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Equicevian points in a triangle

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A *cevian* in a triangle is either a line or a segment connecting a vertex with a point on the opposite side, called the *footpoint* of the cevian. When its length is involved, a cevian clearly means a segment. Special cases of cevians are the triangle's altitudes, medians and angle bisectors. If three cevians from distinct vertices are concurrent, their intersection point is called a *cevian point*, and the triangle determined by their footpoints is a *cevian triangle* of the given triangle. Many triangle centers are introduced or discussed as special cevian points; known examples are *orthocenters*, *centroids* and *incenters*. Cevian triangles are, for instance, the *pedal triangles* of arbitrary triangle points, or *Seebach triangles*. In the literature, one can find various results and interesting problems on cevians that are still a popular area of research today. A small selection of such problems is presented below.

Eine Ecktransversale (Cevane) in einem Dreieck ist eine Gerade, die durch eine seiner Ecken verläuft. Die Länge einer Cevane wird von der Ecke bis zum Schnittpunkt mit der gegenüberliegenden Seite gemessen. Sind die Längen von drei kopunktalen Cevanen alle gleich, so heissen sie Äquicevane und ihr Schnittpunkt Äquicevan-Punkt. Laut einem Satz von Abu-Saymeh, Hajja und Stachel besitzen Dreiecke bis zu zwei Äquicevan-Punkte, die nicht auf den Dreieckseiten liegen. Diese Punkte sind dann just die Brennpunkte der Steiner-Umellipse des Dreiecks. Deren Mittelpunkt ist der Schwerpunkt des Dreiecks. Die Autoren der vorliegenden Arbeit zeigen, dass bis zu sechs weitere Äquicevan-Punkte auf den Dreieckseiten (resp. deren Verlängerungen) liegen. Dabei werden alle Fälle klassifiziert und das allgemeine Resultat für gleichschenklige Dreiecke geometrisch bewiesen.

Comprehensive representations of theorems about cevians are given in [4, Section 1.2] and in the survey [13, Section 1]. Three concurrent cevians of the same length are called *equicevians*, and their intersection is an *equicevian point*. These two notions will be central to our paper, but before getting to them, we discuss a few related, more general results.

The notion of a cevian goes back to the Italian mathematician Ceva, whose famous theorem about concurrent cevians in a triangle was published in 1687; see [4] and [3, Chapter VIII]. Seebach showed that if ABC and $X'Y'Z'$ are arbitrary triangles, then there is a unique triplet of concurrent cevians AX, BY, CZ of ABC such that XYZ is similar to $X'Y'Z'$; see [5, 9] and [13, Subsection 1.3]. Čerin showed in [3] that, for any triangle ABC, an algebraic curve of order 12 bounds the locus S of all points X with the property that the three cevians through them are congruent to the sides of a triangle. It has also been shown that the particular case of the cevians through the centroid of a triangle, the *medians*, always yields a triangle. This theorem and related results from plane geometry lead to intriguing questions about simplices, and these questions are answered by giving a smooth entrance to higher-dimensional geometry; see [7]. More details about this curve are given in [8]. Cevians are also interesting from the viewpoint of convexity; they represent all *affine diameters* of a triangle, a notion also important for general convex sets in higher dimensions; see the survey [17]. Inspired by the *Heronian triangles*, interesting results on rational-sided triangles with triplets of concurrent cevians of rational lengths are obtained in [12]; certain triangles can have infinitely many such triplets, and this is shown by a correspondence to points in certain elliptic surfaces of positive rank. The centroid of a tetrahedron is the intersection of the cevians connecting the vertices with the centroids of the opposite faces, and a similar statement holds for simplices in higher dimensions. In [6], it is shown that this inductive property does not hold for other famous cevian points, such as the Nagel center, the Lemoine center, the orthocenter, the incenter and the circumcenter. The *Steiner–Lehmus theorem*, see [4, Section 1.5], says that any triangle with two equicevians that are angle bisectors is isosceles. One interesting generalization is derived in [15]: if two equicevians intersect on the angle bisector from the remaining vertex, then the triangle is isosceles. More generally, in [14], triangles with two equicevians intersecting at a point on a third cevian are considered. The author studies all possible combinations of external or internal cevians, and also the possibilities of equicevian points. And in [1], it is shown that, besides points on the affine hulls of the sides of a triangle, the real and the imaginary focal points of the Steiner circumellipse are the only equicevian points. Furthermore, non-Euclidean analogues of theorems on equicevians and equicevian points have been used to show that an admissible triangle in the isotropic plane has two equicevian points whose position is centroid-symmetric, and analogously, they are the foci of the Steiner circumellipse; see [11].

Steiner proved in the mid 1800s that each triangle admits a unique circumellipse centered at the centroid. If ABC is the triangle and a, b, c are its side lengths, then the ellipse has semi-axes $\frac{1}{3}\sqrt{a^2+b^2+c^2 \pm 2z^2}$ and focal radius $\frac{2}{3}z$, where

$$z = \sqrt[4]{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}.$$

More properties of Steiner's circumellipse are found in [18, 19].

We set the stage with the following theorem, due to Abu-Saymeh, Hajja and Stachel in [1], on equicevian points of a triangle.

Theorem A. *Let $\mathcal{S}$ be the Steiner circumellipse of a triangle ABC. Then*

(i) *the triangle has either one or two equicevian points that fall outside the sidelines, namely, the foci of $\mathcal{S}$ that fall outside the sidelines;*

(ii) *the common length of the equicevian triplets in part* (i) *is $\frac{3}{2}$ times the major semi-axis of $\mathcal{S}$.*

The proof of Theorem A in [1] is based on complex analytic geometry, namely, on considering the Steiner circumellipse as a curve in the complex plane. This means that the cartesian coordinates (x, y) of points in the plane are viewed as complex coordinates $x + iy$. In this case, Steiner's circumellipse has two real foci on the major semi-axis and two imaginary foci on the minor semi-axis. The main ingredient of this proof is Marden's theorem, asserting that if $p(\zeta)$ is a cubic polynomial whose roots are the complex coordinates A, B, C of the triangle's vertices A, B, C, say $p(\zeta) = (\zeta - A)(\zeta - B)(\zeta - C)$, then the complex coordinates of the foci of the triangle's Steiner circumellipse are the roots of $p'(\zeta)$.

A partial proof of the theorem, using real analytic geometry, for isosceles triangles with a vertex angle of at least 60°, and addressing only one inclusion in part (i), appeared earlier in the Monthly problem [16]. Neither of the two papers addresses the equicevian points on the sidelines.

The goal of this article is to extend the result of Theorem A to a more detailed version for the isosceles triangle that also includes the classification of all equicevian points on the sidelines, and to provide an independent classical geometric proof for the extended theorem. Our extended theorem has two parts.

The following theorem, whose proof is given in Section 1, summarizes the first part of our results on equicevian points of isosceles triangles, namely, those properties that are independent of Steiner's circumellipse. See Figure 1 where, in keeping with our convention that $\angle A$ is the smallest angle and $\angle C$ is the largest, α may be either $\angle A$ or $\angle C$. Here a point is on the extended boundary, or *eboundary*, if it belongs to the support line of a side, and is considered as an *exterior* point of the triangle if it is an exterior point that does not belong to the support line of a side.

Theorem 1. *In every triangle, a vertex is an equicevian point if and only if the triangle is isosceles and the vertex is its top vertex. In an isosceles triangle with top angle α,*

(i) *the distribution of all equicevian points is as follows.*

- *If $\alpha < 53.13°$, there are one interior, three eboundary and one exterior equicevian points.*
- *If $\alpha = 53.13°$, there are one interior, four eboundary and no exterior equicevian points.*
- *If α is between $53.13°$ and $60°$, there are two interior, five eboundary and no exterior equicevian points.*
- *If $\alpha = 60°$, there are one interior, three boundary and no exterior equicevian points.*

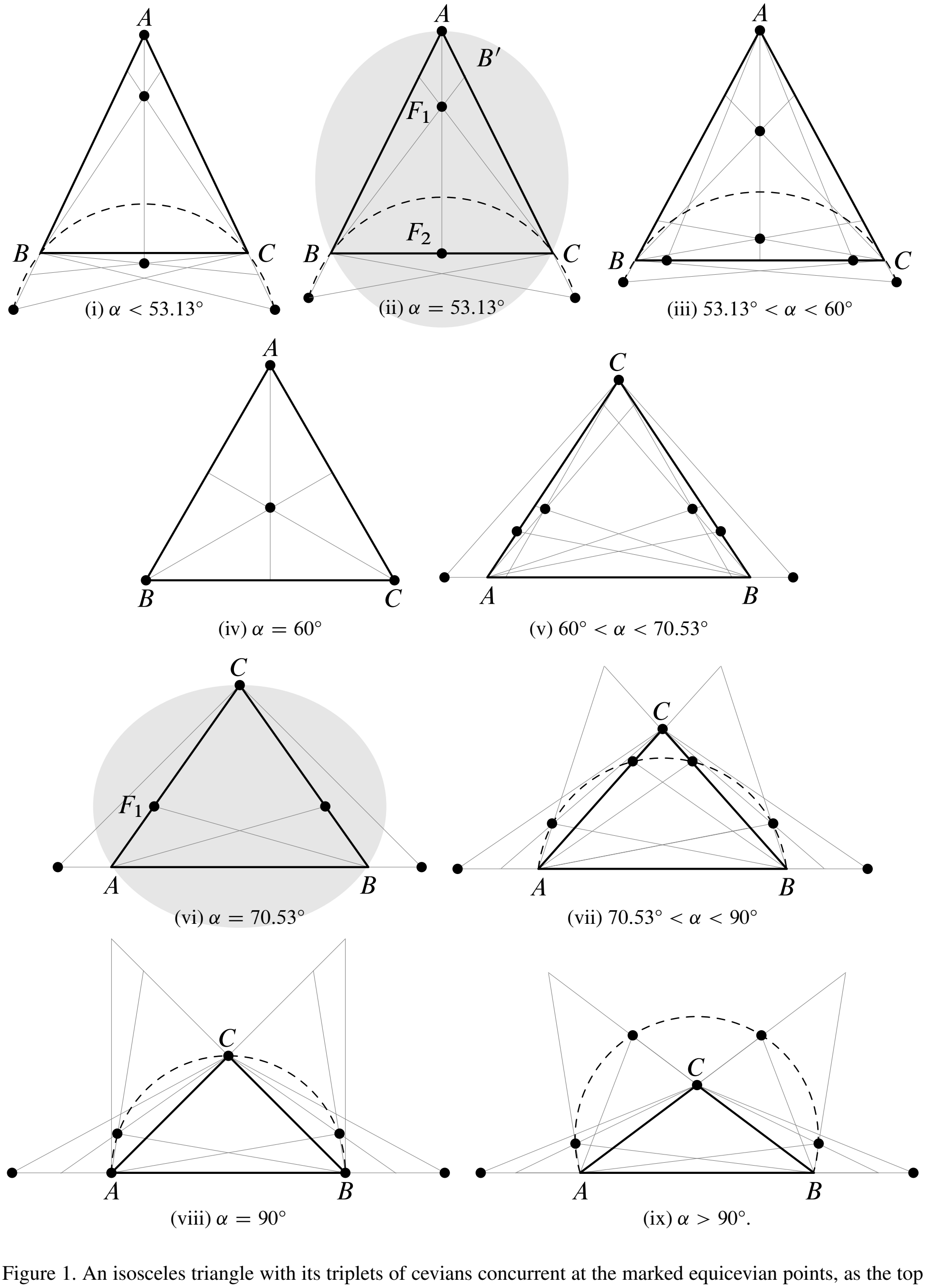

Figure 1. An isosceles triangle with its triplets of cevians concurrent at the marked equicevian points, as the top angle α varies.

- *If α is between $60°$ and $70.53°$, there are two interior, five eboundary and no exterior equicevian points.*
- *If $\alpha = 70.53°$, there are no interior, five eboundary and no exterior equicevian points.*
- *If $\alpha > 70.53°$, there are no interior, five eboundary and two exterior equicevian points.*

(ii) *The common lengths for all triplets of equicevians are distributed as follows.*

- *The longest side lengths appear in all cases, and the shortest occur when $\alpha \geq 53.13°$.*
- *The length of the altitude appears when $\alpha \leq 60°$.*
- *The length of the base times $\frac{1}{2}\sqrt{3}$ appears when $\alpha > 60°$.*

Here the angle $53.13°$ is $2\tan^{-1}(\frac{1}{2})$; it corresponds to the case when $h_a = a$, that is, $b = c = \frac{a}{2}\sqrt{5}$. The angle $70.53°$ is $2\tan^{-1}(\frac{1}{\sqrt{2}})$; it corresponds to the case when $a = b = \frac{c}{2}\sqrt{3}$. All cases in Theorem 1 are illustrated in Figure 1.

The same theorem is complemented by the result of Proposition 7 asserting that, except for some obvious cases, all equicevian points of an isosceles triangle lie on the symmetric of the circumscribed circle relative to the base. Another complement of Theorem 1 is Proposition 2 of Section 1, classifying all equicevian points on the sidelines of a non-isosceles triangle.

The second part of our results on equicevian points in an isosceles triangle is represented by our geometric proof of Theorem A, given in Section 2, where the theorem is restated in an equivalent form for the isosceles triangle as Theorem 6.

Motivation

The remaining part of the introduction includes (1) a basic motivation for Theorem A, by looking at Figure 1; (2) a proof of Theorem A, based on Theorem 1, for the two easier cases illustrated in Figure 1 (ii), (vi), that gives insight into our general geometric proof of Theorem A for the isosceles triangle; and (3) a computation related to the first of the same two cases that relates the result of Theorem 6 to a consequence of Marden's theorem mentioned in [1].

(1) Figure 1 shows that, in almost all cases, an isosceles triangle has two non-eboundary equicevian points. Exception make cases (ii) and (vi) where one or both of the two distinct points belong to the perimeter, and case (iv) where the two points coincide with the centroid of the equilateral triangle. Since, in this last case, the centroid is also the center of the circumscribed circle, one might think that, in general, the two distinct points might be the foci of a circumscribed ellipse.

(2) In case (ii), with the notation from the figure and the known relations between sides,

- in the right triangle ACF_2,

$$\tan C = \frac{AF_2}{CF_2} = \frac{a}{\frac{1}{2}a} = 2,$$

- in the isosceles triangle BCB',

$$\tan B = \tan(180 - 2C) = -\tan 2C = \frac{2\tan C}{\tan^2 C - 1} = \frac{4}{3},$$

- in the right triangle BF_1F_2,

$$F_1F_2 = BF_2 \cdot \tan B = \frac{a}{2} \cdot \frac{4}{3} = \frac{2a}{3} = \frac{2}{3}AF_2,$$

and so the midpoint of F_1F_2 is the centroid. Then

$$AF_1 + AF_2 = \frac{4a}{3} \quad \text{and} \quad BF_2 + BF_1 = \frac{a}{2} + \sqrt{\left(\frac{2a}{3}\right)^2 + \left(\frac{a}{2}\right)^2} = \frac{a}{2} + \frac{5a}{6} = \frac{4a}{3},$$

making F_1 and F_2 the foci of Steiner's circumellipse of major semi-axis $\frac{2a}{3}$. Then the common length of the equicevians through either focus is $a = \frac{3}{2} \cdot \frac{2a}{3}$, validating the remaining part of Theorem A.

In case (vi), by the Law of Cosines, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \left(\frac{3c^2}{2} - c^2\right)\frac{3c^2}{2} = \frac{1}{3}.$$

In the isosceles triangle BCF_1, $CF_1 = 2a\cos C = \frac{2a}{3}$, so the midpoint of F_1F_2 is the centroid. Moreover, $AF_1 + AF_2 = \frac{a}{3} + a = \frac{4a}{3}$ and $CF_1 + CF_2 = 2\frac{2a}{3} = \frac{4a}{3}$ make F_1, F_2 the foci of Steiner's circumellipse of major semi-axis $\frac{1}{2}\frac{4a}{3} = \frac{2a}{3}$. The length of the equicevians through the foci is a, whose expression $\frac{3}{2}\frac{2a}{3}$ validates the result of Theorem A.

(3) A consequence of Marden's theorem mentioned in [1] is that, relative to any cartesian coordinate system centered at the triangle's centroid, the complex coordinates of the two foci of Steiner's circumellipse are

$$F_{1,2} = \pm\sqrt{\frac{2}{3}(A^2 + B^2 + C^2)}.$$

In case (ii) above, the complex coordinates for the vertices of the isosceles triangle with base a and height a, relative to the centroid, are

$$A = \frac{2a}{3}i, \quad B = -\frac{a}{2} - \frac{a}{3}i \quad \text{and} \quad C = \frac{a}{2} - \frac{a}{3}i.$$

We compute

$$A^2 + B^2 + C^2 = -\frac{4a^2}{9} + \frac{2a^2}{4} - \frac{2a^2}{9} = -\frac{a^2}{6}$$

to deduce that

$$F_{1,2} = \pm\sqrt{\frac{2}{3}\left(-\frac{a^2}{6}\right)} = \pm\frac{a}{3}i,$$

which is consistent with the result of Theorem 6 that the focal radius in this case is

$$\frac{2}{3}\sqrt{b^2 - a^2} = \frac{a}{3}.$$

1 Proof of Theorem 1

The proof of Theorem 1 is divided into three parts, by putting the equicevian points of an isosceles triangle in three baskets: those that lie on the supporting lines of the sides, those that lie on the symmetry axis, and neither. The last category is split into two separate cases, depending on how the top angle compares with the 60° angle. We make a few useful observations before proceeding with the announced schedule for the proof.

Let ABC be a triangle with sides $BC = a$, $AC = b$, $AB = c$. Without loss of generality, we may assume that $a \leq b \leq c$; hence the altitudes h_a, h_b, h_c satisfy $h_a \geq h_b \geq h_c$. Suppose the triangle has three concurrent interior cevians of length ℓ. Then ℓ satisfies the following system of inequalities:

$$h_a \leq \ell < \max(b, c) = c, \quad h_b \leq \ell < \max(a, c) = c, \quad h_c \leq \ell < \max(a, b) = b, \tag{1}$$

with solution

$$h_a = \max(h_a, h_b, h_c) \leq \ell < \min(c, c, b) = b. \tag{2}$$

When three equicevians meet at an exterior point, then two of them are exterior cevians and the third is an interior cevian. As the lower bounds of inequalities (1) remain the same for all three such length-ℓ cevians, the two exterior cevians do not have upper bounds. In this way, the only component of the minimum in (2) is the one provided by the interior cevian, or the maximum of its adjacent sides. For an isosceles triangle, this is larger than the upper bound in (2), given by the medium side b, precisely when $a = b < c$ and the interior cevian has c as one of its adjacent sides. In this case, the range for the length ℓ of three equicevians concurrent at an exterior equicevian point is

$$h_a = \max(h_a, h_b, h_c) \leq \ell \leq c, \tag{3}$$

with $\ell > b$ precisely when $a = b < c$ and the interior cevian ℓ is adjacent to the side c.

With the exception of the top vertex of an isosceles triangle, where two triplets of equicevians pass through the point, every equicevian point uniquely determines the triplet of equicevians passing through it. For this reason, counting triplets of equicevians in a triangle is the same as counting equicevian points.

1.1 Equicevian points on the supporting lines of the sides

It is obvious that the top vertex of an isosceles triangle is an equicevian point. Conversely, if a vertex of a triangle is an equicevian point, then the cevians from the other two vertices are equal, and so the equicevian point is the top vertex of an isosceles triangle. In particular, all vertices of an equilateral triangle are equicevian points.

Each altitude of a triangle is shorter than any of the adjacent sides. If it is longer than the opposite side, then that side must be the shortest. If an equicevian point lies on the support line of a side, then two of the cevians are that side, and the third is a cevian from its opposite vertex. In particular, the common length ℓ of the cevians is the length of the side, and this is at least the length of its corresponding altitude. Consequently, the support line of a side has no equicevian points precisely when the side is smaller than its corresponding altitude. For an isosceles triangle, this makes the side the smallest side a, and $a < b = c$; see Figure 1 (i). The condition $a < h_a$ amounts to the top angle being less than $2\tan^{-1}(\frac{1}{2}) \approx 53.13°$.

Since there are up to two cevians of a given length from a vertex, in all cases, the supporting lines of the sides have up to two equicevian points. If they have one, then the cevian from that vertex is an altitude and this has to be equal to the opposite side, with two possibilities: $\alpha = 53.13°$, where the implied equicevian point on line BC, or the midpoint of segment BC, is unique; and $\alpha = 90°$, where the implied equicevian point C is not a unique equicevian point on either AC or BC. See Figure 1 (ii), (viii).

In the leftover cases, each remaining supporting line of a side has precisely two equicevian points. As the top vertex is an equicevian point, and two equal cevians from a vertex are symmetric relative to the altitude from the vertex, the other equicevian points on the supporting lines of the equal sides are the symmetric of the top vertex relative to the altitudes from the base vertices. These are exterior points when $\alpha < 60°$ or $\alpha > 90°$, see Figure 1 (i)–(iii) or (ix), and they fall on the sides when α is between $60°$ and $90°$, see Figure 1 (v)–(vii). Two equicevian points are on the supporting line of the base when $\alpha > 53.13°$. These are interior or exterior to the side, depending on α being smaller or larger than $60°$.

As a parenthesis from the isosceles case, the following proposition completes the classification of all equicevian points on the sidelines of any triangle by providing the result for the non-isosceles triangle. Without loss of generality, the sides a, b, c of such a triangle ABC satisfy $a < b < c$. Let $\bar{a}, \bar{b}, \bar{c}$ denote their respective supporting lines.

Proposition 2. *If ABC is a triangle with side lengths $a < b < c$, then*

(i) *there are two distinct equicevian points on either $\bar{b}$ or $\bar{c}$, and two, one or no equicevian points on $\bar{a}$, depending on the altitude h_a being smaller, equal or greater than a;*

(ii) *both equicevian points on $\bar{c}$ fall outside c. The ones on $\bar{b}$ fall one inside b and the other outside b and opposite to A. The one(s) on $\bar{a}$ fall outside a when $\angle C$ is obtuse, and inside a when $\angle C$ is acute; there are no equicevian points on $\bar{a}$ when $\angle C$ is close to right.*

Proof. (i) When three length-ℓ equicevians meet on the supporting line of a side, then ℓ equals that side. The supporting line of a side ℓ has less than two equicevian points if and only if $\ell \le h_\ell$. As h_ℓ is clearly smaller than the two adjacent sides, the only way that this can possibly happen is when ℓ is the smallest side a.

Part (ii) follows from the comparison between the length ℓ of a cevian through an equicevian point on the support line of a side and the lengths of the other two sides, and the comparison between angle $\angle C$ and the right angle. ■

Back to isosceles, this section finished the proof of the part of Theorem 1 that pertains to eboundary equicevian points. The rest of the proof will describe the non-eboundary equicevian points. These are equicevian points with the property that each belongs to one of the seven open connected two-dimensional regions determined by the sidelines. The next section will show that the number of such regions that a non-eboundary equicevian point may belong to is reduced to up to three and, without loss of generality, to up to two.

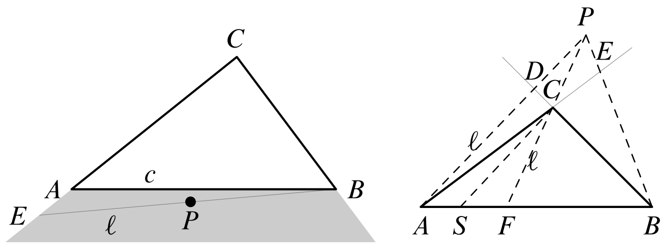

Figure 2. Proof of Lemma 3.

1.2 Equicevian points and the seven open regions determined by the sidelines

We start with the following lemma, showing that a non-eboundary equicevian point of a scalene triangle may lie in only three of the seven open regions determined by the sidelines.

Lemma 3. *Let P be a non-eboundary equicevian point of a scalene triangle. Then P may not belong to either of the following regions:*

(i) *the exterior region bounded by the largest side and the extensions of the other sides;*

(ii) *the interior of either opposite vertical angle.*

Proof. Let ℓ be the length of a cevian through P. (i) Suppose the triangle ABC has sides $a \le b \le c$. Then $\angle A$ is acute. If P is in the exterior region bounded by the largest side c, see Figure 2 (left), then the exterior cevian BE through P is the longest side in the obtuse triangle ABE, so $\ell > c$, a contradiction to (3).

(ii) Without loss of generality, assume that P is an interior point of the vertical angle opposite to $\angle C$, as shown in Figure 2 (left). Then the cevian CF through P is an interior cevian, while the cevians AD and BE are exterior cevians through P. Without loss of generality, we may assume that $\angle CFA \ge 90°$. Since C is between P and F, the parallel from C to PA meets AB at a point S between A and F. The assumption on angle $\angle CFA$ makes it the largest angle in triangle CFS, or its opposite side is the longest side, i.e., $CF < CS$. On the other hand, S between A and B and $CS \parallel AD$ makes $CS < AD$. Putting together, by transitivity, we deduce that $CF < AD$, a contradiction to the assumption that the two cevians through P have the same length ℓ. ∎

The next corollary is the precise version of Lemma 3 for an isosceles triangle with top angle α.

Corollary 4. *Let P be a non-eboundary equicevian point of an isosceles triangle. Then P falls in one of the following three possible cases.*

(i) *If $\alpha < 60°$, then P is in the interior of the top angle.*

(ii) *If $\alpha = 60°$, then P is an interior point of the triangle.*

(iii) *If $\alpha > 60°$, then P is in the interior of a base angle.*

Proof. Note that the three regions in Proposition 3 are uniquely determined by the longest side. If $\alpha < 60°$ or $a < b = c$, then there are two choices for the longest side, b or c, hence two choices of a triplet of open regions for P to belong to. If $\alpha = 60°$ or $a = b = c$, then there are three choices for the longest side, a, b or c, hence three choices of a triplet of open regions for P to belong to. Finally, if $\alpha > 60°$ or $a = b < c$, then c is the unique longest side, and so there is a unique triplet of open regions for P to belong to. Then (i), (ii), (iii) follow easily from here by intersecting the two, three, one triplets of open regions, respectively. ∎

1.3 Equicevian points on the symmetry axis

When P is on the symmetry axis, then one of the cevians through P is the symmetry axis. By (3), this must be the longest altitude, making $\alpha \leq 60°$. The other two cevians through P are reflections of each other in the symmetry axis. Conversely, when $\alpha < 60°$, there are two pairs of reflected cevians from the base vertices that are equal to the symmetry axis; these correspond to two equicevian points on the axis—see Figure 1 (i), (iii). We conclude that there are precisely two equicevian points on the symmetry axis precisely when $\alpha \leq 60°$. This case is clear.

Before continuing with the rest of the proof of Theorem 1, we observe that the cases in Figure 1 (ii), (iv), (vi) are obtained as limiting cases of their adjacent cases. And since the proofs of the limiting cases are nearly identical to the proofs of the adjacent cases, we will not even mention the limiting cases from now on to the end of the section.

1.4 The remaining equicevian points

The remaining non-eboundary equicevian points P are those that do not belong to the symmetry axis. The following lemma shows that such points must belong to the interior of a base angle.

Lemma 5. *There is no non-eboundary equicevian point that lies inside the top angle, outside the triangle and not on the symmetry axis.*

Proof. Let P be such a point, as shown in Figure 3 (left). Without loss of generality, due to symmetry, we may assume that $PB > PC$, hence $\angle BCP > \angle CBP$. Let BB_1 and CC_1 be the exterior equicevians through P. The triangles BCC_1 and CBB_1 have $BC = CB$, obtuse adjacent angles $\angle CBB_1 = \angle BCC_1$ and acute adjacent angles $\angle BCC_1 > \angle CBB_1$. This forces $CC_1 > BB_1$, a contraction with these segments being equicevians. ∎

We finish the proof of Theorem 1 in the remaining case where, by Corollary 4, the non-eboundary non-symmetry axis equicevian point P is interior to one of a base angle. Due to symmetry and without loss of generality, we may assume that this angle is $\angle B$ and that P is in the right open half plane bounded by the symmetry axis, as shown in Figure 3 (right), where, for simplicity, P is chosen to be an interior point of the triangle. The same proof works when P is an exterior point.

Let AD, BE and CF be the three equicevians through P. By scaling and without loss of generality, we may assume that $a = 2$. Denote $CE = x_1$, hence $AE = b - x_1$,

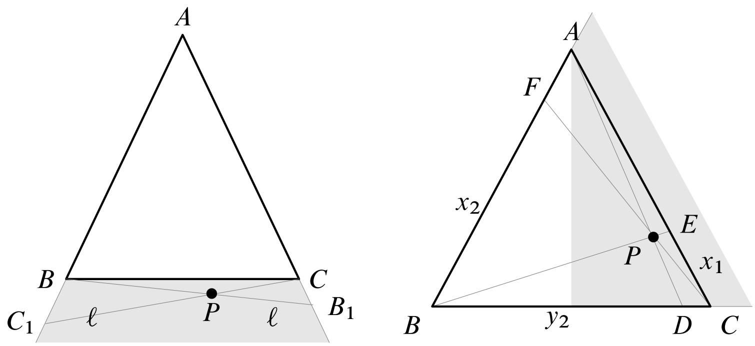

Figure 3. (Left) Proof of Lemma 5. (Right) The remaining case.

and $BF = x_2$, hence $AF = b - x_2$. The condition that P is in the right half plane determined by the symmetry axis makes $x_1 \leq x_2$. The numbers x_1 and x_2 are the roots of the quadratic equation $b^2 x + 4(b - x) - \ell^2 b = bx(b - x)$ induced by Stewart's relations (see [2, Item 308] or [4, Section 1.2 (Exercise 4)]) for the cevians BE and CF. This is the same as

$$bx^2 - 4x + 4b - \ell^2 b = 0, \tag{4}$$

with roots

$$CE = x_1 = \frac{2 - \sqrt{4 - 4b^2 + \ell^2 b^2}}{b} \quad \text{and} \quad BF = x_2 = \frac{2 + \sqrt{4 - 4b^2 + \ell^2 b^2}}{b}.$$

Similarly, $y = y_2 := BD$ satisfies the Stewart relation $b^2 y + b^2(2 - y) - 2\ell^2 = 2y(2 - y)$ corresponding to the cevian AD, which simplifies to the quadratic equation

$$y^2 - 2y + b^2 - \ell^2 = 0, \tag{5}$$

whose largest root is

$$BD = y_2 = 1 + \sqrt{1 - b^2 + \ell^2},$$

and so

$$CD = 2 - y_2 = 1 - \sqrt{1 - b^2 + \ell^2}.$$

The concurrence of the cevians AD, BE and CF is equivalent to Ceva's relation (see, e.g., [4, Section 1.2]), which is given by

$$\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1,$$

which can be expressed in terms of the above notation as

$$\frac{b - x_2}{x_2} \cdot \frac{y_2}{2 - y_2} \cdot \frac{x_1}{b - x_1} = 1.$$

Multiply both sides by the reciprocal of the middle fraction on the left side. Then expand the product of the two fractions on the left side and use the substitution $x_1 x_2 = 4 - \ell^2$, implied by the Viète relation for the quadratic (4), to deduce that

$$\frac{bx_1 - 4 + \ell^2}{bx_2 - 4 + \ell^2} = \frac{2 - y_2}{y_2}.$$

This is equivalent to the derived proportions equation

$$\frac{bx_1 - 4 + \ell^2}{b(x_1 + x_2)/2 - 4 + \ell^2} = \frac{2 - y_2}{1},$$

where the denominators are the averages of the old numerators and denominators. Replace x_1 and y_2 by their exact values obtained earlier, and use the substitution $x_1 + x_2 = \frac{4}{b}$, or the other Viète relation for the quadratic (4), to deduce that

$$\frac{2 - \sqrt{4 - 4b^2 + \ell^2 b^2} - 4 + \ell^2}{2 - 4 + \ell^2} = 1 - \sqrt{1 - b^2 + \ell^2}.$$

By subtracting 1 from both sides, squaring and then multiplying by the denominator, this is equivalent to

$$4 - 4b^2 + \ell^2 b^2 = (\ell^2 - 2)^2(1 - b^2 + \ell^2).$$

This cubic equation in ℓ^2 factors as $\ell^2(\ell^2 - 3)(\ell^2 - b^2) = 0$, with only one acceptable solution, $\ell^2 = 3$. This forces equation (5) to have distinct real roots precisely when its discriminant $4 - b^2$ is positive or $b < 2$. This is equivalent to the top angle $\alpha > 60°$.

To recap, the above proof has shown that a non-eboundary non-symmetry axis equicevian point exists if and only if $\alpha > 60°$, and if it does exist, then it is unique and $\ell = \frac{a}{2}\sqrt{3}$, where a is the base of the isosceles triangle.

By symmetry, this completes the remaining part of the proof of Theorem 1.

2 The geometric proof of Theorem A for the isosceles triangle

The general geometric proof of Theorem A for the isosceles triangle, or rather the part of it that was not covered in Theorem 1, follows along the lines of the proofs for the two simpler cases in Figure 1 (ii), (vi), given in the introduction. These two proofs will be generic for the two possible cases of the general theorem, when the top angle α is either smaller or greater than a $60°$ angle, highlighted by the proof of Theorem 1.

The following is a restatement of Theorem A in a more detailed form for the isosceles triangle, as shown in Figure 1.

Theorem 6. *For an isosceles triangle ABC with sides $a \le b \le c$, the non-eboundary equicevian points and the equicevian points on the open sides when the top angle is $53.13°$ or $70.53°$ are the foci of Steiner's circumellipse. In addition,*

(i) *when $\alpha \le 60°$, the ellipse's focal axis is the symmetry axis and has major semi-axis $\frac{2}{3}\sqrt{b^2 - \frac{1}{4}a^2}$, focal radius $\frac{2}{3}\sqrt{b^2 - a^2}$ and minor semi-axis $\frac{a}{3}\sqrt{3}$;*

(ii) *when $\alpha > 60°$, the ellipse's focal axis is parallel to the base and has minor semi-axis $\frac{2}{3}\sqrt{b^2 - \frac{1}{4}c^2}$, focal radius $\frac{2}{3}\sqrt{c^2 - b^2}$ and major semi-axis $\frac{c}{3}\sqrt{3}$.*

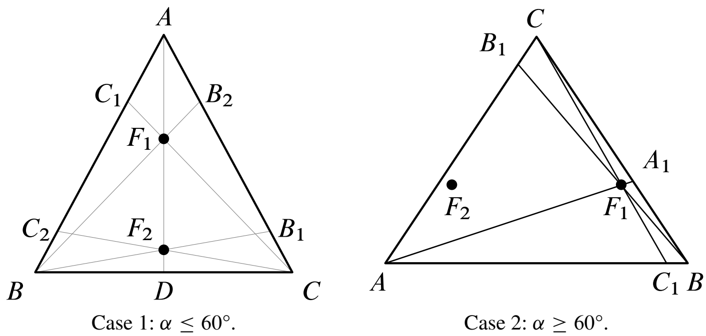

Figure 4. Proof of Theorem 6.

Proof. (i) Without loss of generality, we assume that $a = 2$, hence $b \ge 2$. With the notation from Figure 4 (left), by Section 1.4, the lengths $BC_1 = CB_2$ and the signed lengths $CB_1 = BC_2$ are the roots of equation (4) with $\ell^2 = h_a^2 = b^2 - 1$. These are

$$\frac{2 \pm \sqrt{b^4 - 5b^2 + 4}}{b},$$

with CB_2 taking the positive sign for the radical. We can then compute the ratio

$$\frac{AB_2}{CB_2} = \frac{b - CB_2}{CB_2} = \frac{b^2 - 2 - \sqrt{b^4 - 5b^2 + 4}}{2 + \sqrt{b^4 - 5b^2 + 4}}.$$

Then the van Aubel relation makes

$$\frac{AF_1}{DF_1} = \frac{AB_2}{CB_2} + \frac{AC_1}{BC_1} = \frac{2AB_2}{CB_2},$$

and derived proportions in the equality of the first and last terms yield

$$\frac{AF_1}{h_a} = \frac{2AB_2}{2AB_2 + CB_2} = \frac{b^2 - 2 - \sqrt{b^4 - 5b^2 + 4}}{b^2 - 1 - \frac{1}{2}\sqrt{b^4 - 5b^2 + 4}},$$

which rationalizes to

$$\frac{AF_1}{h_a} = \frac{2}{3} \cdot \frac{b^2 - 1 - \sqrt{b^4 - 5b^2 + 4}}{b^2 - 1}.$$

Similarly,

$$\frac{AF_2}{h_a} = \frac{2}{3} \cdot \frac{b^2 - 1 + \sqrt{b^4 - 5b^2 + 4}}{b^2 - 1}.$$

By averaging these two values and multiplying by h_a, the distance between A and the midpoint of F_1F_2 becomes $\frac{2}{3}h_a$, making the midpoint of F_1F_2 the centroid.

The double of the above distance equation is $AF_1 + AF_2 = \frac{4}{3}h_a$. The last ingredient needed in the proof of Theorem A (i), that $F_{1,2}$ are the foci of the Steiner circumellipse, is $BF_1 + BF_2 = \frac{4}{3}h_a$. To prove this, we compute the lengths BF_1 and BF_2, using the Pythagorean theorem in triangles BDF_1 and BDF_2. Indeed,

$$BF_1 = \sqrt{BD^2 + DF_1^2} = \sqrt{1 + (h_a - AF_1)^2} = \sqrt{1 + h_a^2\Big(\frac{1}{3} - \frac{2}{3}\frac{\sqrt{b^4 - 5b^2 + 4}}{b^2 - 1}\Big)^2}$$
$$= \frac{1}{3}\sqrt{9 + (\sqrt{b^2 - 1} - 2\sqrt{b^2 - 4})^2} = \frac{1}{3}\sqrt{5b^2 - 8 - 4\sqrt{b^4 - 5b^2 + 4}},$$

and similarly,

$$BF_2 = \frac{1}{3}\sqrt{5b^2 - 8 + 4\sqrt{b^4 - 5b^2 + 4}}.$$

Denote $\alpha = 3BF_1$ and $\beta = 3BF_2$. Then, by the above, we have $\alpha^2 + \beta^2 = 10b^2 - 16$ and $\alpha^2\beta^2 = 25b^4 - 80b^2 + 64 - 16(b^4 - 5b^2 + 4) = 9b^4$, or $\alpha\beta = 3b^2$. As a consequence of this,

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = 10b^2 - 16 + 6b^2 = 16(b^2 - 1) = 16h_a^2,$$

hence $\alpha + \beta = 4h_a$, or $BF_1 + BF_2 = \frac{4}{3}h_a$, as needed.

The common length of both triplets of equicevians is h_a, or $\frac{3}{2}$ times the ellipse's major semi-axis. This proves part (ii) of Theorem A.

(ii) As in Section 1, we assume that the triangle has $c = 2$ and $a = b < 2$. With the notation in Figure 4 (right), in Section 1.4, it was proved that the length of the two triplets of equicevians is $\ell = \sqrt{3}$ and that $AC_1 = 1 + \sqrt{4 - b^2}$ and

$$BA_1 = AB_1 = \frac{2 \pm \sqrt{4 - b^2}}{b},$$

where $BA_1 < AB_1$. We apply the van Aubel relation

$$\frac{CF_1}{C_1F_1} = \frac{CA_1}{BA_1} + \frac{CB_1}{AB_1}$$

or

$$\frac{CF_1}{\sqrt{3} - CF_1} = \frac{b - \frac{2-\sqrt{4-b^2}}{b}}{\frac{2-\sqrt{4-b^2}}{b}} + \frac{b - \frac{2+\sqrt{4-b^2}}{b}}{\frac{2+\sqrt{4-b^2}}{b}} = 2.$$

The equation of the first and last terms above has the solution $CF_1 = \frac{2}{3}\sqrt{3}$, thereby locating F_1 on the line through the centroid and parallel to the base. The symmetry makes the centroid to be the midpoint of F_1F_2. Similar computations yield

$$AF_1 = \frac{\sqrt{3}}{3}(2 + \sqrt{4 - b^2}), \quad BF_1 = \frac{\sqrt{3}}{3}(2 - \sqrt{4 - b^2}),$$

hence

$$AF_1 + AF_2 = BF_1 + BF_2 = AF_1 + BF_1 = \frac{4}{3}\sqrt{3} = 2CF_1 = CF_1 + CF_2,$$

making $F_{1,2}$ the foci of Steiner's circumellipse, with major semi-axis $\frac{2}{3}\sqrt{3} = \frac{c}{3}\sqrt{3}$, focal radius $\frac{2}{3}(1 - BC_1) = \frac{2}{3}\sqrt{4 - b^2}$ and minor semi-axis $\frac{2}{3}\sqrt{b^2 - 1}$. ∎

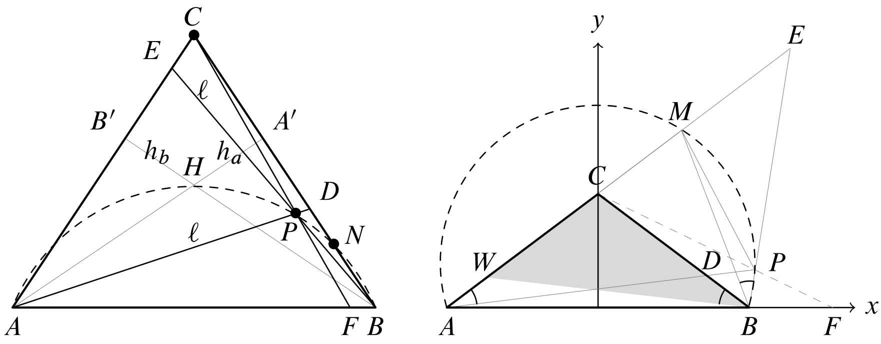

Figure 5. Proof of Proposition 7.

3 Equicevian points and a reflection of the circumcircle

Given a triangle ABC, a point P in the same half plane determined by AB is on the circumcircle if $\angle C = \angle APB$, and P is on the reflection of the circumcircle relative to BC if the two angles are supplementary.

Suppose now that the triangle is isosceles, and let $\mathcal{C}$ be the symmetric of the circumcircle relative to the base of an isosceles triangle. In addition to the base vertices, it is well known (see [2, 10]) that the orthocenter H belongs to $\mathcal{C}$, and we note that the same is true for the second equicevian points on the supporting lines of the equal sides, those other than the top vertex.

The following proposition shows that, except for the obvious cases, all of the remaining equicevian points belong to $\mathcal{C}$ as well—see Figure 1.

Proposition 7. *All equicevian points of an isosceles triangle that do not belong to the circle $\mathcal{C}$ fall in two categories:*

(i) *all equicevian points on the symmetry axis, except for the right angle vertex of an isosceles right triangle, and the centroid of an equilateral triangle;*

(ii) *all equicevian points on the base, except for the base points of an equilateral or an isosceles right triangle.*

Proof. That all equicevian points on the supporting lines of the sides satisfy the result is proved by analyzing Figure 1, which reflects the result of Theorem 1. The same proof is given for all equicevian points on the symmetry axis, using the facts that the only points on the symmetry axis that belong to $\mathcal{C}$ are the orthocenter and the symmetric of the top vertex relative to the base and that two equal cevians from a (base) vertex are symmetric relative to the altitude from the vertex.

It remains to show that all other equicevian points belong to the circle. Note that, by symmetry, Theorem 1, Figure 1 and without loss of generality, each such equicevian point P falls in the shaded region highlighted in Figure 3 (right). This is relabeled in Figure 5 according to the notation in Figure 1, and based on the result in Section 1.4 that $\alpha > 60°$.

The figure is divided into two cases, depending on the point P being either an interior or an exterior point of the triangle.

First, suppose P is an interior equicevian point, as shown in Figure 5 (left), covering the situation in Figure 1 (v). With the notation from the first figure, the congruence of right triangles $AA'D$ and $BB'E$ makes the angles $\angle PDC$ and $\angle PEC$ supplementary, which in turn makes the quadrilateral $PECD$ cyclic. Then $\angle APB = \angle DPE$ is the supplement of $\angle C$, or P belongs to $\mathcal{C}$.

Second, suppose P is at the intersection of the exterior cevians BE, CF and the interior cevian AD, as shown in Figure 5 (right). Symmetries make the triangles ACD and BME congruent to BCW, hence to each other. Then the angles $\angle CAD$ and $\angle MBE$ are congruent, which in turn makes the quadrilateral $ABPM$ cyclic. In particular, P belongs to $\mathcal{C}$. ■

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