Comment. Math. Helv. 78 (2003) 787–804
0010-2571/03/040787-18
DOI 10.1007/s00014-003-0785-6
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Commentarii Mathematici Helvetici

Geodesic flow on the diffeomorphism group of the circle

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Abstract. We show that certain right-invariant metrics endow the infinite-dimensional Lie group of all smooth orientation-preserving diffeomorphisms of the circle with a Riemannian structure. The study of the Riemannian exponential map allows us to prove infinite-dimensional counterparts of results from classical Riemannian geometry: the Riemannian exponential map is a smooth local diffeomorphism and the length-minimizing property of the geodesics holds.

Mathematics Subject Classification (2000). 35Q35, 58B25.

Keywords. Geodesic flow, diffeomorphism group of the circle.

1. Introduction

The group $\mathcal{D}$ of all smooth orientation-preserving diffeomorphisms of the circle $\mathbb{S}$ is the "simplest possible" example of an infinite-dimensional Lie group [1]. Its Lie algebra $T_{Id}\mathcal{D}$ is the space $C^\infty(\mathbb{S})$ of the real smooth periodic maps of period one. Since $C^\infty(\mathbb{S})$ is not provided with a natural inner product, to endow $\mathcal{D}$ with a Riemannian structure we have to define an inner product on each tangent space $T_\eta\mathcal{D},\ \eta \in \mathcal{D}$. For a Lie group the Riemannian exponential map of any two-sided invariant metric coincides with the Lie group exponential map [1]. It turns out that the Lie group exponential map on $\mathcal{D}$ is not locally surjective cf. [12] so that a meaningful[1] Riemannian structure cannot be provided by a bi-invariant metric on $\mathcal{D}$. We are led to define an inner product on $C^\infty(\mathbb{S})$ and produce a right-invariant metric[2] by transporting this inner product to all tangent spaces $T_\eta\mathcal{D},\ \eta \in \mathcal{D}$, by means of right translations.

In this paper we show that certain right-invariant metrics induce noteworthy Riemannian structures on $\mathcal{D}$. Despite the analytical difficulties that are inher-

[1] A prerequisite of a rigorous study aimed at proving infinite-dimensional counterparts of facts established in classical (finite-dimensional) Riemannian geometry is the use of the Riemannian exponential map as a local chart on $\mathcal{D}$.

[2] To carry out the passage from right-invariant metrics to left-invariant ones, note that a right-invariant metric on $\mathcal{D}$ is transformed by the inverse group operation to a left-invariant metric on $\mathcal{D}$ with the reverse law ($\varphi \star \psi = \psi \circ \varphi$).

ent[3], the existence of geodesics is obtained and their length-minimizing property is established. The paper is organized as follows. In Section 2 we discuss the manifold and Lie group structure of $\mathcal{D}$, Section 3 is devoted to basic properties of the Riemannian structures that we construct on $\mathcal{D}$ (existence and local chart property of the Riemannian exponential map), while in Section 4 we prove the length-minimizing property of geodesics. In the last section we present[4] a choice of a right-invariant matric endowing $\mathcal{D}$ with a deficient Riemannian structure (the corresponding Riemannian exponential map is not a local C^1-diffeomorphism), to emphasize the special features of the previously discussed Riemannian structures. Note that for diffeomorphism groups the existence of geodesics is an open question[5] so that it is of interest to have an example ($M = \mathbb{S}$) where an attractive geometrical structure is available.

2. The diffeomorphism group

In this section we discuss the manifold and Lie group structure of $\mathcal{D}$.

If $\xi(x)$ is a tangent vector to the unit circle $\mathbb{S}$ at $x \in \mathbb{S} \subset \mathbb{C}$, then $\Re\,[\overline{x}\;\xi(x)] = 0$ and

$$u(x) = \frac{1}{2\pi i}\;\overline{x}\;\xi(x) \in \mathbb{R}.$$

This allows us to identify the space of smooth vector fields on the circle with $C^\infty(\mathbb{S})$, the space of real smooth maps of the circle. The latter may be thought of as the space of real smooth periodic maps of period one and will be used as a model for the construction of local charts on $\mathcal{D}$. Note that $C^\infty(\mathbb{S})$ is a Fréchet space, its topology being defined by the countable collection of $C^n(\mathbb{S})$-seminorms: a sequence $u_j \to u$ if and only if for all $n \geq 0$ we have $u_j \to u$ in $C^n(\mathbb{S})$ as $j \to \infty$. $\mathcal{D}$ is an open subset of $C^\infty(\mathbb{S};\mathbb{S}) \subset C^\infty(\mathbb{S};\mathbb{C})$, as one can easily see considering the function defined on $C^\infty(\mathbb{S};\mathbb{S})$ by

$$\Theta(\varphi) = \inf_{x \neq y} \frac{\mid \varphi(x) - \varphi(y) \mid}{\mid x - y \mid}.$$

We will describe a Fréchet manifold structure on $\mathcal{D}$. If $t \mapsto \varphi(t)$ is a C^1-path in $\mathcal{D}$ with $\varphi(0) = Id$, we have $\varphi'(0)(x) \in T_x\mathbb{S}$. Therefore $\varphi'(0)$ is a vector field on $\mathbb{S}$ and we can identify $T_{Id}\mathcal{D}$ with $C^\infty(\mathbb{S})$. If $\varphi \in \mathcal{D}$ is such that $\|\varphi - Id\|_{C^0(\mathbb{S})} < 1/2$,

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[3] $\mathcal{D}$ is a Fréchet manifold so that the inverse function theorem and the classical local existence theorem for differential equations with smooth right-hand side are not granted cf. [12]. Moreover, we deal with weak Riemannian metrics (the family of open sets of $\mathcal{D}$ contains but does not coincide with the family of open sets of the topology induced by the metric) so that even the existence of a covariant derivative associated with the right-invariant metric is in doubt cf. [15].

[4] For a detailed analysis see [7].

[5] For any smooth compact manifold M, both the group of smooth diffeomorphisms of M, Diff(M), and its subgroup formed by the volume-preserving diffeomorphisms, have a Lie group structure [16]. Progress towards the existence of geodesics was made but the understanding is still incomplete [1].

we can define

$$u(x) = \frac{1}{2\pi i}\,\mathrm{Log}\Big(\overline{x}\;\varphi(x)\Big) \in C^\infty(\mathbb{S};\mathbb{R}).$$

Note that $u(x)$ is a measure of the angle between x and $\varphi(x)$. Define a lift $F_\varphi : \mathbb{R} \to \mathbb{R}$ of φ such that

$$u \circ \Pi(\tilde{x}) = F_\varphi(\tilde{x}) - \tilde{x}, \quad \tilde{x} \in \mathbb{R},$$

where $\Pi : \mathbb{R} \to \mathbb{S}$ is the cover map. In the neighborhood

$$U_{\varphi_0} = \{\|\varphi - \varphi_0\|_{C^0(\mathbb{S})} < 1/2\}$$

of $\varphi_0 \in \mathcal{D}$ we are led to define

$$u(x) = \frac{1}{2\pi i}\,\mathrm{Log}\Big(\overline{\varphi_0(x)}\;\varphi(x)\Big), \quad x \in \mathbb{S},$$

and

$$u \circ \Pi(\tilde{x}) = F_\varphi(\tilde{x}) - F_{\varphi_0}(\tilde{x}), \quad \tilde{x} \in \mathbb{R}.$$

For $\varphi \in U_{\varphi_0}$, let $\Psi_{\varphi_0}(\varphi) = u$. We obtain the local charts $\{U_{\varphi_0}, \Psi_{\varphi_0}\}$, with the change of charts given by[6]

$$\Psi_{\varphi_2} \circ \Psi_{\varphi_1}^{-1}(u_1) = u_1 + \frac{1}{2\pi i}\,\mathrm{Log}(\overline{\varphi_2}\;\varphi_1).$$

The previous transformation being just a translation on the vector space $C^\infty(\mathbb{S})$, the structure described above endows $\mathcal{D}$ with a smooth manifold structure based on the Fréchet space $C^\infty(\mathbb{S})$.

A direct computation shows that the composition and the inverse are both smooth maps from $\mathcal{D} \times \mathcal{D}$ to $\mathcal{D}$, respectively from $\mathcal{D}$ to $\mathcal{D}$, so that the group $\mathcal{D}$ is a Lie group. Note that the derivatives of the left-translation

$$L_\eta : \mathcal{D} \to \mathcal{D}, \quad L_\eta(\varphi) = \eta \circ \varphi, \qquad \eta \in \mathcal{D},$$

and right-translation

$$R_\eta : \mathcal{D} \to \mathcal{D}, \quad R_\eta(\varphi) = \varphi \circ \eta, \qquad \eta \in \mathcal{D},$$

are given by

$$L_{\eta^*} : T_{Id}\mathcal{D} \to T_\eta\mathcal{D}, \quad u \mapsto \eta_x \cdot u, \qquad \eta \in \mathcal{D},$$

respectively

$$R_{\eta^*} : T_{Id}\mathcal{D} \to T_\eta\mathcal{D}, \quad u \mapsto u \circ \eta, \qquad \eta \in \mathcal{D}.$$

The Lie bracket on the Lie algebra $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ of $\mathcal{D}$ is

$$[u, v] = -(u_x v - u v_x), \qquad u, v \in C^\infty(\mathbb{S}). \tag{2.1}$$

[6] With $u_1 = \Psi_{\varphi_1}(\varphi)$ and $u_2 = \Psi_{\varphi_2}(\varphi)$ for $\varphi \in U_{\varphi_1} \cap U_{\varphi_2}$, we have

$$(2\pi i)\,u_2 = \mathrm{Log}(\overline{\varphi_2}\varphi) = \mathrm{Log}(\overline{\varphi_2}\varphi_1\overline{\varphi_1}\varphi) = \mathrm{Log}(\overline{\varphi_2}\varphi_1) + \mathrm{Log}(\overline{\varphi_1}\varphi) = \mathrm{Log}(\overline{\varphi_2}\varphi_1) + (2\pi i)\,u_1.$$

Hence $\Psi_{\varphi_2}(\varphi) = \Psi_{\varphi_1}(\varphi) + \frac{1}{2\pi i}\,\mathrm{Log}(\overline{\varphi_2}\;\varphi_1)$ and the change of charts is plain.

Each $v \in T_{Id}\mathcal{D}$ gives rise to a one-parameter group of diffeomorphisms $\{\eta(t,\cdot)\}$ obtained solving

$$\eta_t = v(\eta) \quad \text{in} \quad C^\infty(\mathbb{S}) \tag{2.2}$$

with data $\eta(0) = Id \in \mathcal{D}$. Conversely, each one-parameter subgroup $t \mapsto \eta(t) \in \mathcal{D}$ is determined by its infinitesimal generator $v = \frac{\partial}{\partial t}\,\eta(t)\Big|_{t=0} \in T_{Id}\mathcal{D}$. Evaluating the flow $t \mapsto \eta(t,\cdot)$ of (2.2) at $t = 1$ we obtain an element $\exp_L(v)$ of $\mathcal{D}$. The Lie-group exponential map $v \to \exp_L(v)$ is a smooth map of the Lie algebra to the Lie group. Although the derivative of $\exp_L$ at $0 \in C^\infty(\mathbb{S})$ is the identity, $\exp_L$ is not locally surjective cf. [16]. This failure, in contrast with the case of Hilbert manifolds (see [15]), is due to the fact that the inverse function theorem does not necessarily hold in Fréchet spaces cf. [12]. This inconvenience might seem to indicate that it is preferable to work with the Hilbert manifolds[7]

$$\mathcal{D}^k = \{\eta \in H^k(\mathbb{S}) \text{ is bijective, orientation-preserving, and } \eta^{-1} \in H^k(\mathbb{S})\}, \quad k \geq 2,$$

instead of $\mathcal{D}$. However, $\mathcal{D}^k$ is only a topological group and not a Lie group since for $\eta \in \mathcal{D}^k$ the right composition $R_\eta : \mathcal{D}^k \to \mathcal{D}^k$ is smooth but both the left composition L_η and the inverse map $\varphi \mapsto \varphi^{-1}$ are merely continuous on $\mathcal{D}^k$, without being smooth (see [2]). Therefore, in order to obtain a Lie group structure, we have to consider the Fréchet manifold $\mathcal{D}$.

Let $\mathcal{F}(\mathcal{D})$ be the ring of smooth real-valued functions defined on $\mathcal{D}$ and $\mathcal{X}(\mathcal{D})$ be the $\mathcal{F}(\mathcal{D})$-module of smooth vector fields on $\mathcal{D}$. For $X \in \mathcal{X}(\mathcal{D})$ and $f \in \mathcal{F}(\mathcal{D})$, the Lie derivative $\mathcal{L}_X f$ is defined in a local chart as

$$\mathcal{L}_X f(\varphi) = \lim_{h\to 0} \frac{f(\varphi + h\,X(\varphi)) - f(\varphi)}{h}, \quad \varphi \in \mathcal{D}.$$

If $U \subset \mathcal{D}$ is open and $X, Y : U \to C^\infty(\mathbb{S})$ are smooth, let

$$D_X Y(\varphi) = \lim_{h\to 0} \frac{Y(\varphi + h\,X(\varphi)) - Y(\varphi)}{h}, \quad \varphi \in \mathcal{D}.$$

This leads to a covariant definition of the Lie bracket of $X, Y \in \mathcal{X}(\mathcal{D})$,

$$\mathcal{L}_X Y = [X, Y] = D_X Y - D_Y X.$$

Note that if $\mathcal{X}^R(\mathcal{D})$ is the space of all right-invariant smooth vector fields[8] on $\mathcal{D}$, then the bracket $[X, Y]$ of X, $Y \in \mathcal{X}^R(\mathcal{D})$ is a right-invariant vector field and

[7] $H^k(\mathbb{S})$, $k \geq 0$, is the space of all $L^2(\mathbb{S})$-functions (square integrable functions) f with distributional derivatives up to order k, $\partial_x^i f$ with $i = 0, \dots, k$, in $L^2(\mathbb{S})$. Endowed with the norm

$$\|f\|_k^2 = \sum_{i=0}^{k} \int_{\mathbb{S}} (\partial_x^i f)^2(x)\,dx,$$

$H^k(\mathbb{S})$ becomes a Hilbert space. Note that if $\{\hat{f}(j)\}_{j\in\mathbb{Z}}$ is the Fourier series of $f \in H^k(\mathbb{S})$, then

$$\|f\|_k^2 = \sum_{j\in\mathbb{Z}} \Big(1 + (2\pi j)^2 + \cdots + +(2\pi j)^{2k}\Big) |\ \hat{f}(j)\ |^2\ .$$

[8] $X \in \mathcal{X}^R(\mathcal{D})$ is determined by its value u at Id, $X_\eta = R_{\eta *} u$ for $\eta \in \mathcal{D}$.

$[X, Y]_{Id} = [u, v]$, where $u = X_{Id}$, $v = Y_{Id}$ cf. [16]. This feature explains the minus sign entering formula (2.1) – the commutation operation is defined by this construction carried out with right-invariant vector fields.

3. Riemannian structures on $\mathcal{D}$

We define an inner product on the Lie algebra $T_{Id}\mathcal{D} \equiv C^\infty(\mathcal{D})$ of $\mathcal{D}$, and extend it to $\mathcal{D}$ by right-translation. The resulting right-invariant metric on $\mathcal{D}$ will be a weak Riemannian metric. In this section we discuss the existence of the geodesic flow associated with this metric.

Consider on $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ the $H^k(\mathbb{S})$ inner product

$$\langle u,\, v\rangle_k = \sum_{i=0}^{k} \int_{\mathbb{S}} (\partial_x^i u)\,(\partial_x^i v)\,dx, \quad u, v \in C^\infty(\mathbb{S}),$$

and extend this inner product to each tangent space $T_\eta\mathcal{D}$, $\eta \in \mathcal{D}$, by right-translation, i.e.

$$\langle V, W\rangle_k := \Big\langle R_{\eta^{-1}*}V,\, R_{\eta^{-1}*}W \Big\rangle_k, \quad V, W \in T_\eta\mathcal{D}. \tag{3.1}$$

We have thus endowed $\mathcal{D}$ with a smooth right-invariant metric. Note that the right-invariant metric (3.1) defines a weak topology on $\mathcal{D}$ so that the existence of a covariant derivative which preserves the inner product (3.1) is not ensured on general grounds cf. [15]. We will give a constructive proof of the existence of such a covariant derivative. Let us first note that

$$\langle u,\, v\rangle_k = \int_{\mathbb{S}} A_k(u)\, v\, dx, \quad u, v \in H^k(\mathbb{S}), \qquad k \geq 0,$$

where for every $n \geq 0$, $A_k : H^{n+2k}(\mathbb{S}) \to H^n(\mathbb{S})$ is the linear continuous isomorphism

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}}. \tag{3.2}$$

This enables us to define the bilinear operator $B_k : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$,

$$B_k(u, v) = -\,A_k^{-1}\Big(2v_x A_k(u) + v A_k(u_x)\Big), \qquad u, v \in C^\infty(\mathbb{S}), \tag{3.3}$$

with the property that

$$\langle B_k(u, v),\, w\rangle_k = \langle u,\, [v, w]\rangle_k, \qquad u, v, w \in C^\infty(\mathbb{S}).$$

We can extend B_k to a bilinear map B_k on the space $\mathcal{X}^R(\mathcal{D})$ of smooth right-invariant vector fields on $\mathcal{D}$ by

$$B_k(X, Y)_\eta = R_{\eta*} B_k(X_{Id}, Y_{Id}), \quad \eta \in \mathcal{D},\ X, Y \in \mathcal{X}^R(\mathcal{D}).$$

For $X \in \mathcal{X}(\mathcal{D})$, let us denote by X_η^R the smooth right-invariant vector field on $\mathcal{D}$ whose value at η is X_η.

Theorem 1. *Let $k \geq 0$. There exists a unique Riemannian connection ∇^k on $\mathcal{D}$ associated to the right-invariant metric* (3.1)*, with*

$$(\nabla_X^k Y)_\eta = [X, Y - Y_\eta^R]_\eta + \frac{1}{2}\Big([X_\eta^R, Y_\eta^R]_\eta - B_k(X_\eta^R, Y_\eta^R)_\eta - B_k(Y_\eta^R, X_\eta^R)_\eta\Big),$$

for smooth vector fields X, Y on $\mathcal{D}$.

Proof. The uniqueness of ∇^k is obtained like in classical Riemannian geometry (see e.g. [15]) and all the required properties can be checked from its explicit representation, using the defining identity for B_k. □

The existence of ∇^k enables us to define parallel translation along a curve on $\mathcal{D}$ and to derive the geodesic equation of the metric defined by (3.1). Throughout the discussion, let $J \subset \mathbb{R}$ be an open interval with $0 \in J$. For a C^1-curve $\alpha : J \to \mathcal{D}$, let $\mathrm{Lift}(\alpha)$ be the set of lifts of α to $T\mathcal{D}$. The derivation $D_{\alpha_t} : \mathrm{Lift}(\alpha) \to \mathrm{Lift}(\alpha)$ along α is given in local coordinates by

$$D_{\alpha_t}\gamma = \gamma_t - Q_k(\alpha_t \circ \alpha^{-1}, \gamma \circ \alpha^{-1}) \circ \alpha, \qquad \gamma \in \mathrm{Lift}(\alpha), \tag{3.4}$$

where $Q_k : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is the bilinear operator

$$Q_k(u, v) = \frac{1}{2}\Big(u_x v + u v_x + B_k(u, v) + B_k(v, u)\Big), \quad u, v \in C^\infty(\mathbb{S}).$$

For a C^1-curve $\alpha : J \to \mathcal{D}$ we have

$$\frac{d}{dt}\langle\gamma_1, \gamma_2\rangle_k = \langle D_{\alpha_t}\gamma_1, \gamma_2\rangle_k + \langle\gamma_1, D_{\alpha_t}\gamma_2\rangle_k, \qquad t \in J, \tag{3.5}$$

for all $\gamma_1, \gamma_2 \in \mathrm{Lift}(\alpha)$.

If $\alpha : J \to \mathcal{D}$ is a C^2-curve, a lift $\gamma : J \to T\mathcal{D}$ is called α-parallel if $D_{\alpha_t}\gamma \equiv 0$ on J. This is equivalent to requiring that

$$v_t = \frac{1}{2}\Big(v u_x - v_x u + B_k(u, v) + B_k(v, u)\Big), \tag{3.6}$$

where $u, v \in C^1(J; C^\infty(\mathbb{S}))$ are defined as $\alpha_t \circ \alpha^{-1} = u$, respectively $\gamma \circ \alpha^{-1} = v$. A C^2-curve $\varphi : J \to \mathcal{D}$ satisfying $D_{\varphi_t}\varphi_t \equiv 0$ on J is called a **geodesic**. If $u = \varphi_t \circ \varphi^{-1} \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$, then u satisfies the equation

$$u_t = B_k(u, u), \qquad t \in J. \tag{3.7}$$

Equation (3.7), called the *Euler equation*, is the geodesic equation transported by right-translation to the Lie algebra $T_{Id}\mathcal{D}$. Equations of type (3.7) arise in mathematical physics.

Example 1. For $k = 0$, that is, for the L^2 right-invariant metric, equation (3.7) becomes the inviscid Burgers equation

$$u_t + 3uu_x = 0.$$

This equation is a simplified model for the occurence of shock waves in gas dynamics and can be studied quite explicitly [13]: all solutions but the constant functions have a finite life span. □

Example 2. For $k = 1$, that is, for the H^1 right-invariant metric, equation (3.7) becomes cf. [17] the Camassa–Holm equation

$$u_t + uu_x + \partial_x (1 - \partial_{x^2})^{-1} \left(u^2 + \frac{1}{2} u_{x^2} \right) = 0.$$

This equation is a model for the unidirectional propagation of shallow water waves [3] as well as a model for axially symmetric waves on hyperelastic rods [9]. It has a bi-Hamiltonian structure [11] and is completely integrable [8]. Some solutions of the Camassa–Holm equation exist globally in time [5], whereas others develop singularities in finite time [6]. The blowup phenomenon can be interpreted as a simplified model for wave/rod breaking – the solution (representing the surface water wave or the deformation of the hyperelastic rod) stays bounded while its slope becomes vertical in finite time [4]. □

In a local chart the geodesic equation (3.7) is

$$\varphi_{tt} = P_k(\varphi, \varphi_t),$$

where P_k is an operator that will be specified in the proof of Theorem 2. Assuming for the moment the local existence of geodesics on $\mathcal{D}$ for the metric (3.1), proved below, let us derive a conservation law for the geodesic flow[9]. Observe that any $v \in C^\infty(\mathbb{S}) \equiv T_{Id}\mathcal{D}$ defines a one-parameter group of diffeomorphisms $h^s : \mathcal{D} \to \mathcal{D}$, $h^s(\varphi) = \varphi \circ \exp_L(sv)$, where $\exp_L$ is the Lie-group exponential map. The metric being by construction invariant under the action of h^s, Noether's theorem ensures that if $g : T\mathcal{D} \to \mathbb{R}$ stands for the right-invariant metric, then

$$\frac{\partial g}{\partial \varphi_t}(\varphi, \varphi_t) \left[\frac{dh^s(\varphi)}{ds}\Big|_{s=0} \right]$$

is preserved along the geodesic curve $t \mapsto \varphi(t)$ with $\varphi(0) = Id$ and $\varphi_t(0) = u_0 \in T_{Id}\mathcal{D}$. We compute

$$\frac{dh^s(\varphi)}{ds}\Big|_{s=0} = \varphi_x \cdot v, \qquad \frac{\partial g}{\partial v}(\varphi, v)\,[w] = 2\,\langle v \circ \varphi^{-1}, w \circ \varphi^{-1} \rangle_k,$$

[9] For finite-dimensional Lie groups the geodesic flow of a one-sided invariant metric the angular momentum is preserved [1]. This is a consequence of the invariance of the metric by the action of the group on itself, in view of Noether's theorem. The same reasoning can be carried over to the present infinite-dimensional case.

obtaining that

$$\langle \varphi_t \circ \varphi^{-1}, \varphi_x \circ \varphi^{-1} \cdot v \circ \varphi^{-1} \rangle_k = \langle u_0, v \rangle_k, \qquad v \in C^\infty(\mathbb{S}).$$

Therefore

$$\int_{\mathbb{S}} A_k(u) \cdot \varphi_x \circ \varphi^{-1} \cdot v \circ \varphi^{-1}\, dx = \int_{\mathbb{S}} A_k(u_0) \cdot v\, dx, \qquad v \in C^\infty(\mathbb{S}),$$

where, as before, $u = \varphi_t \circ \varphi^{-1}$. A change of variables yields

$$\int_{\mathbb{S}} A_k(u) \circ \varphi \cdot \varphi_x^2 \cdot v\, dx = \int_{\mathbb{S}} A_k(u_0) \cdot v\, dx, \qquad v \in C^\infty(\mathbb{S})$$

so that, denoting

$$m_k = A_k(u) \circ \varphi \cdot \varphi_x^2, \tag{3.8}$$

we obtain

$$m_k(t) = m_k(0), \qquad t \in [0, T), \tag{3.9}$$

where $\varphi \in C^2([0,T); \mathcal{D})$ is the geodesic for the metric (3.1), starting at $\varphi(0) = Id \in \mathcal{D}$ in the direction $u_0 = \varphi_t(0) \in T_{Id}\mathcal{D}$; as before, $u = \varphi_t \circ \varphi^{-1}$.

To prove the existence of geodesics, we proceed as follows. The classical local existence theorem for differential equations with smooth right-hand side, valid for Hilbert spaces (see [15]), does not hold in $C^\infty(\mathbb{S})$ cf. [12]. However, note that $C^\infty(\mathbb{S}) = \cap_{n \geq 2k+1} H^n(\mathbb{S})$. We use the classical approach to prove that for every $n \geq 2k+1$, the geodesic equation has, on some maximal interval $[0, T_n)$ with $T_n > 0$, a unique solution in $H^n(\mathbb{S})$, depending smoothly on time. A priori $T_n \leq T_{2k+1}$. It turns out that $T_n = T_{2k+1}$ for all $n \geq 2k+1$, a fact that will ensure the existence of geodesics on $\mathcal{D}$ endowed with the right-invariant metric (3.1) for every $k \geq 1$. The peculiarities of the special case $k = 0$ (where this approach is not applicable) are discussed in the last section.

Theorem 2. *Let $k \geq 1$. For every $u_0 \in C^\infty(\mathbb{S})$, there exists a unique geodesic $\varphi \in C^\infty([0,T); \mathcal{D})$ for the metric (3.1), starting at $\varphi(0) = Id \in \mathcal{D}$ in the direction $u_0 = \varphi_t(0) \in T_{Id}\mathcal{D}$. Moreover, the solution depends smoothly on the initial data $u_0 \in C^\infty(\mathbb{S})$.*

Proof. Note that

$$u A_k(u_x) = A_k(u u_x) + C_k^0(u), \quad u \in H^n(\mathbb{S}), \quad n \geq 2k+1,$$

where $C_k^0 : H^n(\mathbb{S}) \to H^{n-2k}(\mathbb{S})$ is a C^∞-operator depending quadratically on $u, u_x, \ldots, \partial_x^{2k} u$. Denoting by $C_k : H^n(\mathbb{S}) \to H^{n-2k}(\mathbb{S})$ the C^∞-operator

$$C_k(u) = -C_k^0(u) - 2u_x A_k(u),$$

we obtain that

$$B_k(u,u) = A_k^{-1} C_k(u) - u u_x, \quad u \in H^n(\mathbb{S}),\ n \geq 2k+1.$$

The geodesic equation (3.7) becomes

$$u_t + uu_x = A_k^{-1} C_k(u),$$

where $u = \varphi_t \circ \varphi^{-1} \in C^\infty(\mathbb{S})$. Letting $v = u \circ \varphi = \varphi_t$, we can write the geodesic equation in a local chart U in $C^\infty(\mathbb{S})$ as

$$\begin{cases} \varphi_t = v, \\ v_t = P_k(\varphi, v), \end{cases} \tag{3.10}$$

where

$$P_k(\varphi, v) = \Big[A_k^{-1} C_k(v \circ \varphi^{-1})\Big] \circ \varphi.$$

The operator

$$(\varphi, v) \mapsto (\varphi,\ P_k(\varphi, v))$$

can be decomposed into $Q_k \circ E_k$ with

$$E_k(\varphi, v) = \Big(\varphi,\ R_\varphi \circ C_k \circ R_{\varphi^{-1}}(v)\Big),$$

and

$$Q_k(\varphi, v) = \Big(\varphi,\ R_\varphi \circ A_k^{-1} \circ R_{\varphi^{-1}}(v)\Big).$$

Specifying the explicit form of $E_k(\varphi, v)$, we see that this operator extends to the space $U_n \times H^n(\mathbb{S})$, where U_n is the open subset of $H^n(\mathbb{S})$ of all functions having a strictly positive derivative.[10] The same argument can be pursued in the case of the operator $G_k : U_n \times H^n(\mathbb{S}) \to U_n \times H^{n-2k}(\mathbb{S})$,

$$G_k(\varphi, v) = \Big(\varphi,\ R_\varphi \circ A_k \circ R_{\varphi^{-1}}(v)\Big),$$

the inverse of Q_k (as a map). Direct calculations confirm that E_k and G_k are both smooth maps from $U_n \times H^n(\mathbb{S})$ to $U_n \times H^{n-2k}(\mathbb{S})$. The regularity of G_k ensures that its Fréchet differential can be computed by calculating directional derivatives. One finds that

$$DG_k(\varphi, v) = \begin{pmatrix} Id & 0 \\ * & \sum_{i=0}^{2k} a_i(\varphi)\, \partial_x^i \end{pmatrix},$$

with

$$a_{2k} = \frac{(-1)^k}{\varphi_x^{2k}}$$

and $a_0, \dots, a_{2k-1}$, all of the form

$$\frac{p_k(\varphi, \varphi_x, \dots, \partial_x^{2k}\varphi)}{\varphi_x^{4k}}$$

[10] Note that $n \geq 2k+1 \geq 3$ so that $H^n(\mathbb{S})$-functions are of class C^2. Explicit calculations show that if $\eta \in U_n$, then η is a $H^n(\mathbb{S})$-homeomorphism of the circle with $\eta^{-1} \in H^n(\mathbb{S})$.

for a polynomial p_k with constant coefficients, while $*$ is a linear differential operator of order $2k$ with coefficients rational functions of the form

$$\frac{q(\varphi, v, \varphi_x, v_x, \dots, \partial_x^{2k}\varphi, \partial_x^{2k}v)}{\varphi_x^{4k}}$$

for some constant coefficient polynomial q. Observe[11] that for every $f \in H^{n-2k}(\mathbb{S})$ there is a unique solution $u \in H^n(\mathbb{S})$ of the ordinary linear differential equation with $H^{n-2k}(\mathbb{S})$-coefficients

$$\sum_{i=0}^{2k} a_i(\varphi)\, \partial_x^i v = f.$$

Taking into account the form of DG_k, we infer that

$$DG_k(\varphi, v) \in \mathrm{Isom}\Big(U^n \times H^n(\mathbb{S}),\, U^n \times H^{n-2k}(\mathbb{S})\Big).$$

Since the differential of the smooth map $G_k : U_n \times H^n(\mathbb{S}) \to U_n \times H^{n-2k}(\mathbb{S})$ is invertible at every point, from the inverse function theorem on Hilbert spaces [15] we deduce that its inverse $Q_k : U_n \times H^{n-2k}(\mathbb{S}) \to U_n \times H^n(\mathbb{S})$ is also smooth. The regularity properties that we just proved for the maps Q_k, E_k, show that P_k is a smooth map from $U_n \times H^n(\mathbb{S})$ to $H^n(\mathbb{S})$.

Regard (3.10) as an ordinary differential equation on $U_n \times H^n(\mathbb{S})$, with a smooth right-hand side, viewed as a map from $U_n \times H^n(\mathbb{S})$ to $U_n \times H^n(\mathbb{S})$. The Lipschitz theorem for differential equations in Banach spaces [15] ensures that for every ball $B(0, \varepsilon_n) \subset H^n(\mathbb{S})$ there exists $T_n = T_n(\varepsilon_n) > 0$ such that for every $u_0 \in B(0, \varepsilon_n)$, the equation (3.10) with data $\varphi(0) = Id$ and $v(0) = u_0$ has a unique solution $(\varphi, v) \in C^\infty([0, T_n); U_n \times H^n(\mathbb{S}))$. Moreover, this solution (φ, v) depends smoothly on the initial data u_0 and can be extended to some maximal existence time $T_n^* > 0$. If $T_n^* < \infty$, we have either that $\limsup_{t \uparrow T_n^*} \|v(t)\|_n = \infty$ or there is a sequence $t_j \uparrow T_n^*$ such that $\varphi(t_j)$ accumulates at the boundary of U_n as $j \to \infty$.

Choose some ball $B(0, \varepsilon_{2k+1}) \subset H^{2k+1}(\mathbb{S})$. We prove now that for any $u_0 \in B(0, \varepsilon_{2k+1}) \cap C^\infty(\mathbb{S})$ there exists a unique geodesic $\varphi \in C^\infty([0, T_{2k+1}); \mathcal{D})$ for the metric (3.1), starting at $\varphi(0) = Id$ in the direction u_0. Since $u_0 \in H^n(\mathbb{S})$ for every $n \geq 2k+1$, it suffices to prove that the solution (φ, v) of equation (3.10) on each $U_n \times H^n(\mathbb{S})$, with data $\varphi(0) = Id$ and $v(0) = u_0$, has the maximal existence time $T_n = T_{2k+1}$. Assuming that $T_{2k+2} < T_{2k+1}$, note that $(\varphi(T_{2k+2}), v(T_{2k+2}))$ is defined in $U_{2k+1} \times H^{2k+1}(\mathbb{S})$ and $\varphi(T_{2k+2})$ is a C^1-diffeomorphism of the circle. Recall the notation $u = v \circ \varphi^{-1}$.

To prove that $\varphi(t)$ converges in $U_{2k+2}(\mathbb{S})$ as $t \uparrow T_{2k+2}$, let us use $\varphi_t = u \circ \varphi$ to compute $\partial_x^{2k}\varphi_t$, $t \in (0, T_{2k+1})$. We obtain

$$\varphi_x \cdot \partial_x^{2k}\varphi_t - \varphi_{tx} \cdot \partial_x^{2k}\varphi = (-1)^k \varphi_x^2 \Big[\varphi_x^{2k-3} m_k(t) + \mathcal{E}_k(v, \varphi)\Big], \quad t \in (0, T_{2k+1}),$$

[11] This can be proved using the Fourier representation of functions in $H^j(\mathbb{S})$, $j \geq 0$.

where $\mathcal{E}_k(v,\varphi)$ is a smooth expression containing only x-derivatives of φ of order $i \le 2k-1$ and x-derivatives of v of order $j \le 2k-1$. Hence

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial_x^{2k}\varphi}{\varphi_x}\Big) &= (-1)^k\,\Big[\varphi_x^{2k-3}\cdot m_k(t)+\mathcal{E}_k(v,\varphi)\Big]\\ &= (-1)^k\,\Big[\varphi_x^{2k-3}\cdot m_k(0)+\mathcal{E}_k(v,\varphi)\Big], \qquad t\in(0,T_{2k+1}),\end{aligned}$$

in view[12] of (3.9). For $t \in (0, T_{2k+1})$ we obtain that

$$\partial_x^{2k}\varphi(t) = (-1)^k\,\varphi_x(t)\int_0^t \Big[\varphi_x^{2k-3}\cdot m_k(0)+\mathcal{E}_k(v,\varphi)\Big]\,ds. \tag{3.11}$$

Since $m_k(0) \in C^\infty(\mathbb{S})$ and

$$(\varphi, v) \in C^\infty([0,T_{2k+1}); U_{2k+1}\times H^{2k+1}(\mathbb{S}))\ \cap\ C^\infty([0,T_{2k+2}); U_{2k+2}\times H^{2k+2}(\mathbb{S})),$$

differentiating (3.11) twice with respect to x, we infer that $(\varphi(t),\varphi_t(t))$ converges in $U_{2k+2}(\mathbb{S})\times H^{2k+2}(\mathbb{S})$ as $t \uparrow T_{2k+2}$. The limit can only be $(\varphi(T_{2k+2}), v(T_{2k+2}))$. Therefore $T_{2k+2} = T_{2k+1}$. This procedure can be repeated for $n = 2k+3$ etc. and the existence of the smooth geodesics on $\mathcal{D}$ is now plain. □

The previous results enable us to define the Riemannian exponential map $\mathfrak{exp}$ for the H^k right-invariant metric ($k \ge 1$). If $\varphi(t;u_0)$ is the geodesic on $\mathcal{D}$, starting at Id in the direction $u_0 \in C^\infty(\mathbb{S})$, note the homogeneity property

$$\varphi(t; su_0) = \varphi(ts; u_0) \tag{3.12}$$

valid for all $t,\ s \ge 0$ such that both sides are well-defined. In the proof of Theorem 2 we saw that there exists $\delta > 0$ so that all geodesics $\varphi(t;u_0)$ are defined on the same time interval $[0,T]$ with $T>0$, for all $u_0 \in \mathcal{D}$ with $\|u_0\|_{2k+1} < \delta$. Hence, we can define $\mathfrak{exp}(u_0) = \varphi(1;u_0)$ on the open set

$$\Big\{u_0 \in \mathcal{D}: \quad \|u_0\|_{2k+1} < \frac{2\,\delta}{T}\Big\}$$

of $\mathcal{D}$, and the map $u_0 \mapsto \mathfrak{exp}(u_0)$ is smooth.

Theorem 3. *The Riemannian exponential map for the H^k right-invariant metric on $\mathcal{D}$, $k \ge 1$, is a smooth local diffeomorphism from a neighborhood of zero on $T_{Id}\mathcal{D}$ to a neighborhood of Id on $\mathcal{D}$.*

Let us first establish

[12] Relation (3.9) was derived assuming the existence of geodesics on $\mathcal{D}$. In the present context, it can be proved as follows. Define m_k by (3.8) and note that it is a polynomial expression in $v, \varphi, v_x, \varphi_x, \dots, \partial_x^{2k}v, \partial_x^{2k}\varphi$, divided by some power of φ_x. Therefore $m_k \in C^\infty([0,T_{2k+1}); H^1(\mathbb{S}))$ and, using (3.10), it can be checked by differentiation with respect to t that $m_k(t) = m_k(0)$ for all $t \in [0, T_{2k+1})$.

Lemma 1. *Let $n \geq 2k+1$ and let (φ, v) be a solution of (3.10) with data $(Id, u_0) \in U_n \times H^n(\mathbb{S})$, defined on $[0,T)$. If there exists $t \in [0,T)$ such that $\varphi(t) \in U_{n+1}$ then $u_0 \in H^{n+1}(\mathbb{S})$.*

Proof. From (3.11) we get

$$\frac{\partial_x^{2k}\varphi(t)}{\varphi_x(t)} = (-1)^k \; m_k(0) \cdot \int_0^t \varphi_x^{2k-3}\,ds + (-1)^k \int_0^t \mathcal{E}_k(v,\varphi)\,ds, \qquad t \in [0,T).$$

If for some $t \in [0,T)$ we have $\varphi(t) \in H^{n+1}(\mathbb{S})$, the fact that φ_x is strictly positive forces $m_k(0) \in H^{n+1-2k}(\mathbb{S})$ and therefore $u_0 \in H^{n+1}(\mathbb{S})$. □

Proof of Theorem 3. Viewing $\mathfrak{exp}$ as a smooth map from a small neighborhood of $0 \in H^n(\mathbb{S})$ to U_n, $n \geq 2k+1$, its differential at $0 \in H^n(\mathbb{S})$, $D\mathfrak{exp}_0$, is the identity map. Indeed, for $v \in H^n(\mathbb{S})$ we have by (3.12) that $\mathfrak{exp}(tv) = \varphi(t;v)$ so that

$$\frac{d}{dt}\,\mathfrak{exp}(tv)\Big|_{t=0} = \frac{d}{dt}\,\varphi(t;v)\Big|_{t=0} = v.$$

As a consequence of the inverse function theorem on Hilbert spaces, we can find open neighborhoods V_{2k+1} and O_{2k+1} of $0 \in H^{2k+1}(\mathbb{S})$ and $Id \in U_{2k+1}$, respectively, such that $\mathfrak{exp} : V_{2k+1} \to O_{2k+1}$ is a C^∞ diffeomorphism with $D\mathfrak{exp}_{u_0} : H^{2k+1}(\mathbb{S}) \to H^{2k+1}(\mathbb{S})$ bijective for every $u_0 \in V_{2k+1}$. We already know from the proof of Theorem 2 that

$$\mathfrak{exp}\Big(V_{2k+1} \cap C^\infty(\mathbb{S})\Big) \subset O_{2k+1} \cap C^\infty(\mathbb{S}),$$

while Lemma 1 ensures that $\mathfrak{exp}$ is a local bijection between these open sets. It remains to show that $\mathfrak{exp}$ is a smooth diffeomorphism from $V_{2k+1} \cap C^\infty(\mathbb{S})$ to $O_{2k+1} \cap C^\infty(\mathbb{S})$.

Let $u_0 \in V_{2k+1} \cap C^\infty(\mathbb{S})$. We know that $D\mathfrak{exp}_{u_0}$ is a bounded linear operator from $H^n(\mathbb{S})$ to $H^n(\mathbb{S})$ for every $n \geq 2k+1$ and we will prove that it is actually a bijection. Then, in view of the inverse function theorem on Hilbert spaces, both $\mathfrak{exp}$ and its inverse are smooth maps on small $H^n(\mathbb{S})$-neighborhoods of $u_0 \in V_{2k+1} \cap C^\infty(\mathbb{S})$, respectively $\mathfrak{exp}(u_0) \in O_{2k+1} \cap C^\infty(\mathbb{S})$. Letting $n \uparrow \infty$, this would show that $\mathfrak{exp}$ is locally a smooth diffeomorphism.

To prove this last step, we use an inductive argument. To start with, $D\mathfrak{exp}_{u_0}$ is a bijection from $H^{2k+1}(\mathbb{S})$ to $H^{2k+1}(\mathbb{S})$ as $u_0 \in V_{2k+1}$. For a fixed $n \geq 2k+1$, assume that the map $D\mathfrak{exp}_{u_0}$ is a bijection from $H^j(\mathbb{S})$ to $H^j(\mathbb{S})$ for all $j = 2k+1, \dots, n$, and let us show that $D\mathfrak{exp}_{u_0}$ is a bijection from $H^{n+1}(\mathbb{S})$ to $H^{n+1}(\mathbb{S})$. First of all, $D\mathfrak{exp}_{u_0}$ is injective as a bounded linear map from $H^{n+1}(\mathbb{S})$ to $H^{n+1}(\mathbb{S})$ since its extension to $H^n(\mathbb{S})$ is injective. To prove its surjectivity as a map from $H^{n+1}(\mathbb{S})$ to $H^{n+1}(\mathbb{S})$, it suffices[13] to see that there is no $w \in H^n(\mathbb{S})$, $w \notin H^{n+1}(\mathbb{S})$, with $D\mathfrak{exp}_{u_0}(w) \in H^{n+1}(\mathbb{S})$. Assume there is such a w. For $\varepsilon > 0$ small enough let $\varphi^\varepsilon(t)$

[13] Since $\mathfrak{exp}$ is a smooth map on $V_{2k+1} \cap H^{n+1}(\mathbb{S})$ we have $D\mathfrak{exp}_{u_0}(H^{n+1}(\mathbb{S})) \subset H^{n+1}(\mathbb{S})$ while the inductive assumption ensures $D\mathfrak{exp}_{u_0}(H^n(\mathbb{S})) = H^n(\mathbb{S})$.

be the solution of (3.10) on U_n starting at Id in the direction $u_0 + \varepsilon w$, with the corresponding $v^\varepsilon \in H^n(\mathbb{S})$). We know that the map $(\varphi^\varepsilon(t), v^\varepsilon(t)) \in U_n \times H^n(\mathbb{S})$ depends smoothly on ε and $t \in [0,1]$. From (3.11) we obtain

$$(-1)^k \frac{\partial_x^{2k}\varphi^\varepsilon(1)}{\varphi_x^\varepsilon(1)} = \Big(m_k(0;u_0) + \varepsilon m_k(0;w)\Big)\int_0^1 (\varphi_x^\varepsilon)^{2k-3}\,ds + \int_0^1 \mathcal{E}_k(v^\varepsilon,\varphi^\varepsilon)\,ds.$$

Differentiating with respect to ε, a calculation shows that $D\mathfrak{exp}_{u_0}(w) = \frac{d}{d\varepsilon}\varphi^\varepsilon(1)\Big|_{\varepsilon=0} \in H^{n+1}(\mathbb{S})$ is possible only if $m_k(0;w) \in H^{n-2k+1}(\mathbb{S})$, i.e. $w \in H^{n+1}(\mathbb{S})$. The obtained contradiction concludes the proof. □

4. Minimizing property of the geodesics

Throughout this section we prove the length minimizing property for the geodesics of the right-invariant metric (3.1) on $\mathcal{D}$ for some fixed $k \geq 1$.

Let V_0 be a vector tangent at $\alpha(0) = \alpha_0$ to a C^2-curve $\alpha : J \to \mathcal{D}$. The parallel transport of V along the curve α is defined as a curve $\gamma \in \mathrm{Lift}(\alpha)$ with $\gamma(0) = V_0$ and $D_{\alpha_t}\gamma \equiv 0$ on J.

Lemma 2. *Let $\alpha : J \to \mathcal{D}$ be a C^2 curve. Given $V_0 \in T_{\alpha_0}\mathcal{D}$, $\alpha_0 = \alpha(0) \in \mathcal{D}$, there exists a unique lift $\gamma : J \to T\mathcal{D}$ which is α-parallel and such that $\gamma(0) = V_0$. Moreover, if γ_1, γ_2 are the unique α-parallel lifts of α with $\gamma_i(0) = V_i \in T_{\alpha_0}\mathcal{D}$, $i = 1,2$, then*

$$\langle \gamma_1(t),\, \gamma_2(t)\rangle_k = \langle V_1, V_2\rangle_k, \qquad t \in J.$$

Proof. In view of (3.3) and (3.6), the equation of parallel transport is

$$v_t = \frac{1}{2}(vu_x - uv_x) - A_k^{-1}\Big[v_xA_k(u) + u_xA_k(v) + \frac{1}{2}vA_k(u_x) + \frac{1}{2}uA_k(v_x)\Big],$$

where $u = \alpha_t \circ \alpha^{-1}$ and $v = \gamma \circ \alpha^{-1}$. Note that the operators

$$(u,v) \mapsto A_k^{-1}[v_xA_k(u) + u_xA_k(v)],$$
$$(u,v) \mapsto \frac{1}{2}A_k^{-1}\Big[vA_k(u_x) + uA_k(v_x)\Big] - \frac{1}{2}(vu_x + uv_x)$$

are smooth from $H^n(\mathbb{S}) \times H^n(\mathbb{S})$ to $H^n(\mathbb{S})$ for every $n \geq 2k+1$. Denote by $\Theta_k(u,v)$ their sum. The equation of parallel transport can be written as

$$v_t + uv_x + \Theta_k(u,v) = 0. \tag{4.1}$$

For a fixed $u \in C^1(J;\mathcal{D})$, the map $v \mapsto \Theta_k(u,v)$ is a bounded linear operator from $H^n(\mathbb{S})$ to $H^n(\mathbb{S})$ for every $n \geq 2k+1$. Viewing (4.1) as linear hyperbolic evolution equation in v with fixed $u \in C^1(J;\mathcal{D})$, it is known (see [14]) that, given $V_0 \in H^n(\mathbb{S})$, $n \geq 2k+1$, there exists a unique solution

$$v \in C(J;\, H^n(\mathbb{S})) \cap C^1(J; H^{n-1}(\mathbb{S}))$$

of (4.1) with initial data $v(0) = V_0$. Letting $n \uparrow \infty$, we infer that, given $V_0 \circ \alpha_0^{-1} \in T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$, there exists a unique solution $v \in C^1(J;\mathcal{D})$ to (4.1) with $v(0) = V_0 \circ \alpha_0^{-1}$.

From (3.5) we deduce that $\langle \gamma_1(t), \gamma_2(t)\rangle_k$ is constant for any α-parallel lifts and the second assertion follows. □

Choose open neighborhoods $\mathcal{W}$ of $0 \in C^\infty(\mathbb{S})$, respectively $\mathcal{U}$ of $Id \in \mathcal{D}$, such that $D\mathfrak{exp}_{u_0} : H^{2k+1}(\mathbb{S}) \to H^{2k+1}(\mathbb{S})$ is bijective for every $u_0 \in \mathcal{W}$ and $\mathfrak{exp}$ is a smooth diffeomorphism from $\mathcal{W}$ onto $\mathcal{U}$, cf. Theorem 3. The map

$$G : \mathcal{D} \times \mathcal{W} \to \mathcal{D} \times \mathcal{D}, \quad (\eta, u) \mapsto \Big(\eta, R_\eta\, \mathfrak{exp}(u)\Big),$$

is a smooth diffeomorphism onto its image. Let $\mathcal{U}(\eta) = R_\eta \mathcal{U} = R_\eta\, \mathfrak{exp}(\mathcal{W})$. If $\varphi \in \mathcal{U}(\eta) - \{\eta\}$, then $\varphi = \mathfrak{exp}(v) \circ \eta$ for some $v \in \mathcal{W}$. Let $v = r\,w$, where $\langle w, w\rangle_k = 1$ and $r \in \mathbb{R}_+$ to define the polar coordinates (r, w) of $\varphi \in \mathcal{U}(\eta)$.

If $\sigma : J_1 \times J_2 \to \mathcal{D}$ is a map such that $\dfrac{\partial^2\sigma}{\partial r^2}$, $\dfrac{\partial^2\sigma}{\partial t\,\partial r}$ and $\dfrac{\partial^2\sigma}{\partial r\,\partial t}$ are continuous, denote by $\partial_1\sigma$ the partial derivative with respect to r and define similarly $\partial_2\sigma$. Both curves $r \mapsto \partial_1\sigma(r,t)$ and $r \mapsto \partial_2\sigma(r,t)$ are lifts of $r \mapsto \sigma(r,t)$. Generally, if γ is a lift of $r \mapsto \sigma(r,t)$, let $(D_1\gamma)(r,t) = (D_{\partial_1\sigma}\gamma)(r)$ and define $D_2\gamma$ similarly. In a local chart we have by (3.4) that

$$D_1\partial_2\sigma = \partial_1\partial_2\sigma - Q_k(\partial_1\sigma, \partial_2\sigma) = D_2\partial_1\sigma \tag{4.2}$$

since Q_k is symmetric. On the other hand, from (3.5) we infer

$$\partial_2 \langle\, \partial_1\sigma,\, \partial_1\sigma\,\rangle_k = 2\,\langle\, D_2\partial_1\sigma,\, \partial_1\sigma\,\rangle_k.$$

The previous relation combined with (4.2) yields

$$\partial_2 \langle\, \partial_1\sigma,\, \partial_1\sigma\,\rangle_k = 2\,\langle\, D_1\partial_2\sigma,\, \partial_1\sigma\,\rangle_k. \tag{4.3}$$

Lemma 3. *Let $\gamma : [a,b] \to \mathcal{U}(\eta) - \{\eta\}$ be a piecewise C^1-curve. Then*

$$l(\gamma) \geq |r(b) - r(a)|,$$

where $l(\gamma)$ is the length of the curve and $(r(t), w(t))$ are the polar coordinates of $\gamma(t)$. Equality holds if and only if the function $t \mapsto r(t)$ is monotone and the map $t \mapsto w(t) \in \mathcal{W}$ is constant.

Proof. We may assume without loss of generality that γ is C^1 (in the general case, break γ up into pieces that are C^1) and that $\eta = Id$ (in view of the right-invariance property of the metric). Observe that $w(t)$ is obtained in a chart by the inversion of $\mathfrak{exp}$ followed by a projection so that the functions $t \mapsto r(t)$ and $t \mapsto w(t)$ are of class C^1.

Let $\sigma(r,t) = \mathfrak{exp}(r\,w(t))$. Let $\varphi(s;z)$ be the solution of (3.10) starting at Id in the direction $z \in C^\infty(\mathbb{S})$. Relation (3.12) yields $\sigma(r,t) = \varphi(r; w(t))$, while the proof of Theorem 2 ensures for every $n \geq 2k+1$ the smooth dependence of $\varphi(s;z)$

on s as well as the smooth dependence of (φ, φ_s) on z in $H^n(\mathbb{S})$. Therefore $\dfrac{\partial^2\sigma}{\partial r^2}$ and $\dfrac{\partial^2\sigma}{\partial t\,\partial r}$ are continuous in the $H^n(\mathbb{S})$-setting for every $n \geq 2k+1$. Furthermore, since

$$\varphi(s; z) = Id + \int_0^s \frac{\partial\varphi}{\partial s}(\xi; z)\, d\xi \quad \text{in} \quad H^n(\mathbb{S}),$$

we have

$$\frac{\partial\varphi}{\partial z}(s; z) = \int_0^s \frac{\partial^2\varphi}{\partial z\partial s}(\xi; z)\, d\xi \quad \text{in} \quad \mathcal{L}(H^n(\mathbb{S}), H^n(\mathbb{S})),$$

thus $\dfrac{\partial^2\varphi}{\partial z\partial s} = \dfrac{\partial^2\varphi}{\partial s\partial z}$. But $t \mapsto w(t) \in H^n(\mathbb{S})$ is a C^1-map so that $\dfrac{\partial^2\sigma}{\partial r\,\partial t}$ is also continuous in the $H^n(\mathbb{S})$-setting for every $n \geq 2k+1$. Letting $n \uparrow \infty$ we obtain that $\dfrac{\partial^2\sigma}{\partial r^2}$, $\dfrac{\partial^2\sigma}{\partial r\,\partial t}$ and $\dfrac{\partial^2\sigma}{\partial t\,\partial r}$ are all continuous in the $C^\infty(\mathbb{S})$-topology.

Note that

$$\gamma'(t) = \frac{\partial\,\sigma}{\partial\,r} \cdot r'(t) + \frac{\partial\,\sigma}{\partial\,t}, \qquad t \in J. \tag{4.4}$$

Since $r \mapsto \sigma(r, t)$ is a geodesic, we obtain by Lemma 2 that

$$\left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,r} \right\rangle_k = \langle w(t), w(t)\rangle_k \equiv 1. \tag{4.5}$$

Let us now show that

$$\left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k \equiv 0. \tag{4.6}$$

Indeed, from (4.3) and (4.5) we obtain that

$$\left\langle D_1 \frac{\partial\,\sigma}{\partial\,t}, \frac{\partial\,\sigma}{\partial\,r} \right\rangle_k = \frac{1}{2}\,\partial_t \left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,r} \right\rangle_k \equiv 0.$$

This, in combination with (3.5), leads to

$$\partial_r \left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k = \left\langle D_1 \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k + \left\langle \frac{\partial\,\sigma}{\partial\,r}, D_1 \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k \equiv 0,$$

since $(D_1 \frac{\partial\,\sigma}{\partial\,r}) = 0$ as $r \mapsto \sigma(r, t)$ is a geodesic. The previous relation yields

$$\left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k (r, t) = \left\langle \frac{\partial\,\sigma}{\partial\,r}, \frac{\partial\,\sigma}{\partial\,t} \right\rangle_k (0, t).$$

But $\sigma(0, t) = Id$ forces $\dfrac{\partial\,\sigma}{\partial\,r}(0, t) = 0$ and therefore (4.6) holds.

Combining (4.4)–(4.6), we obtain

$$\|\gamma'(t)\|_k^2 = |r'(t)|^2 + \left\| \frac{\partial\,\sigma}{\partial\,t} \right\|_k^2 \geq |r'(t)|^2, \qquad t \in [a, b],$$

so that the length of γ is estimated by

$$l(\gamma) \geq \int_a^b |r'(t)|\, dt \geq |r(b) - r(a)|.$$

Since $\|\frac{\partial \sigma}{\partial t}\|_k \equiv 0$ forces $w'(t) = 0$ as $D\mathfrak{exp}_{rw(t)}$ is a bijection from $H^{2k+1}(\mathbb{S})$ to $H^{2k+1}(\mathbb{S})$, the characterization of the equality case follows at once. □

Let us now prove

Theorem 4. *If $\eta, \varphi \in \mathcal{D}$ are close enough, more precisely, if $\varphi \circ \eta^{-1} \in \mathcal{U}$, then η and φ can be joined by a unique geodesic in $\mathcal{U}(\eta)$. Among all piecewise C^1-curves joining η to φ on $\mathcal{D}$, the geodesic is length minimizing.*

Proof. Observe that if $v = \mathfrak{exp}^{-1}(\varphi \circ \eta^{-1})$, then $\alpha(t) = \mathfrak{exp}(tv) \circ \eta$ is the unique geodesic joining η to φ in $\mathcal{U}(\eta)$ cf. Theorem 3.

To prove the second statement, let $\varphi \circ \eta^{-1} = \mathfrak{exp}(r\, w)$ with $\|w\|_k = 1$ and choose $\varepsilon \in (0, r)$. If γ is any piecewise C^1-curve on $\mathcal{D}$ joining η to φ, then γ contains an arc of curve γ^* such that, after reparametrization,

$$\|\mathfrak{exp}^{-1}(\gamma^*(0))\|_k = \varepsilon, \quad \|\mathfrak{exp}^{-1}(\gamma^*(1))\|_k = r,$$

and

$$\varepsilon \leq \|\mathfrak{exp}^{-1}(\gamma^*(t))\|_k \leq r, \quad t \in [0, 1].$$

Lemma 3 yields $l(\gamma^*) \geq r - \varepsilon$, thus $l(\gamma) \geq l(\gamma^*) \geq r - \varepsilon$. The arbitraryness of $\varepsilon > 0$ ensures $l(\gamma) \geq r$. But $l(\alpha) = r$ in view of Lemma 3 and the minimum is attained if and only if the curve is a reparametrization of a geodesic. □

Remark. Specializing $k = 1$ in Theorem 4 we obtain that for the Camassa–Holm model (Example 2 in Section 3) the Least Action Principle holds. That is, a state of the system is transformed to another nearby state through a uniquely determined flow that minimizes the kinetic energy cf. [7].

5. Comments

This section is devoted to a discussion of the $L^2(\mathbb{S})$ right-invariant metric on $\mathcal{D}$, case when the geodesic equation (3.7) is the inviscid Burgers equation

$$u_t + 3uu_x = 0$$

cf. [1]. The crucial difference from the case of a $H^k(\mathbb{S})$ right-invariant metric (with $k \geq 1$) lies in the fact that the inverse of the operator A_k, defined by (3.2), is not regularizing. This feature makes the previous approach inapplicable but the existence of geodesics can be proved by the method of characteristics.

Proposition 1 [7]**.** *For the $L^2(\mathbb{S})$ right-invariant metric on $\mathcal{D}$ there exists a unique smooth geodesic on $\mathcal{D}$ starting at Id in the direction $u_0 \in T_{Id}\mathcal{D}$.*

This result enables one to define the Riemannian exponential map of the $L^2(\mathbb{S})$ right-invariant metric on $\mathcal{D}$, in analogy to the cases considered in the present paper. However,

Proposition 2 [7]**.** *The Riemannian exponential map of the $L^2(\mathbb{S})$ right-invariant metric on $\mathcal{D}$ is not a C^1-diffeomorphism from a neighborhood of zero in $T_{Id}\mathcal{D} \equiv C^\infty(\mathbb{S})$ to a neighborhood of the identity on $\mathcal{D}$.*

The question whether another right-invariant metric could provide $\mathcal{D}$ with a nice Riemannian structure has been positively answered in this paper.

Acknowledgements. We thank L. Hörmander for his helpful interest and V. Schroeder for stimulating conversations.

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(Received: March 28, 2002)

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