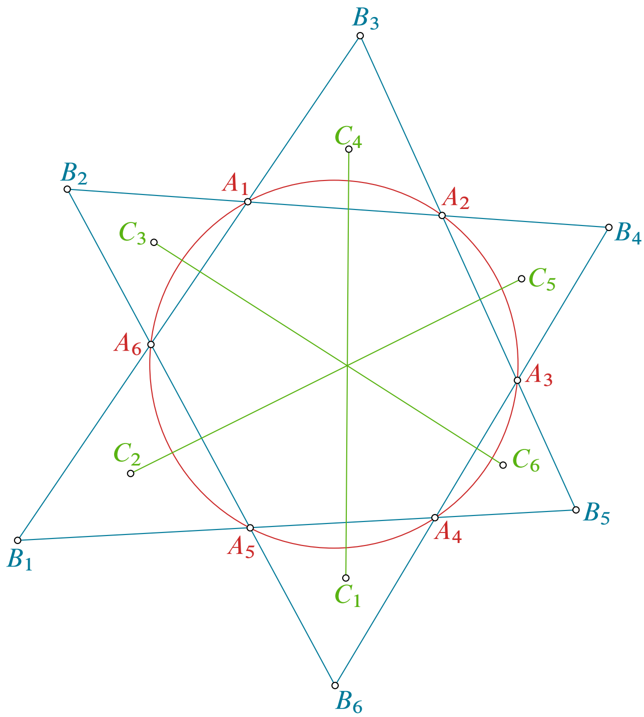

Figure 1. Illustration of Dao's theorem.

a similar theorem on three collinear points (instead of three concurrent lines). Our theorem is as follows.

Theorem 2 (Main theorem). *Let $A_1A_2A_3A_4A_5A_6$ be a circumscribed hexagon on a plane. Taking subscripts modulo 6, we denote, for $i = 1, 2, \dots, 6$, the intersection of the lines A_iA_{i+1} and $A_{i+2}A_{i+3}$ by B_{i+1}, and the incenter of the triangle $A_iB_iA_{i+1}$ by I_i, and the intersection of the lines I_iI_{i+3} and B_iB_{i+3} by X_i. Then the three points X_1, X_2, and X_3 are collinear. (See Figure 2.)*

We shall give a solution to Theorem 2 using complex coordinates in the next section.

2 Proof of Theorem 2 using complex numbers

In this section, we shall use complex numbers to approach the theorem as in [3]. An interesting observation is that, when we replace the secant line (of cyclic quadrilateral) with the tangents (of circumscribed quadrilateral), we get a similar formula for complex numbers. This formula will also lead to another complex number solution to our new theorem on the circumscribed hexagon.

We first introduce and prove a lemma for tangents (for a similar lemma, see [3]).

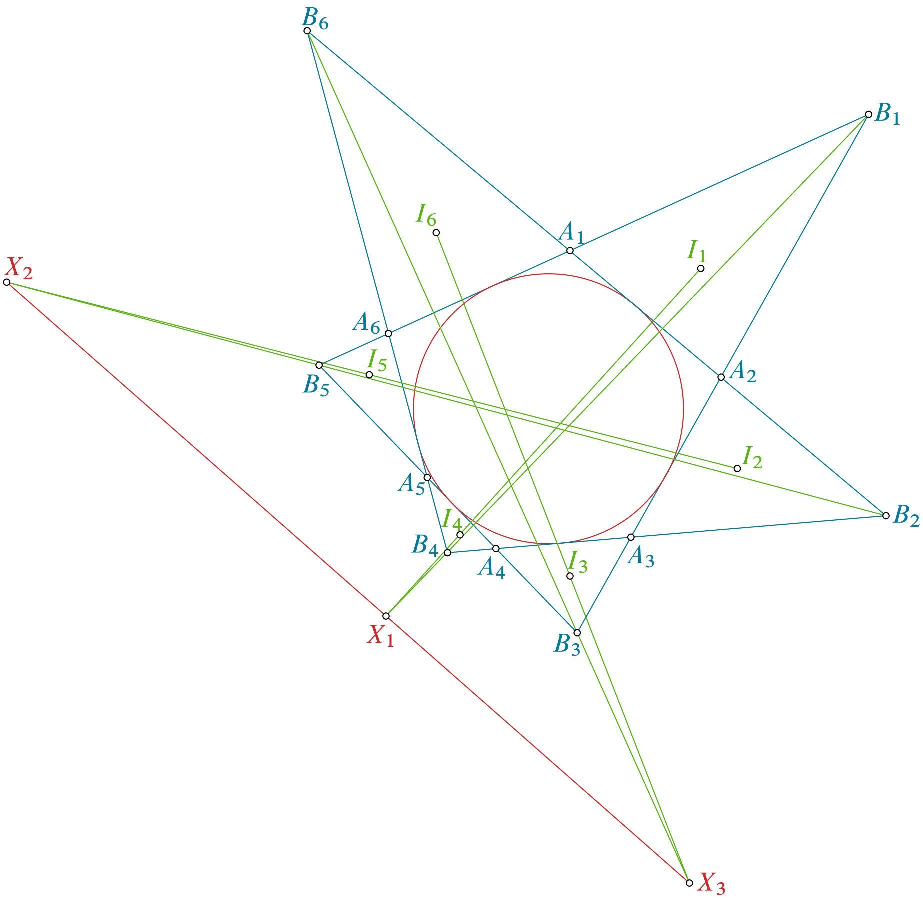

Figure 2. Illustration of the main theorem.

Lemma 1. *On the unit complex circle (centered at O), consider the consecutive points b, a, and c. Let the tangents at these points meet at the points A_1, B_1, and C_1 such that O is the A_1-excenter of triangle $A_1B_1C_1$. Then the incenter K of the triangle $A_1B_1C_1$ has the complex coordinate*

$$k = \frac{4abc}{(a+b)(a+c)}.$$

Proof. (See Figure 3.) The tangents at the points b, c of the unit circle have the equations $z + b^2\overline{z} - 2b = 0$ and $z + c^2\overline{z} - 2c = 0$. Therefore, the complex coordinate of the intersection A_1 is the solution of the system

$$\begin{cases} z + b^2\overline{z} - 2b = 0, \\ z + c^2\overline{z} - 2c = 0. \end{cases} \tag{1}$$

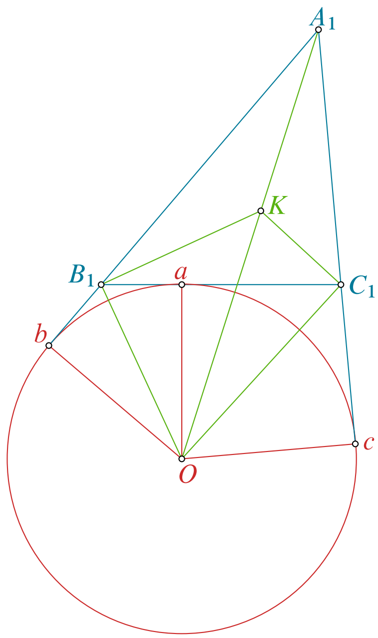

Figure 3. Proof of Lemma 1.

Solving system (1) yields

$$a_1 = \frac{2bc}{b+c}. \tag{2}$$

Similarly, we have

$$b_1 = \frac{2ca}{c+a} \tag{3}$$

and

$$c_1 = \frac{2ab}{a+b}.$$

Since O is the A_1-excenter of triangle $A_1B_1C_1$, the perpendicular to OB_1 at B_1 is the angle bisector of $\angle A_1B_1C_1$. From (3), this line has the equation

$$(a+b)z + ab(a+b)\overline{z} - 4ab = 0.$$

The angle bisector of $\angle B_1A_1C_1$ is the line OA_1. From (2), line OA_1 has the equation $z - bc\overline{z} = 0$. Hence the complex coordinate of the incenter K of the triangle $A_1B_1C_1$ is the solution of the system

$$\begin{cases} (a+b)z + ab(a+b)\overline{z} - 4ab = 0, \\ z - bc\overline{z} = 0. \end{cases} \tag{4}$$

Solving system (4), we get $k = \frac{4abc}{(a+b)(a+c)}$. This completes the proof of Lemma 1. ■

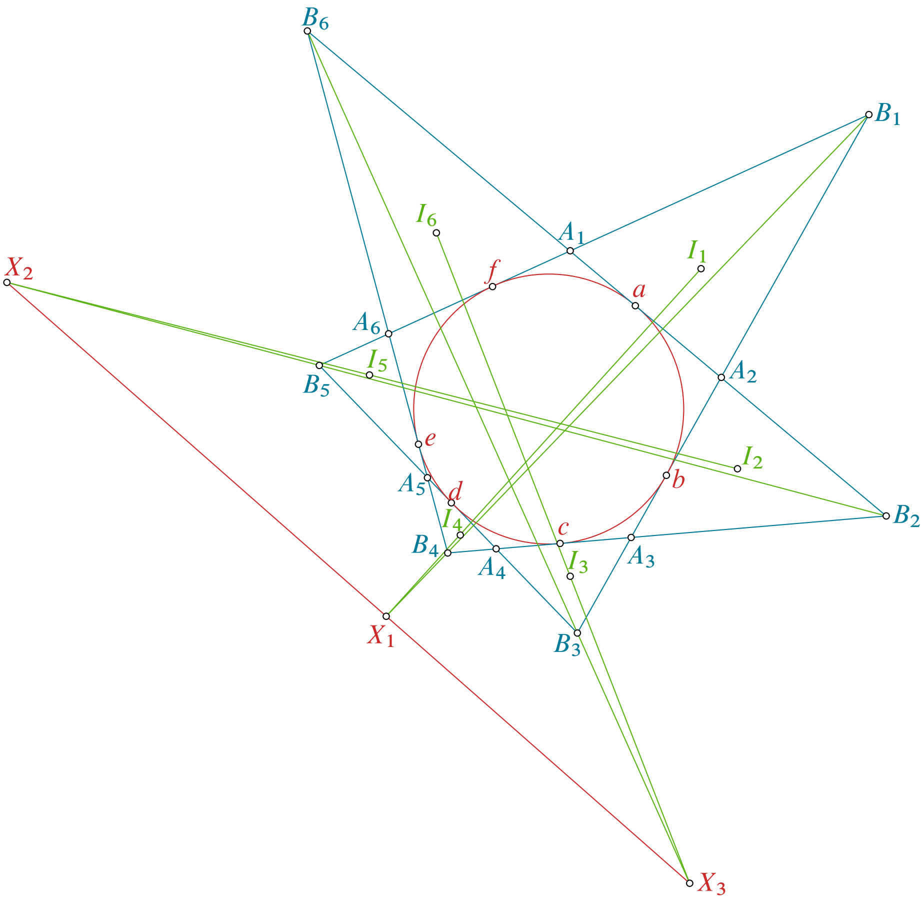

Figure 4. Illustration of the proof of the main theorem.

Proof of Theorem 2. (See Figure 2.) Assume that the incircle of the hexagon is the complex unit circle. Let a, b, c, d, e, and f be the complex coordinates of the contact points of the sides A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, and A_6A_1, respectively, with the unit circle (as shown in Figure 4). It follows from the proof of Lemma 1 that we have

$$b_1 = \frac{2fb}{f+b}, \quad b_2 = \frac{2ac}{a+c}, \quad b_3 = \frac{2bd}{b+d},$$
$$b_4 = \frac{2ce}{c+e}, \quad b_5 = \frac{2df}{d+f}, \quad b_6 = \frac{2ea}{e+a}.$$

On the other hand, we deduce from Lemma 1 that

$$i_1 = \frac{4fab}{(a+b)(a+f)}, \quad i_2 = \frac{4abc}{(b+c)(b+a)}, \quad i_3 = \frac{4bcd}{(b+c)(c+d)},$$

$$i_4 = \frac{4cde}{(c+d)(d+e)}, \quad i_5 = \frac{4def}{(d+e)(e+f)}, \quad i_6 = \frac{4efa}{(e+f)(f+a)},$$

where i_1, i_2, i_3, i_4, i_5, and i_6 are the complex coordinates of the incenters of the triangles $A_1A_2B_1$, $A_2A_3B_2$, $A_3A_4B_3$, $A_4A_5B_4$, $A_5A_6B_5$, and $A_6A_1B_6$, respectively. From the coordinates of the points B_1 and B_4, the line B_1B_4 has the equation

$$\frac{z-b_1}{z-b_4} = \overline{\left(\frac{z-b_1}{z-b_4}\right)}$$

or

$$(c+e-f-b)z + (bce+cef-bcf-bef)\overline{z} + 2(bf-ce) = 0. \tag{5}$$

Similarly, from the coordinates of the points I_1 and I_4, the line I_1I_4 has the equation

$$\frac{z-i_1}{z-i_4} = \overline{\left(\frac{z-i_1}{z-i_4}\right)}$$

or

$$\begin{aligned} &(cda+cea+d^2a+dea-da^2-dab-daf-dbf)z \\ &\quad + (cdea^2+cdeab+cdeaf+cdebf-cdabf-ceabf-d^2abf-deabf)\overline{z} \\ &\quad + 4(dabf-cdea) = 0. \end{aligned} \tag{6}$$

Thus, the intersection X_1 of the two lines B_1B_4 and I_1I_4 is the solution of the system of equations (5) and (6). Solving this system, we get

$$x_1 = \frac{2(a^2cde-abcde+abcdf-abd^2f-abcef+abdef-acdef+bcdef)}{a^2cd-abd^2-abce+a^2de+bcdf-ad^2f-acef+bdef}.$$

Similarly,

$$x_2 = \frac{2(b^2def-bcdef+abcde-abce^2-abcdf+abcef-abdef+acdef)}{b^2de-bce^2-bcdf+b^2ef+acde-abe^2-abdf+acef},$$

$$x_3 = \frac{2(ac^2ef-acdef+bcdef-bcdf^2-bcdef+abcdf-abcef+abdef)}{c^2ef-cdf^2-acde+ac^2f+bdef-bcf^2-abce+abdf}.$$

The points X_1, X_2, and X_3 are collinear if and only if

$$\begin{vmatrix} x_1 & \overline{x_1} & 1 \\ x_2 & \overline{x_2} & 1 \\ x_3 & \overline{x_3} & 1 \end{vmatrix} = 0 \tag{7}$$

(see [1,2]). Since

$$x_1 = \frac{2p_1}{r_1}, \quad \overline{x_1} = \frac{2q_1}{r_1}, \quad x_2 = \frac{2p_2}{r_2}, \quad \overline{x_2} = \frac{2q_2}{r_2}, \quad x_3 = \frac{2p_3}{r_3}, \quad \overline{x_3} = \frac{2q_3}{r_3}$$

and after elimination of denominators, equation (7) is equivalent to

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0, \tag{8}$$

where

$$\begin{aligned}
p_1 &= a^2cde - abcde + abcdf - abd^2f - abcef + abdef - acdef + bcdef,\\
p_2 &= b^2def - bcdef + abcde - abce^2 - abcdf + abcef - abdef + acdef,\\
p_3 &= ac^2ef - acdef + bcdef - bcdf^2 - bcdef + abcdf - abcef + abdef,\\
q_1 &= a^2d - abd + acd - ad^2 - ace + ade - adf + bdf,\\
q_2 &= b^2e - bce + bde - be^2 - bdf + bef - abe + ace,\\
q_3 &= c^2f - cdf + cef - cf^2 - ace + acf - bcf + bdf,\\
r_1 &= a^2cd - abd^2 - abce + a^2de + bcdf - ad^2f - acef + bdef,\\
r_2 &= b^2de - bce^2 - bcdf + b^2ef + acde - abe^2 - abdf + acef,\\
r_3 &= c^2ef - cdf^2 - acde + ac^2f + bdef - bcf^2 - abce + abdf.
\end{aligned}$$

By determinant transformation, (8) is equivalent to

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 + p_2 & q_1 + q_2 & r_1 + r_2 \\ p_3 - p_2 & q_3 - q_2 & r_3 - r_2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ ace - bdf & a + c + e - b - d - f & ac + ae + ce - bd - bf - df \\ ace - bdf & a + c + e - b - d - f & ac + ae + ce - bd - bf - df \end{vmatrix} = 0,$$

which is true. This completes the proof. ■

3 Conclusion

We seem to see a duality in the statement of our new theorem on circumscribed hexagons with Dao's theorem on inscribed hexagons. Furthermore, the complex number tool once again shows its usefulness in both theorems. Both theorems are an interesting complement to each other and have a common view through complex numbers as we have seen.

References

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