
Regular Hexagons Associated to Triangles with Equal Centroids

Zvonko Čerin

Zvonko Čerin, born 1950 in Zagreb, began his studies at the University of Zagreb. The years from 1973 to 1976 he spent at the Louisiana State University in Baton Rouge where he received his PhD. He has published numerous papers on various topics in topology and is now Professor at the University of Zagreb. His recent interest in geometry began under the influence of Professor Baldi from the Università di Torino who introduced him to the world of computers. In his spare time he likes to play rather competitive recreational tennis.

1 Introduction

Let r denote a real number. For points A and B , let r_A^B be the point A when $A = B$, the point B when $A \neq B$ and $r = -1$, and the point P on the line AB such that $|AP|/|PB| = r$ when $A \neq B$ and $r \neq -1$.

In Problem 1493 in the February 1996 issue of *Mathematics Magazine*, Jiro Fukuta observed the following method of generating regular hexagons associated with any triangle ABC .

Fukuta's Method. Let $0 \leq r < 1$. Let $P_1 = r_B^C$, $P_2 = r_C^B$, $P_3 = r_C^A$, $P_4 = r_A^C$, $P_5 = r_A^B$, and $P_6 = r_B^A$. Let $P_7 = P_1$. Let $Q_i P_i P_{i+1}$, ($i = 1, 2, \dots, 6$), be the equilateral triangles built outwards on the sides of the hexagon $P_1 P_2 \dots P_6$. Let $Q_0 = Q_6$ and $Q_7 = Q_1$. Let F_i be the centroid of the triangle $Q_{i-1} Q_i Q_{i+1}$, $i = 1, 2, \dots, 6$. Then $F_1 F_2 \dots F_6$ is a regular hexagon whose center coincides with the centroid of the triangle ABC .

Geometrische Konstruktionen, die auf eine reguläre Figur führen, üben einen besonderen ästhetischen Reiz aus. Zvonko Čerin liefert uns hier ein Beispiel. Er gibt sich zwei reelle Zahlen p und q vor und zwei Dreiecke mit gemeinsamem Schwerpunkt S . Ausgehend von einander entsprechenden Ecken der beiden Dreiecke führt er eine Anzahl von Konstruktionsschritten durch, die im einzelnen von p und q abhängig sind. Es resultiert schliesslich ein Sechseck. Und siehe da: Das Sechseck ist regulär und besitzt den Mittelpunkt S . *ust*

The solution by O.P. Lossers from the Technical University in Eindhoven (The Netherlands) has appeared in the February 1997 issue. His solution uses complex numbers and is about two printed pages long. The aim of this note is to propose the following more general method of creating regular hexagons associated with two triangles with common centroid.

New Method. Let ABC and XYZ be two triangles with common centroid, and let p and q be any real numbers. Let $P_1 = p_A^X$, $P_2 = q_X^A$, $P_3 = p_B^Y$, $P_4 = q_Y^B$, $P_5 = p_C^Z$, and $P_6 = q_Z^C$. Let $P_7 = P_1$. For $i = 1, 2, \dots, 6$, let $Q_i P_i P_{i+1}$ be the six equilateral triangles built on the sides of the hexagon $P_1 P_2 \dots P_6$ all with the same orientation. Let $Q_0 = Q_6$ and $Q_7 = Q_1$. Let G_i be the centroid of the triangle $Q_{i-1} Q_i Q_{i+1}$, $i = 1, 2, \dots, 6$. Then $G_1 G_2 \dots G_6$ is a regular hexagon whose center coincides with the centroid of the triangle ABC .

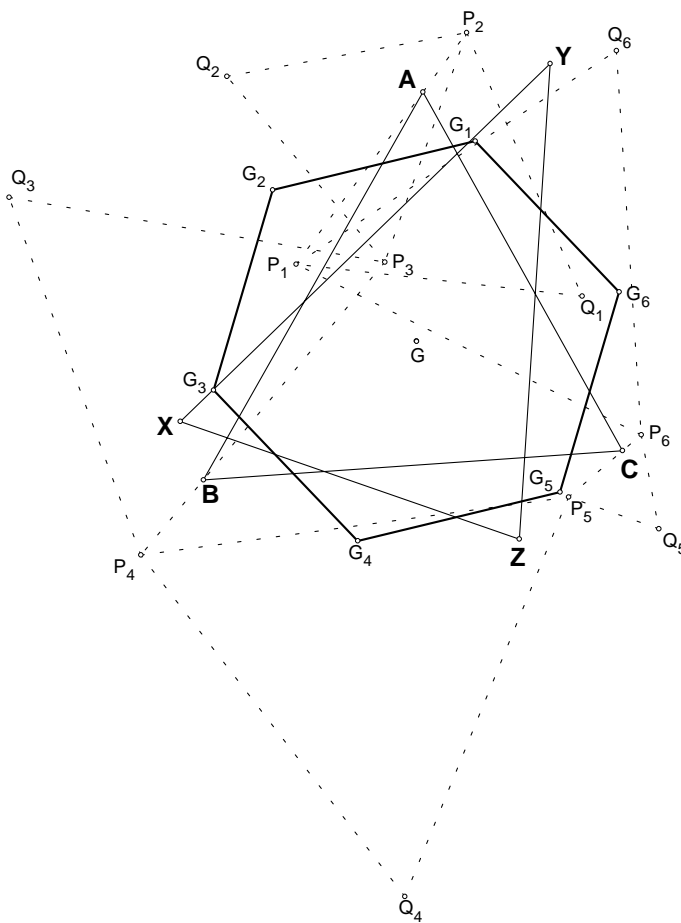


Fig. 1 The New Method: General situation.

Observe that there is no condition on the numbers p and q and that there are two regular hexagons obtained in this way because there are two orientation classes. Moreover, the new method reduces to Fukuta's method if we consider a pair ABC and BCA and put $p = q = r$.

There are many triangles associated with the triangle ABC which have the same centroid. For example, the complementary triangle $A_m B_m C_m$ whose vertices are midpoints of sides, the anticomplementary triangle $A_a B_a C_a$ whose vertices are intersections of parallels through vertices to opposite sides, the first Brocard triangle $A_b B_b C_b$ of ABC , the triangles whose vertices are vertices (centers) of equilateral triangles built on sides either outwards or inwards, and the degenerate triangle GGG , where G is the centroid of ABC .

2 Verification of the new method

In our proof we shall work in the complex plane, and therefore we shall allow ourselves to regard points as complex numbers. In the references we have listed several books that give nice introductions into this technique of proof in geometry.

For an expression Ex that depends on the set of variables S and a permutation b of S let $b(Ex)$ denote the expression obtained by permuting the variables according to b . For example, if $Ex = 3uv^5w^3$ and $b(u) = v$, $b(v) = w$, and $b(w) = u$, then $b(Ex) = 3u^3vw^5$.

Let s and t denote the substitutions $\{A = B, B = C, C = A, X = Y, Y = Z, Z = X\}$ and $\{A = C, B = A, C = B, X = Z, Y = X, Z = Y\}$. We prove the following amusing theorem which could be regarded as the main fact implying the new method.

Theorem 1. *Let the triangles ABC and XYZ have the common centroid G . Let $P_1 = A$, $P_2 = X$, $P_3 = B$, $P_4 = Y$, $P_5 = C$, and $P_6 = Z$. Let the points G_1, G_2, \dots, G_6 be constructed as in the description of the new method. Then $G_1 G_2 \cdots G_6$ is a regular hexagon with center G .*

Proof. Without loss of generality, we can assume that G is the origin of the complex plane, so that the common centroid of ABC and XYZ corresponds to the complex number zero. It follows that $A + B + C = 0$ and $X + Y + Z = 0$.

Set $\Omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Then ω is the complex conjugate of Ω , and one has $\Omega = 1 - \omega$, $\Omega^2 = -\omega$, and $\Omega\omega = 1$. It follows that $Q_1 = \Omega A + \omega X$ and $Q_2 = \Omega X + \omega B$ while $Q_3 = s(Q_1)$, $Q_4 = s(Q_2)$, $Q_5 = t(Q_1)$, and $Q_6 = t(Q_2)$. Also,

$$G_1 = \frac{1}{3}(\Omega A + B + X + \omega Y), \quad G_2 = \frac{1}{3}(\Omega X + Y + B + \omega C),$$

$G_3 = s(G_1)$, $G_4 = s(G_2)$, $G_5 = t(G_1)$, and $G_6 = t(G_2)$.

Since

$$\Omega G_1 = \frac{1}{3}(-\omega A + \Omega B + \Omega X + Y) = \frac{1}{3}(\Omega X + Y + B + \omega(-A - B)) = G_2,$$

and similarly $\Omega G_2 = G_3$, $\Omega G_3 = G_4$, $\Omega G_4 = G_5$, $\Omega G_5 = G_6$, and $\Omega G_6 = G_1$, we conclude that the points G_1, \dots, G_6 are the vertices of a regular hexagon with center at the centroid of ABC and XYZ because rotating by $\frac{\pi}{3}$ around the origin maps the point G_i to a point G_{i+1} for $1 \leq i < 6$ and G_6 to G_1 .

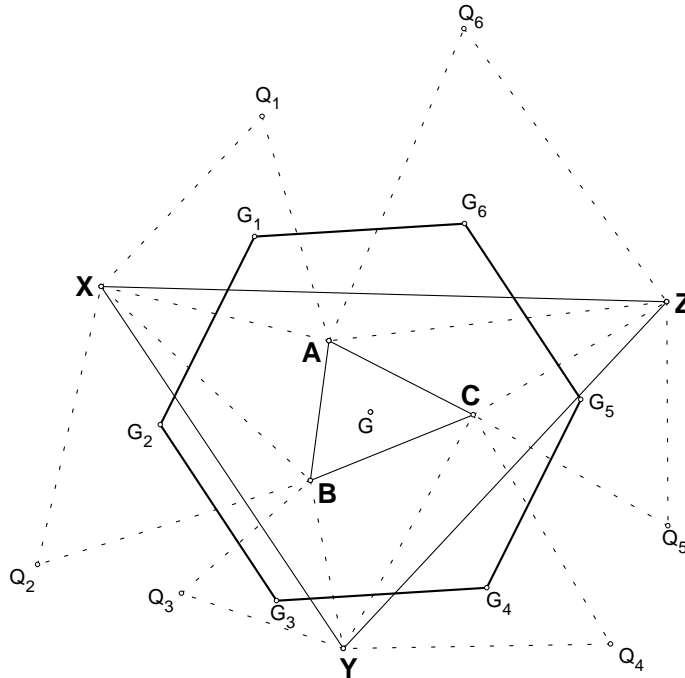


Fig. 2 In Theorem 1 we build equilateral triangles on the sides of the hexagon $AXBYCZ$

In the above calculations we have assumed that the triangle $\omega\Omega J$ is positively oriented, where J is the point corresponding to the complex number -1 , and that we have constructed on the sides of the hexagon $P_1P_2 \cdots P_6$ six positively oriented equilateral triangles. We obtain similar conclusions in the situation when negatively oriented triangles are used.

The fact that $G_1G_2 \cdots G_6$ is a regular hexagon can also be proved using the following idea. Assuming that G is the origin of the complex plane, it is sufficient to show that (i) the triangles $G_1G_3G_5$ and $G_2G_4G_6$ are symmetric to each other with respect to the origin, (ii) the triangle $G_1G_3G_5$ is equilateral, and (iii) its centroid is the origin. Each of these three assertions can be proved in a few lines. \square

The new method follows from the Theorem 1 and the following result.

Theorem 2. *Let the triangles ABC and XYZ have the common centroid G , and let r be any real number. Let $P = r_A^X$, $Q = r_B^Y$, and $R = r_C^Z$. Then G also is the centroid of the triangle PQR .*

Proof. Under the same simplification as in the proof of the Theorem 1, for $r \neq -1$, we have $P = q(A + rX)$, $Q = q(B + rY)$, and $R = q(C + rZ)$, where $q = 1/(r + 1)$. The centroid of the triangle PQR is $q(A + B + C + r(X + Y + Z))/3$. But, this number is zero because $A + B + C = 0$ and $X + Y + Z = 0$. Hence, the centroid of PQR is the origin. For $r = -1$, the proof is trivial. \square

Here are two more results analogous to the Theorem 2 that give us new methods for selecting points P_1, P_2, \dots, P_6 . They use the following notion of a (p, q) -point of a triangle ABC .

Let p and q be any real numbers. Let $(p, q)_{ABC}$ be a short notation for the point p_A^W , where $W = q_B^C$. Notice that $(1, 1)_{ABC}$ is the centroid of ABC .

Theorem 3. *Let p and q be any real numbers. Let $X = (p, q)_{ABC}$, $Y = (p, q)_{BCA}$, and $Z = (p, q)_{CAB}$. Then the triangles ABC and XYZ have the same centroid.*

Proof. Let $f = 1/(p+1)$ and $g = 1/(q+1)$. Obviously we have $X = fA + fgpB + fgpqC$, $Y = s(X)$, and $Z = t(X)$. It follows that the triangle XYZ has the centroid at the complex number $(A+B+C)/3$. Hence, ABC and XYZ have the same centroid. This proof assumes that both p and q are different from -1 . In these exceptional cases the proof is even simpler, and we therefore leave it to the reader. \square

Theorem 4. *Let p and q be any real numbers, and let ABC and XYZ be triangles with the common centroid G . Let $P = (p, q)_{AYZ}$, $Q = (p, q)_{BZX}$, and $R = (p, q)_{CXY}$. Then G also is the centroid of the triangle PQR .*

Proof. We can assume that G is the origin. Then $P = fA + fgpY + fgpqZ$, $Y = s(X)$, and $Z = t(X)$, where $f = 1/(p+1)$ and $g = 1/(q+1)$. It follows that the triangle PQR has centroid $(A+B+C+p(X+Y+Z))/3$. Hence, PQR , ABC and XYZ have the same centroid. This proof again assumes that both p and q are different from -1 . The proof for the cases, when this is not true is easy, and left to the reader. \square

3 Diagonal method

In this section we shall describe another method of associating a regular hexagon with two triangles having the same centroid. This time the points Q_1, Q_2, \dots, Q_6 , instead of being the vertices of the equilateral triangles built on sides of $P_1P_2 \dots P_6$, are the centers of equilateral triangles built on small diagonals of $P_1P_2 \dots P_6$.

Diagonal Method. *Let ABC and XYZ be two triangles with common centroid, and let p and q be any real numbers. Let $P_1 = p_A^X$, $P_2 = q_X^A$, $P_3 = p_B^Y$, $P_4 = q_Y^B$, $P_5 = p_C^Z$, and $P_6 = q_Z^C$. Let $P_7 = P_1$ and $P_8 = P_2$. For $i = 1, 2, \dots, 6$, let Q_i be the centroids of the six equilateral triangles built on the small diagonals P_iP_{i+2} of the hexagon $P_1P_2 \dots P_6$ all with the same orientation. Let $Q_0 = Q_6$ and $Q_7 = Q_1$. Let D_i be the centroid of the triangle $Q_{i-1}Q_iQ_{i+1}$, $i = 1, 2, \dots, 6$. Then $D_1D_2 \dots D_6$ is a regular hexagon whose center coincides with the centroid of the triangle ABC .*

This method is a simple consequence of the following theorem.

Theorem 5. *Let ABC and XYZ be two triangles with common centroid G . Let $P_1 = A$, $P_2 = X$, $P_3 = B$, $P_4 = Y$, $P_5 = C$, and $P_6 = Z$. Let points D_1, D_2, \dots, D_6 be constructed as above from centroids of equilateral triangles built on small diagonals of $P_1P_2 \dots P_6$. Then $D_1D_2 \dots D_6$ is a regular hexagon with center G .*

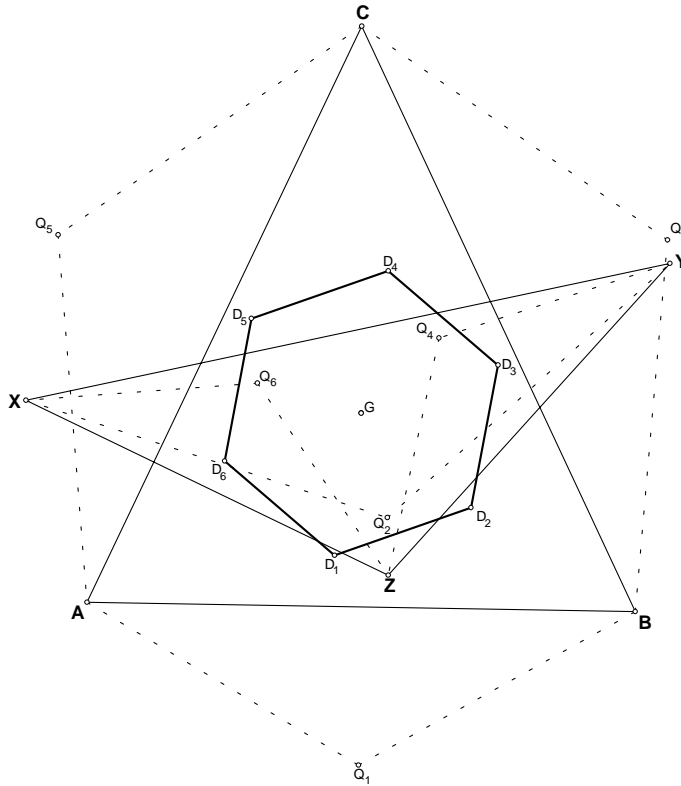


Fig. 3 In Theorem 5 we take centers of equilateral triangles on small diagonals of the hexagon \$AXBYCZ\$.

Proof. Let $\lambda = i \frac{\sqrt{3}}{3}$, $\mu = \frac{1}{2}(1 + \lambda)$, and $\nu = \frac{1}{2}(1 - \lambda)$. Under the same simplifications as in the proof of the Theorem 1, we can show that $Q_1 = \mu A + \nu B$, $Q_2 = \mu X + \nu Y$, $Q_3 = -\nu A + \lambda B$, $Q_4 = -\nu X + \lambda Y$, $Q_5 = -\lambda A - \mu B$, and $Q_6 = -\lambda X - \mu Y$. Also, one has

$$\begin{aligned} D_1 &= (\lambda A + \mu B + \mu X + \nu Y)/3, & D_4 &= -D_1, \\ D_5 &= (\nu A - \lambda B - \lambda X - \mu Y)/3, & D_2 &= -D_5, \\ D_6 &= (\mu A + \nu B + \nu X - \lambda Y)/3, & \text{and } D_3 &= -D_6. \end{aligned}$$

Notice that $\Omega \lambda = -\nu$, $\Omega \mu = \lambda$, and $\Omega \nu = \mu$. It is now obvious that $\Omega D_1 = D_2$, $\Omega D_2 = D_3$, $\Omega D_3 = D_4$, $\Omega D_4 = D_5$, $\Omega D_5 = D_6$, and $\Omega D_6 = D_1$. We conclude that the points D_1, D_2, \dots, D_6 are vertices of a regular hexagon whose center is the common centroid of ABC and XYZ because the rotation by $\frac{\pi}{3}$ around the origin maps each point D_i to a point D_{i+1} for $1 \leq i < 6$ and D_6 to D_1 .

In the above calculations we have again assumed that the triangle $\omega \Omega J$ is positively oriented and that we have constructed on small diagonals of the hexagon $P_1 P_2 \dots P_6$ six positively oriented equilateral triangles. We obtain similar conclusions in the situation where negatively oriented triangles are used. \square

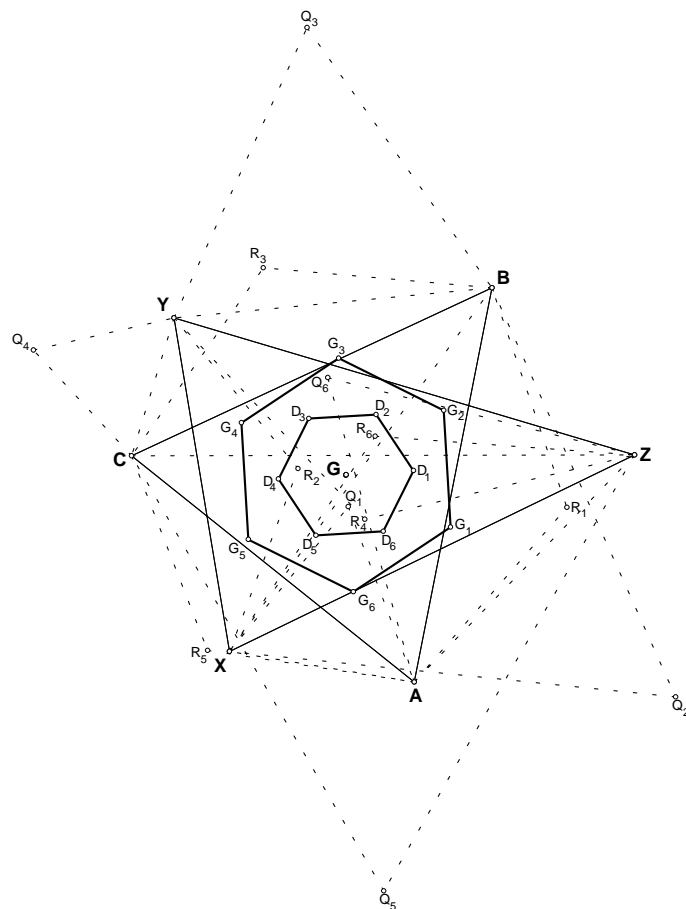


Fig. 4 Regular hexagons from Theorems 1 and 5 together.

We can now easily compute the lengths of sides of regular hexagons from Theorems 1 and 5 and check that their quotient is precisely $\sqrt{3}$.

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