Elemente der Mathematik

Dynamic Geometry of Polygons

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1 Introduction

For a given n-sided plane polygon P_0, one may construct a sequence of polygons via
an iterative procedure. A simple example of this sort of construction is the so-called
Kasner polygons [20] – construct a second polygon P_1 whose vertices are the midpoints
of the sides of P_0; construct a third polygon P_2 whose vertices are the midpoints of the
sides of P_1; continuing this process one obtains a sequence of n-gons. An interesting
question is: what can we say about the limit of this sequence? It is obvious that the
size of the polygons will get smaller rapidly, but what about the limiting behavior of
their shapes? For instance, if we consider a triangle, joining the mid-points on each side

Verbindet man die Seitenmittelpunkte eines vorgelegten Dreiecks, so erhält man ein
zum Ausgangsdreieck ähnliches Dreieck. Fährt man in dieser Weise fort, so erhält man
eine Folge immer kleiner werdender ähnlicher Dreiecke, welche zum Schwerpunkt des
Ausgangsdreiecks konvergiert. In der vorliegenden Arbeit untersuchen L.R. Hitt und
X.-M. Zhang analog konstruierte Folgen von beliebigen n-Ecken in der Ebene. Im
Gegensatz zum Fall $n = 3$ ist aber die Beschreibung des Grenzwerts der entsprechen-
den Folge von Polygonen für $n > 3$ im allgemeinen nicht einfach. Auch wenn sich
die Autoren deshalb auf das Studium von Polygonen, welche einen Umkreis besitzen,
beschränken müssen, finden sie eine Fülle interessanter Ergebnisse und stellen da-
bei mannigfache und unvorhergesehene Zusammenhänge zu anderen mathematischen
Gebieten her. *jk*

yields a similar triangle. Consequently all triangles in the resulting sequence are similar triangles. If we consider a quadrilateral, joining the midpoints on each of the four sides produces a parallelogram. All subsequent quadrilaterals are parallelograms. However, when we consider an arbitrary pentagon and produce the sequence of pentagons by this midpoint-construction, the limiting situation can be very complicated.

These kinds of questions have been investigated by many mathematicians including P.J. Davis, G.R. Maclane, A.M. Gleason, G.C. Shephard, J. Douglas, and I.J. Schoenberg [3, 4, 5, 7, 9, 20]. Besides the Kasner polygons, constructing sequences of triangles and studying the geometry of their limits has repeatedly appeared in undergraduate math journals and books as well as in proposed problems in American Mathematical Monthly [1, 6, 10, 12, 21].

When one tries to generalize the problem for triangles to n-sided polygons, there is a great deal of difficulty. Part of the reason might be the fact that every triangle is uniquely associated with two circles – an inscribed circle and a circumscribed circle – whereas a general n-sided polygon ($n \geq 4$) does not always have this property.

In this paper, we will be concerned with the construction of sequences of polygons that are inscribed in a circle (they are called "cyclic" in this case.) We find this kind of construction and the underlying analysis and algebra to be interesting. It is known that the class of cyclic polygons plays an important role in isoperimetric problems for plane polygons. We hope that they can bridge the gap of difficulty between the constructions of sequences of triangles and sequences of general polygons.
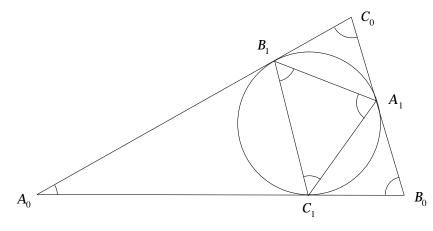
2 Nesting Triangles

We will begin with some well-known simple examples.

Example 2.1   Take any scalene triangle T_0 and construct the inscribed circle. The points of tangency form a second triangle; denote it by T_1. Then construct the inscribed circle for T_1. The points of tangency on the three sides of T_1 form a third triangle T_2. Continuing this process one gets a sequence of triangles $\{T_n\}_{n=0}^{\infty}$. What does the shape of T_n look like as n gets bigger and bigger? The answer is that T_n will be closer to an equilateral triangle! (Of course, if T_0 is equilateral, then every subsequent T_n, $n \geq 1$, will be equilateral.) To confirm the answer, observe from Figure 1 that

$$A_1 = \frac{\pi - A_0}{2}, \quad A_2 = \frac{\pi - A_1}{2} = \left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{A_0}{4},$$

$$A_3 = \left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right) - \frac{A_0}{8}, \quad A_4 = \ldots,$$

$$A_n = \pi \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{2^k} + (-1)^n \frac{A_0}{2^n}, \qquad \text{and} \qquad \lim_{n \to \infty} A_n = \frac{\pi}{3}.$$

Similarly, we have $\lim_{n \to \infty} B_n = \lim_{n \to \infty} C_n = \pi/3$.

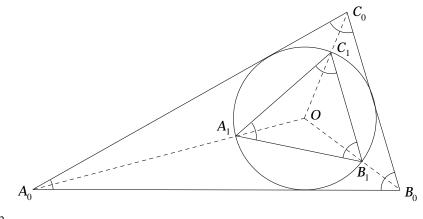This problem has been discussed in [1, 12] using slightly different arguments.

Fig. 1

Example 2.2 A variation of the first example is the following. Let $T_0 = \triangle A_0 B_0 C_0$ be any scalene triangle circumscribing a circle Γ_0 with center O. The line segments $A_0 O$, $B_0 O$, and $C_0 O$ (the angle bisectors of T_0) intersect Γ_0 at points A_1, B_1, and C_1, respectively, and form a second triangle $T_1 = \triangle A_1 B_1 C_1$ that circumscribes a circle Γ_1 with center O_1. Construct a third triangle from T_1 in the same manner, and so on. We have a new sequence of triangles that are nested in a coherent manner. The triangle T_n in this sequence is also approaching the equilateral one as n increases. To see this, notice that from Figure 2

$$A_n = \sum_{k=1}^{n} \frac{\pi}{4^k} + \frac{A_0}{4^n}, \quad n \geq 1,$$

and the rest of the argument is the same as Example 2.1. This problem was posed in [10].

Fig. 2

Remark 2.3

(i) There are certainly other ways to construct nesting triangles whose members become closer and closer to equilateral. For instance, let T_0 again be a scalene triangle with vertices A, B, and C, and let T_1 be the triangle formed by the intersection points of the angle bisectors of T_0 on its three sides. Construct T_2, T_3,... in the same way. Then it can be proved that the shape of T_n will be closer and closer to equilateral as n gets bigger and bigger. Trimble gave a proof for the special case when T_0 is an isosceles triangle in [21]. He claimed that the proof for the general case was rather messy, and hoped for an elegant simple proof.

(ii) Since the limiting triangles in the examples above are actually points, it might be improper to talk about the shapes of the limits of these sequences. However, we may rescale the triangles after they have been constructed. To be more specific, notice that in Example 2.1, the initial triangle T_0 is always inscribed in a circle Γ with radius r. After the construction of the second triangle T_1 in terms of the "incircle" Γ_0 of T_0, we may rescale the circle Γ_0 to the same size as Γ. Consequently, the triangle T_1 will be rescaled too by a similarity transformation S. That is, we change the size of T_1 and preserve its shape! Continue this rescaling every time for the subsequent triangles. It is clear that we are actually dealing with a sequence of triangles that are all inscribed in the same circle (if we identify all congruent circles). See Figure 3.
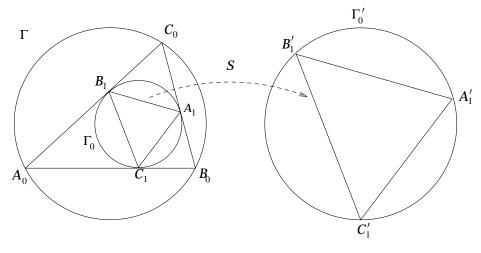

Fig. 3

(iii) From the point of view of dynamic systems, if we denote by $\mathscr{P}$ the set of all triangles that are inscribed in the circle Γ, and let F be a map from $\mathscr{P}$ to itself defined by the construction and rescaling of triangles mentioned above, then the sequence of triangles can be viewed as the orbit of T_0 under the iteration of F. We are interested in the limiting behavior of the orbit of a given triangle. How complicated F can be depends on the construction. In the next section, we will be

concerned with constructions and rescaling of various polygons and will present the map F analytically and algebraically.

3 Sequences of Polygons

We now turn to the general n-sided polygons. Let P be a convex n-sided plane polygon and let TP be the polygon whose vertices are the midpoints of the sides of P. We are interested in the sequence of polygons $\{P, TP, T^2P, \ldots\}$, and especially in the change of the shapes of these polygons. From elementary geometry, it is well-known that if P is a triangle, then TP is similar to P. Therefore, all T^mP's, $m = 1, 2, \ldots$, are similar despite their sizes. If P is a quadrilateral, then TP is a parallelogram, and T^mP is similar to TP if m is odd, to T^2P if m is even. However, when P is a convex k-sided polygon with $k \geq 5$, the problem is far from simple. There have been a number of articles that discussed the possible change of the shape of T^mP with different emphases [3, 4, 5, 7, 20]. Since a sequence of polygons obtained by the midpoint-construction always converges to a point (the centroid of the initial polygon [7]) it might be awkward to talk about the "limiting figure" which is degenerate. Inspired by Remark 2.3(ii) in Section 2, we will make a slight change to the midpoint-construction. We will be concerned with only the sequences of cyclic polygons and will study the limits of these sequences.

Let P be an n-sided polygon with vertices $z_1, z_2, \ldots, z_n$, inscribed in a unit circle Γ centered at O. Assume that O is in the interior of P. Join O to each midpoint on the sides of P and extend these line segments to meet the circle Γ at points $v_1, v_2, \ldots, v_n$, respectively. These new points form a second n-sided polygon that is inscribed in the same circle as P. Denote the second polygon by TP where T represents a transformation on the set of all n-sided polygons inscribed in Γ. We are interested in the sequence of polygons $\{P, TP, T^2P, \ldots\}$ and the limit of T^mP as $m \to \infty$. Since we have stretched every midpoint on the sides of P to the circumference of Γ, the resulting polygons are no longer nested like those discussed in [4, 7]. We will call the sequence of polygons $\{T^mP\}_{m=0}^{\infty}$ the *midpoint-stretching* polygons generated by P.
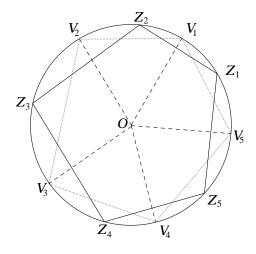

Fig. 4

Theorem A *Every sequence of midpoint-stretching polygons converges to the regular polygon.*

Proof. Let $a_i = z_i z_{i+1}$ be the ith side of P, $i = 1, 2, \ldots, n$ (where $z_{n+1} = z_1$) and let θ_i denote the central angle of Γ subtended by a_i where $1 \leq i \leq n$ and $\sum_{i=1}^{n} \theta_i = 2\pi$. Since we are concerned only with cyclic polygons inscribed in the same circle and interested only in their shapes, we may characterize the polygon P by an n-tuple of real numbers $\Theta = [\theta_1, \theta_2, \ldots, \theta_n]$ and denote by $P(\Theta)$ the corresponding congruence class of cyclic polygons. From the construction of TP whose vertices are $v_1, v_2, \ldots, v_n$, we see that the central angles subtended by the sides of TP are $(\theta_1 + \theta_2)/2, (\theta_2 + \theta_3)/2, \ldots, (\theta_{n-1} + \theta_n)/2$. Setting

$$T = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

we may represent TP by $P(T\Theta)$, and inductively, $T^m P$ by $P(T^m \Theta)$. Because the matrix T is a doubly stochastic matrix and $\sum_{i=1}^{n} \theta_i = 2\pi$, it can be proved (see Section 5) that
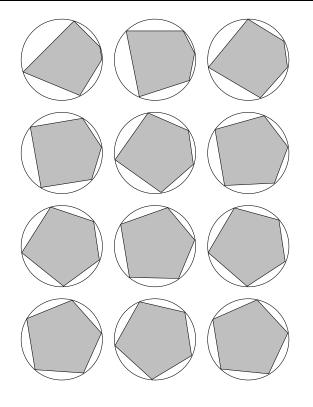
$$\lim_{m \to \infty} [T^m \Theta] = [2\pi/n, 2\pi/n, \ldots, 2\pi/n]. \tag{1}$$

Therefore, the sequence of midpoint-stretching polygons converges to the regular polygon. □

Figure 5 shows the deformation of a cyclic 5-gon $P(\Theta)$ under the iteration of the midpoint-stretching, where $\Theta = (0.1\pi, 0.3\pi, 0.7\pi, 0.55\pi, 0.35\pi)$

Remark 3.1

(i)   Intuitively, T averages every two consecutive central angles of the polygon eventually "evening up" all of them.

(ii) Doubly stochastic matrices are a very important class of matrices. We will briefly review some of their properties that are relevant to our problems in the next section.

(iii) Cyclic polygons play important roles in isoperimetric inequalities. It is well-known that among all n-sided plane polygons with the given set of n-sides, the cyclic one encloses the largest area [13, 16, 23]. Thereafter the study of the isoperimetric inequalities for plane polygons can be reduced to the study of isoperimetric inequalities for cyclic polygons. We have discussed this problem in articles [23, 24] and have introduced the "degree of irregularity" for cyclic polygons which has some probabilistic interpretations. We find the same idea can also be used in the discussion of midpoint-stretching polygons. In particular, we will prove the above limit (1) in Section 5.

Fig. 5

In general, we choose an arbitrary point on the ith side of P and stretch the line segment joining the center of Γ with this point to meet the circumference of Γ at a point v_i, $i = 1, 2, \ldots, n$. Denote the polygon with vertices $v_1, v_2, \ldots, v_n$ by TP, then we may characterize this polygon by the n-tuple of its central angles $\Phi = [\phi_1, \phi_2, \ldots, \phi_n]$, where
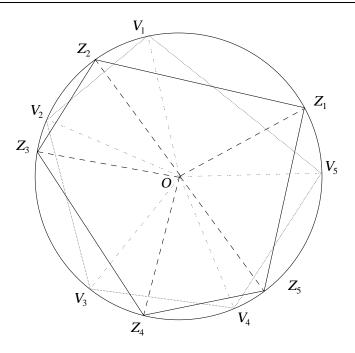
$$\phi_1 = (1 - \lambda_1)\theta_1 + \lambda_2\theta_2, \quad \phi_2 = (1 - \lambda_2)\theta_2 + \lambda_3\theta_3, \ldots, \phi_n = (1 - \lambda_n)\theta_n + \lambda_1\theta_1,$$

and each λ_i ($i = 1, 2, \ldots, n$) is a real number between 0 and 1. We may rewrite TP as $P(\Phi) = P(T\Theta)$ where the transformation T can be expressed as the following matrix:

$$T = \begin{bmatrix} 1 - \lambda_1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \lambda_1 & 0 & 0 & \cdots & 1 - \lambda_n \end{bmatrix}.$$

It is clear that T is a doubly stochastic matrix if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Let us set $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and call the sequence of polygons constructed by the iteration of T, $\{T^m P\}_{m=0}^{\infty}$, the Λ-*stretching* polygons generated by P under T.

Theorem B *The Λ-stretching polygons converge to the regular polygon if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = t$ where $0 < t < 1$.*

Fig. 6

Proof. See Section 5.   □

Note 3.2 Theorem A is simply a special case of Theorem B when $t = 1/2$. Figure 7 shows the convergence of a sequence of 6-gons $P(T^m\Theta)$ with

$$\Theta = (0.25\pi, 0.1\pi, 0.35\pi, 0.75\pi, 0.45\pi, 0.1\pi)$$

under the iteration of the Λ-stretching where $\Lambda = (0.27, 0.27, \ldots, 0.27)$.

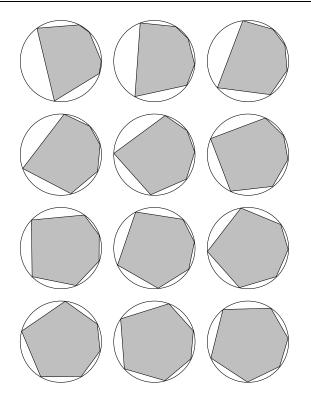## 4 Doubly Stochastic Matrices and Schur-Convex Functions

In what follows, we will briefly review some basic concepts about doubly stochastic matrices and Schur-convex functions. A more detailed study of these topics and their fruitful applications can be found in [17, 19].

Definition 4.1 An $n \times n$ matrix $S = [s_{ij}]$ is called *doubly stochastic* if $s_{ij} \geq 0$ for $1 \leq i, j \leq n$, and

$$\sum_{j=1}^{n} s_{ij} = 1, \quad i = 1, 2, \ldots, n; \qquad \sum_{i=1}^{n} s_{ij} = 1, \quad j = 1, 2, \ldots, n.$$

Example 4.2

  (i) A permutation matrix is a doubly stochastic matrix. (A permutation matrix is a matrix obtained by permuting the rows of the identity matrix.)

 (ii) $S = [s_{ij}]$ with $s_{ij} = \frac{1}{n}$, $1 \leq i, j \leq n$ is a doubly stochastic matrix.

Fig. 7

Let $I^n = I \times I \times \cdots \times I$ (*n* copies), where I is an interval on the real number line $\mathbb{R}$. We are interested in the following special multivariable functions.

Definition 4.3   A function $f : I^n \longrightarrow \mathbb{R}$ ($n > 1$) is called *Schur-convex* if for every doubly stochastic matrix S,

$$f(S\mathbf{x}) \leq f(\mathbf{x}) \tag{2}$$

for all $\mathbf{x} \in I^n$. It is called *strictly Schur-convex* if the inequality is strict and *Schur-concave* (respectively, *strictly Schur-concave*) if the inequality (2) is reversed.

Definition 4.4   A function $f : I^n \longrightarrow \mathbb{R}$ ($n > 1$) is called *symmetric* if for every permutation matrix P,

$$f(P\mathbf{x}) = f(\mathbf{x})$$

for all $\mathbf{x} \in I^n$.

Every Schur-convex function is a symmetric function. Because if P is a permutation matrix, so is its inverse P^{-1}. Hence if f is Schur-convex, then

$$f(\mathbf{x}) = f(P^{-1}(P\mathbf{x})) \leq f(P(\mathbf{x})) \leq f(\mathbf{x}).$$

This shows that $f(P\mathbf{x}) = f(\mathbf{x})$ for every permutation matrix P, so f is symmetric. On the other hand, it is not hard to see that not every symmetric function is a Schur-convex

function (cf. [19, p. 258]). Similarly, every Schur-concave function is symmetric, but not conversely. However, we have the following so-called Schur's condition.

Lemma 4.5 ([19, p. 259]) *Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be symmetric and have continuous partial derivatives on I^n where I is an open interval. Then $f : I^n \longrightarrow \mathbb{R}$ is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{3}$$

on I^n. It is strictly Schur-convex if (3) is a strict inequality for $x_i \neq x_j$, $1 \leq i, j \leq n$.

Since $f(\mathbf{x})$ is symmetric, Schur's condition can be reduced to ([17, p. 57])

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0, \tag{4}$$

and f is strictly Schur-convex if (4) is a strict inequality for $x_1 \neq x_2$. The Schur's condition that guarantees a symmetric function being Schur-concave is the same as (3) or (4) except for the direction of the inequality.

Example 4.6 Let $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$, where $0 < \theta_i < \pi$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \theta_i = \pi$. Define

$$F(\Theta) = n \sin\frac{\pi}{n} - \sum_{i=1}^{n} \sin \theta_i,$$

then it is easy to verify by (4) that F is a strictly Schur-convex function because

$$(\theta_1 - \theta_2)\left(\frac{\partial F}{\partial \theta_1} - \frac{\partial F}{\partial \theta_2}\right) = (\theta_1 - \theta_2)^2 \sin \theta^* > 0 \qquad \text{for} \quad \theta_1 \neq \theta_2,$$

where θ^* is between θ_1 and θ_2. Moreover,

$$F(\Theta) \geq 0, \text{ with equality holding if and only if } \theta_1 = \theta_2 = \cdots = \theta_n = \pi/n. \tag{5}$$

5 Degree of Irregularity and Isoperimetric Inequalities

Let us recall the classical isoperimetric inequality for an n-sided plane polygon P_n:

$$L_n^2 - 4d_n A_n \geq 0, \tag{6}$$

where L_n is the perimeter of P_n, A_n is the area of the domain enclosed by P_n, and $d_n = n \tan \frac{\pi}{n}$. Equality holds if and only if P_n is regular ([13, 16, 18, 23]).

In geometry, we call $L_n^2 - 4d_n A_n$ the *isoperimetric deficit* of the polygon P_n. It measures the deviation of P_n from the "regularity". Among all n-sided plane polygons with given n sides, the cyclic polygon encloses the largest area ([13, 16, 23]). So to investigate the isoperimetric inequalities for plane polygons, we need to pay attention to cyclic polygons only. Now, for a cyclic polygon P_n with lengths of sides $a_1, a_2, \ldots, a_n$, if $\theta_1, \theta_2, \ldots, \theta_n$ are the central angles subtended by the n sides respectively, and $0 < \theta_i < \pi$ for $i = 1, 2, \ldots, n$, we denote P_n by $P_n(\Theta)$ where $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$, $\sum_{i=1}^{n} \theta_i = 2\pi$, and introduce the *degree of irregularity* for $P_n(\Theta)$.

Definition 5.1 Let $P_n(\Theta)$ be an *n*-sided cyclic polygon. The degree of irregularity of $P_n(\Theta)$ is defined as

$$\text{Deg}[P_n(\Theta)] = n \sin \frac{\pi}{n} - \sum_{i=1}^{n} \sin(\theta_i/2). \qquad (7)$$

It is clear that $\text{Deg}[P_n(\Theta)] \geq 0$ for all $P_n(\Theta)$, and $\text{Deg}[P_n(\Theta)] = 0$ only when $P_n(\Theta)$ is regular. If $P_n(\Theta)$ is inscribed in a circle of radius r, then $\text{Deg}[P_n(\Theta)]$ times $2r$ is simply the difference between the perimeters of $P_n(\Theta)$ and the regular *n*-sided polygon inscribed in the same circle. We have noticed that the equality in (6) distinguishes only the regular polygon among all isoperimetric polygons with the same number of sides. The following theorem allows us to compare any two *n*-sided cyclic polygons in terms of their degrees of irregularity.

Theorem 5.2 *Let $P_n(\Theta)$ and $Q_n(\bar{\Theta})$ be two n-sided cyclic polygons. If there is a doubly stochastic matrix S such that $\bar{\Theta} = S\Theta$, then*

$$\text{Deg}[Q_n(\bar{\Theta})] \leq \text{Deg}[P_n(\Theta)].$$

Equality holds if and only if Θ and $\bar{\Theta}$ differ by a permutation.

Proof. Let $F(\Theta) = \text{Deg}[P_n(\Theta)]$, then $F(\bar{\Theta}) = \text{Deg}[Q_n(\bar{\Theta})]$. From Example 4.6 we see that F is Schur-convex, hence

$$F(\bar{\Theta}) = F(S\Theta) \leq F(\Theta). \qquad \square$$

Remark 5.3

(i) $\text{Deg}[P_n(\Theta)]$ is a homothetic invariant for cyclic polygons. Hence if $P_n(\Theta)$ and $Q_n(\bar{\Theta})$ are two *n*-sided homothetic cyclic polygons, that is, $\Theta = \bar{\Theta}$ (they have the same shape but may have different size), then they have the same degree of irregularity. On the other hand, $\text{Deg}[P_n(\Theta)] = \text{Deg}[Q_n(\bar{\Theta})]$ does not necessarily imply the homotheticity of $P_n(\Theta)$ and $Q_n(\bar{\Theta})$ because Θ and $\bar{\Theta}$ may differ by a permutation so that the two cyclic polygons may not be obtained from each other by a homothetic transformation. If we consider the class of all *n*-sided cyclic polygons ($n > 3$), the degree of irregularity can be used to classify the polygons according to their deviations from the regularity.

(ii) There are many different ways to define the degree of irregularity for a polygon. For more examples see [23, 24].

Theorem C *Let $P(\Theta)$ be an n-sided cyclic polygon, where $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$ is the set of the central angles subtended by the sides of $P(\Theta)$. If T is an n by n doubly stochastic matrix other than a permutation matrix, then $P(T^m\Theta)$ converges to the regular polygon as $m \to \infty$.*

Proof. Let $F(\Theta)$ denote the degree of irregularity of the initial polygon $P(\Theta)$. If the matrix T is doubly stochastic other than a permutation, then

$$F(\Theta) > F(T\Theta) > F(T^2\Theta) > \cdots = 0.$$

That is, the degree of irregularity of $T^m P$ is monotonically decreasing and bounded from below by 0. Therefore the limit of $\{F(T^m\Theta)\}_{m=0}^{\infty}$ exists, and it is not hard to see that we must have

$$\lim_{m \to \infty} F(T^m\Theta) = 0.$$

This shows that the limit of the sequence $\{T^m P\}$ has to be the regular polygon. $\qquad\square$

Proof of Theorem B. The matrix T used in a Λ-stretching is a doubly stochastic matrix if all the components of Λ are the same real number t, where $0 < t < 1$. Theorem B follows from Theorem C immediately. $\qquad\square$

Remark 5.4

(i) In fact, the degree of irregularity quantifies the shapes of cyclic polygons as the isoperimetric deficit does for general polygons. Theorems A and B provide an algorithm for changing a cyclic polygon to the regular polygon by reducing its degree of irregularity.

(ii) If we view the vertices of a cyclic polygon P as a set of points $[z_1, z_2, \ldots, z_n]$ that are distributed on the circle at random, Λ-stretching is simply a redistribution of these points. If one continues the process of this redistribution, then these n points will be evenly spread on the circle. For some probability problems that involve inscribed or circumscribed polygons, refer to [17] and references therein.

Let us recall the nesting triangles that were discussed in Section 2. Up to a rescaling of their circumscribed circles to the same size, those triangles can be viewed as special cases of sequences of cyclic n-sided polygons when $n = 3$. Since we are concerned with only their shapes and the rescaling involves only simple similarity transformations, we may treat them uniformly by the conclusion of Theorem C.

First of all, for Example 2.1, let $P(\Theta)$ be the initial triangle T_0 and $P(\Phi) = P(T\Theta)$ be the second triangle T_1 upon a rescaling. Then from Figure 8 it is clear that

$$A_0 = \theta_1/2, \; B_0 = \theta_2/2, \; C_0 = \theta_3/2, \qquad A_1 = \phi_1/2, \; B_1 = \phi_2/2, \; C_1 = \phi_3/2.$$
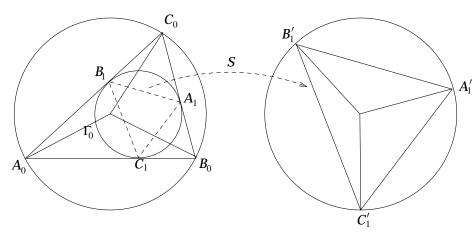
From the earlier observation in Section 2, we have

$$\phi_1 = 2A_1 = \pi - A_0 = \pi - \frac{\theta_1}{2} = \frac{\theta_2 + \theta_3}{2}.$$

Similarly, $\phi_2 = (\theta_3 + \theta_1)/2$ and $\phi_3 = (\theta_1 + \theta_2)/2$.

That is, the nesting triangles in Example 2.1, up to similarity transformations, are actually midpoint-stretching triangles. By Theorem A, they converge to the equilateral one.

Secondly, we give an alternate proof to Example 2.2. Although they might not be regarded as special Λ-stretching triangles, they can be viewed as a simple case of Theorem C.

Fig. 8

Let $P(\Theta)$ be the initial triangle and $P(\Phi) = P(T\Theta)$ be the the second triangle (upon a rescaling) in Example 2.2, where $\Theta = (\theta_1, \theta_2, \theta_3)$ and $\Phi = (\phi_1, \phi_2, \phi_3)$ are the sets of central angles of the two triangles, respectively. A direct calculation shows that the matrix T is the following:

$$T = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix},$$

which is clearly a doubly stochastic matrix. Hence the limiting shape of the triangles in Example 2.2 must be an equilateral triangle by Theorem C.

In elementary geometry, there are some equilateral triangles associated with a given triangle. Perhaps two of the most well-known examples are the so-called *Morley triangle* and *Napoleon triangle* (refer to [6, 8]). From a dynamic systems point of view, there exists a transformation T acting on the set of all triangles $\{P(\Theta)\}$ such that $TP(\Theta) = P(T\Theta)$ is equilateral for any triangle $P(\Theta)$. It is clear that such a transformation T is the special doubly stochastic matrix

$$T = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

That is, we could summarize those particular transformations that change a given triangle to an equilateral one as a result of the matrix T acting on the triangle.

Philip J. Davis has done an extensive study on the problems of nested polygons by using circulant matrices. Although we have approached these problems from a different perspective, all the matrices that we have used so far are circulant matrices, too. For a detailed study of circulant matrices and related geometry problems, refer to the work of Davis [5, 8, 9].

6 Dynamic Properties of Cyclic Polygons

In this section we will discuss a few interesting dynamic properties of cyclic polygons. For simplicity, let $\mathcal{P}_n$ be the set of all n-sided polygons that are inscribed in a circle Γ of radius r, and let T_Λ be the Λ-stretching. A polygon $P(\Theta) \in \mathcal{P}_n$ is called a *fixed polygon* of T_Λ if $T_\Lambda P(\Theta) = P(\Theta)$ (up to a rotation about the center of Γ). It is called a *periodic polygon with period m* if, up to a rotation about the center of Γ, $T_\Lambda^m P(\Theta) = P(\Theta)$ and $T_\Lambda^k P(\Theta) \neq P(\Theta)$ for $k = 1, 2, \ldots, m-1$.

Example 6.1 Let $n = 3$, $\Theta = (3\pi/5, 2\pi/3, 11\pi/15)$, $\Lambda = (1/3, 1/2, 4/11)$. Then the triangle $P(\Theta)$ is a fixed polygon of the transformation T_Λ. In fact, if we let C be the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then $C\Theta$ represents a rotation on the components of Θ. Set the transformation $T = C \circ T_\Lambda$. It is easy to verify that $TP(\Theta) = P(\Theta)$. In general, we have the following result.

Theorem 6.2 *Every Λ-stretching transformation has at least one fixed polygon.*

Proof. For a given $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, if all λ_i's are equal, where $0 < \lambda_i < 1$ and $1 \leq i \leq n$, then the regular polygon is its fixed polygon and any other polygons will converge to the regular one by Theorem B. Suppose these λ_i's are different and consider the following $(n+1) \times (n+1)$ system of linear equations

$$\theta_1 + \cdots + \theta_n = 2\pi$$
$$(1 - \lambda_1)\theta_1 + \lambda_2\theta_2 - \theta_3 = 0$$
$$(1 - \lambda_2)\theta_2 + \lambda_3\theta_3 - \theta_4 = 0$$
$$\cdots \cdots = 0$$
$$-\theta_1 + (1 - \lambda_{n-1})\theta_{n-1} + \lambda_n\theta_n = 0$$
$$\lambda_1\theta_1 - \theta_2 + (1 - \lambda_n)\theta_n = 0.$$

From the coefficient matrix A of the last n equations where

$$A = \begin{bmatrix} 1 - \lambda_1 & \lambda_2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 - \lambda_2 & \lambda_3 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 - \lambda_{n-1} & \lambda_n \\ \lambda_1 & -1 & 0 & 0 & \cdots & 0 & 1 - \lambda_n \end{bmatrix},$$

we see that $\mathrm{rank}(A) < n$ since the sum of its n rows is a 0 row. Therefore, the last n homogeneous linear equations have non-trivial solutions with at least one free variable. Substituting them into the first equation we get a solution for that system. Geometrically, the system represents a Λ-stretching transformation T such that $\Phi = (\phi_1, \phi_2, \ldots, \phi_n) = T\Theta$ is obtained by rotating the components of Θ. $\square$

Remark 6.3 Unlike the study of ordinary matrix iterations in which the eigenvalues and eigenvectors play the essential roles when one tries to compute T^m, our transformation T is supposed to act on the n-tuple $\Theta = (\theta_1, \ldots, \theta_n)$ whose components are constrained by the condition $\sum_{i=1}^{n} \theta_i = 2\pi$. Thus $T\Theta = \mu\Theta$ yields $\mu = 1$ since we must have $\sum_{i=1}^{n} \mu\theta_i = 2\pi$.

In general, let $P(\Theta)$ be an n-sided cyclic polygon with the set of central angles $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$, and let $T = [t_{ij}]$ be an $n \times n$ matrix acting on Θ. Since

$$\sum_{i=1}^{n} \theta_i = 2\pi \qquad \text{and} \qquad \sum_{i=1}^{n} \left(\sum_{j=1}^{n} t_{ij}\theta_j \right) = 2\pi,$$

a necessary condition for $TP(\Theta) = P(T\Theta)$ to be well-defined is

$$\sum_{i=1}^{n} t_{ij} = 1, \qquad j = 1, 2, \ldots, n. \tag{8}$$

That is, the sum of the entries in each column of T must be 1. When T is a doubly stochastic matrix, we have seen that $T^m P(\Theta) = P(T^m\Theta)$ converges to the regular polygon for any given cyclic polygon $P(\Theta)$, and the only fixed polygon of T is the regular polygon. When T is a Λ-stretching transformation, then the proof of Theorem 6.2 shows that T could have many different fixed polygons. However, if we only assume that T satisfies equation (8), then the dynamic behavior of $\{T^m P(\Theta)\}_{m=0}^{\infty}$ could be very complex. To illustrate this and to conclude this paper, we will briefly mention the well-known *sequence of pedal triangles*.

Let P_0 be a triangle with inner angles A_0, B_0, and C_0. Construct a second triangle P_1 whose vertices are the feet of the altitudes of P_0 and with inner angles A_1, B_1, and C_1. Construct a third triangle P_2 in the same way, and so on. One obtains a sequence of triangles $\{P_m\}_{m=0}^{\infty}$ called pedal triangles that were studied more than a century ago [11]. In late 1980's, Kingston and Synge revisited this problem [14]. They discovered many surprising properties of such sequences and also fixed some errors that occurred in the earlier literature. The limiting shape of the sequence $\{P_m\}_{m=0}^{\infty}$ appeared to be almost any shape of triangle if one chooses an appropriate initial triangle P_0. Soon after their work, a number of articles made nice connections between the sequence of pedal triangles and symbolic dynamic systems and ergodic theory [2, 15, 22]. Since triangles are always cyclic, if we are concerned only with the shape of its limit, we may treat the sequence of pedal triangles as a particular case of sequences of cyclic polygons with necessary rescaling. From [14], it is known that if P_0 is an acute triangle, then the inner angles of P_1 are given by

$$A_1 = \pi - 2A_0, \qquad B_1 = \pi - 2B_0, \qquad C_1 = \pi - 2C_0. \tag{9}$$

If P_0 is an obtuse triangle, say, $\pi > A_0 > \pi/2$, then

$$A_1 = 2A_0 - \pi, \qquad B_1 = 2B_0, \qquad C_1 = 2C_0. \tag{10}$$

Therefore, if $\Theta = (\theta_1, \theta_2, \theta_3)$ and $\Phi = (\phi_1, \phi_2, \phi_3)$ are the sets of central angles of P_0 and P_1 respectively, then we have:

(i) if the center of the circumscribed circle is inside P_0 (it is an acute triangle), then

$$\phi_1 = -\theta_1 + \theta_2 + \theta_3, \quad \phi_2 = \theta_1 - \theta_2 + \theta_3, \quad \phi_3 = \theta_1 + \theta_2 - \theta_3;$$

that is, $P_1 = TP_0 = P(T\Theta)$ where

$$T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix};$$

(ii) if the center of the circumscribed circle is outside of P_0 (it is an obtuse triangle), then

$$\phi_1 = \theta_1 - \theta_2 - \theta_3, \qquad \phi_2 = 2\theta_2, \qquad \phi_3 = 2\theta_3;$$

that is, $P_1 = SP_0 = P(S\Theta)$ where

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

In general, we cannot represent the pedal triangles as iterations of a single transformation matrix. For instance, if $A_0 = 36°$, $B_0 = 72°$, and $C_0 = 72°$ ($\theta_1 = 72°$, $\theta_2 = 144°$, $\theta_3 = 144°$), we use the matrix T to obtain the second triangle $P_1 = P(T\Theta)$. But notice that $A_1 = 108°$, $B_1 = 36°$, and $C_1 = 36°$, and we have to use the matrix S to get the third triangle P_2 because P_1 is an obtuse triangle! Using two different matrices to describe the limiting shape of a sequence of pedal triangles causes a more chaotic situation. See [14] for more details. That is why we have limited our previous discussion about sequences of cyclic polygons to some rather special cases involving only doubly stochastic matrices and Λ-stretching. However, inspired by the abstract treatment of the sequence of pedal triangles in terms of symbolic dynamic systems in [2], it seems very likely that one could approach the problem of sequences of cyclic polygons from a more abstract point of view and generalize our discussion here further to include more complicated situations. Even within the scope of the Λ-stretching transformation, there are still some questions to be answered, such as what conditions on T or Θ would ensure a convergent sequence $\{T^m P(\Theta)\}_{m=0}^{\infty}$ (not necessarily having a regular polygon as the limit)? When does it have chaotic behavior? How can the appropriate definition of entropy be introduced? These questions seem to be interesting in geometry, algebra, and dynamic systems. On the other hand, by virtue of computer graphics, it should not be too difficult to simulate a Λ-stretching construction on a computer and to visualize the convergent or chaotic transformations generated by iterations of T_Λ. We will leave these questions to the interested reader and hope that the ideas and examples illustrated in this paper will bring a deeper and broader discussion on the dynamic geometry of polygons on the surfaces of constant curvature.

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