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## Short note      The generalized Binet formula for $k$ -bonacci numbers

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**Abstract.** Using Vandermonde determinants, we give a simple proof of the generalization of the Binet formula to the  $k$ -bonacci numbers.

For  $k \geq 2$ , define the  $k$ -bonacci numbers  $F_n^{(k)}$  by the initial values and recursion

$$F_n^{(k)} = \begin{cases} 0, & n \leq k - 2, \\ 1, & n = k - 1, \\ \sum_{i=1}^k F_{n-i}^{(k)}, & n \geq k. \end{cases} \quad (1)$$

For any linear recurrence, it is always possible to express the  $n$ th term as a linear combination of the  $n$ th powers of the roots of the characteristic equation of the recurrence. For the particular case of the  $k$ -bonacci numbers, the formula is the following:

$$\begin{aligned} F_n^{(k)} &= \sum_{i=1}^k \frac{\phi_i^n}{(\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_i - \phi_{i+1}) \cdots (\phi_i - \phi_k)} \\ &= \sum_{i=1}^k \frac{\phi_i^n}{\prod_{j \neq i} (\phi_i - \phi_j)}, \end{aligned} \quad (2)$$

where the  $\phi_i$  are the roots of  $x^k - x^{k-1} - \cdots - 1 = 0$ , the characteristic equation of the  $k$ -bonacci recurrence.<sup>1</sup> Note that, when  $k = 2$ , the  $k$ -bonacci numbers are the usual Fibonacci numbers and (2) reduces to the well-known Binet formula for the Fibonacci numbers.

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<sup>1</sup>The roots  $\phi_i$  are all distinct because, if the characteristic polynomial had a multiple root, then that same multiple root would be a multiple root of  $(x-1)(x^k - x^{k-1} - \cdots - 1) = x^{k+1} - 2x^k + 1$  and thus a root of  $d(x^{k+1} - 2x^k + 1)/dx = (k+1)x^k - 2kx^{k-1}$ . This last polynomial has only rational roots, and the Rational Root Theorem tells us that they are not roots of  $x^k - x^{k-1} - \cdots - 1$ .

Our goal in this paper is to give a clear and simple proof of (2) that relies solely on Vandermonde determinants. Indeed, we hope our proof is as clear and simple as is possible, considering the attempts by many other papers in this area and environs, e.g. [1–11].

## 1 Vandermonde determinants

The Vandermonde determinant is the  $k \times k$  determinant appearing in the next equation. The important elementary fact is that the determinant has the value given on the right-hand side of the equation:

$$\mathcal{V}(\phi, k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & \phi_1^{k-1} \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & \phi_2^{k-1} \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & \phi_3^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & \phi_k^{k-1} \end{vmatrix} = \prod_{1 \leq m < n \leq k} (\phi_n - \phi_m).$$

The preceding equation holds for all choices of the  $\phi_i$  in any field, not only the roots of the characteristic equation.

Similarly, if one forms a minor by omitting the rightmost column and the  $i$ th row of the above Vandermonde determinant, one obtains another Vandermonde determinant for which the next equation holds:

$$\mathcal{V}_i(\phi, k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{i-1} & \phi_{i-1}^2 & \cdots & \phi_{i-1}^{k-2} \\ 1 & \phi_{i+1} & \phi_{i+1}^2 & \cdots & \phi_{i+1}^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} \end{vmatrix} = \prod_{\substack{1 \leq m < n \leq k \\ m, n \neq i}} (\phi_n - \phi_m).$$

Again, the preceding equation holds for all choices of the  $\phi_i$  in any field, not only the roots of the characteristic equation.

**Lemma 1.** *Let the  $\phi_i$  be elements of any field and let  $f_i$ , for  $1 \leq i \leq k$ , be elements in the same field. If the  $\phi_i$  are all distinct, then*

$$\sum_{i=1}^k \frac{f_i}{\prod_{j \neq i} (\phi_i - \phi_j)} = \frac{1}{\mathcal{V}(\phi, k)} \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \end{vmatrix} \quad (3)$$

holds.

*Proof.* Expanding the determinant on the right-hand side of (3) along the last column, we see that

$$\begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \end{vmatrix} = \sum_{i=1}^k (-1)^{k-i} f_i \cdot \mathcal{V}_i(\phi, k).$$

For each  $i$ , observe that

$$\prod_{1 \leq m < n \leq k} (\phi_n - \phi_m) = (-1)^{k-i} \prod_{\substack{1 \leq m < n \leq k \\ m, n \neq i}} (\phi_n - \phi_m) \prod_{j \neq i} (\phi_i - \phi_j),$$

so

$$\sum_{i=1}^k \frac{f_i}{\prod_{j \neq i} (\phi_i - \phi_j)} = \sum_{i=1}^k \frac{(-1)^{k-i} f_i \cdot \mathcal{V}_i(\phi, k)}{\mathcal{V}(\phi, k)}$$

holds, and the result follows.  $\blacksquare$

**Corollary 2.** *If the  $\phi_i$  are distinct elements of any field, then*

$$\sum_{i=1}^k \frac{\phi_i^n}{\prod_{j \neq i} (\phi_i - \phi_j)} = \begin{cases} 0, & n < k, \\ 1, & n = k - 1, \end{cases} \quad (4)$$

*holds.*

*Proof.* For  $n = 0, 1, \dots, k - 2$ , Lemma 1 tells us that the left-hand side of (4) equals  $\mathcal{V}(\phi, k)^{-1}$  times the value of a determinant with a repeated column; thus the right-hand side of (4) equals 0. When  $n = k - 1$ , Lemma 1 tells us that the left-hand side of (4) equals  $\mathcal{V}(\phi, k)^{-1}$  times  $\mathcal{V}(\phi, k)$ .  $\blacksquare$

## 2 Completion of the proof of equation (2)

To prove that the recurrence in (1) holds, we must now require that the  $\phi_i$  be the solutions of the characteristic equation of that recurrence. That is, we require that  $\phi_i^k = \phi_i^{k-1} + \dots + 1$  holds for each  $i = 1, 2, \dots, k$ .

Supposing that  $n \geq k$ , we argue inductively, as we may, since (1) and Corollary 2 show that (2) holds for  $n = 0, 1, \dots, k - 1$ . We have

$$\begin{aligned} \sum_{i=1}^k \frac{\phi_i^n}{\prod_{j \neq i} (\phi_i - \phi_j)} &= \sum_{i=1}^k \frac{\phi_i^{n-k} \phi_i^k}{\prod_{j \neq i} (\phi_i - \phi_j)} = \sum_{i=1}^k \frac{\phi_i^{n-k} (\phi_i^{k-1} + \dots + 1)}{\prod_{j \neq i} (\phi_i - \phi_j)} \\ &= \sum_{m=1}^k \sum_{i=1}^k \frac{\phi_i^{n-m}}{\prod_{j \neq i} (\phi_i - \phi_j)} = \sum_{m=1}^k F_{n-m}^{(k)} = F_n^{(k)}. \end{aligned}$$

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