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# A Property of Euler's Elastic Curve

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Die Fourierentwicklung glatter, periodischer Funktionen dürfte den meisten Leserinnen und Lesern bekannt sein. Das Studium komplexer, doppeltperiodischer Funktionen führt auf die elegante Theorie der Weierstrass'schen $\wp$-Funktion. Deren Umkehrfunktionen geben Anlass zu den elliptischen Integralen, welche – historisch gesehen – am Anfang der Entwicklung standen. Fagnano, Euler, Legendre und Gauss haben wesentliche Beiträge dazu geleistet. Erst Abel und Jacobi führten – unabhängig voneinander, wie die Korrespondenz zwischen A.-M. Legendre und C.G.J. Jacobi belegt – die elliptischen Funktionen ein. Der vorliegende Beitrag gibt zunächst einen Überblick über die Untersuchungen von Euler und Legendre über elliptische bzw. lemniskatische Integrale und schliesst mit einer Verallgemeinerung der klassischen Formel von Legendre. *jk*

## 1 Introduction

During the first two decades of the 19th century, Legendre developed the theory of elliptic integrals. His work [5] appeared in 1811 and his monumental treatise [6] in 1825. Shortly after that, Abel published his work [1] on the inversion of elliptic integrals and on the properties of the elliptic functions defined by this procedure. One of Legendre's most elegant formulae appears on [5] page 61. This is his famous relation:

$$\begin{aligned} &\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \times \int_0^1 \sqrt{\frac{1-(k')^2x^2}{1-x^2}}\,dx + \\ &\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(k')^2x^2)}} \times \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx - \\ \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} &\times \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(k')^2x^2)}} = \frac{\pi}{2}. \end{aligned} \tag{1.1}$$

The terms in (1.1) are the classical elliptic integrals that made their debut in the calculation of the length of the ellipse and the lemniscate. The reader is referred to [7] for details on this topic and to [2] for the history of Legendre's relation (1.1).

The lemniscatic integral ((1.3), below) appears in the calculation of the arclength of the lemniscate of equation $(x^2+y^2)^2 = a^2(x^2-y^2)$. Siegel [8] makes this example his starting point in his book on abelian functions. The parametrization of the lemniscate

$$x = \sqrt{\frac{r^2+r^4}{2}} \quad \text{and} \quad y = \sqrt{\frac{r^2-r^4}{2}}, \tag{1.2}$$

with $r = \sqrt{x^2+y^2}$, yields the expression

$$L = \int_0^1 \frac{dx}{\sqrt{1-x^4}} \tag{1.3}$$

for the total arclength. This lemniscatic integral was studied by Euler in [4] and is the special case $k = \sqrt{-1}$ of the *elliptic integral of the first kind*

$$K(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

later studied by Legendre in [6]. In this case (1.1) becomes

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \frac{\pi}{4}. \tag{1.4}$$

In this paper we describe Euler's method to prove (1.4) and establish a generalization that deals with the elastic curve

$$f_n(x) := \int_0^x \frac{t^n}{\sqrt{1-t^{2n}}}\,dt$$

for which we prove that

$$R_n \times L_n = \frac{\pi}{2n},$$

where $R_n = f_n(1)$ is the so-called *main radius*, and $L_n$ is the length of the curve from $x = 0$ to $x = 1$. The special case $n = 2$ yields Euler's result.

Section 2 recalls a standard proof of (1.1) based on the fact that the Legendre integrals satisfy a differential equation. Section 3 describes Euler's original proof, its generalization and discusses the issue of convergence, a fact that Euler was happy to ignore. Although Euler did not explicitly address the issue of convergence in [3], his familiarity with Stirling's formula dates from at least 1736.

## 2 Legendre's proof

The first proof of Legendre's relation (1.1) is based on a differential equation satisfied by the elliptic integrals

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx.$$

Among the many identities satisfied by these functions we employ an expression for their derivatives.

**Proposition 2.1** *The functions $K(k)$ and $E(k)$ satisfy*

$$\begin{aligned} k(k')^2 \frac{dK}{dk} &= E - (k')^2 K \\ k\frac{dE}{dk} &= E - K, \end{aligned} \tag{2.1}$$

*where $k' = \sqrt{1-k^2}$ is the conjugate modulus.*

*Proof.* This follows directly from the definitions. □

**Proposition 2.2** *Let $K'(k) = K(k')$ and $E'(k) = E(k')$. Then the function $KE' + EK' - KK'$ is constant.*

*Proof.* Employ Proposition 2.1 to check that the derivative is identically 0. □

Legendre then evaluates the constant at the modulus $k = \frac{1}{2}\sqrt{2-\sqrt{3}}$ and its complement $k' = \frac{1}{2}\sqrt{2+\sqrt{3}}$. In this paper we complete Legendre's proof by using the modulus $k = \sqrt{-1}$. This is explained in the next section.

## 3 Euler's direct proof

In [3] Euler developed his theory of infinite products and used it in [4] to prove the relation

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \frac{\pi}{4}. \tag{3.1}$$

In this paper we generalize Euler's method and prove the following result.

**Theorem 3.1** *The generalized elastic curve*

$$f_n(x) := \int_0^x \frac{t^n}{\sqrt{1-t^{2n}}}\,dt \tag{3.2}$$

*satisfies*

$$R_n \times L_n = \frac{\pi}{2n},$$

*$R_n$ is the main radius, the value $f_n(1)$, and $L_n$ is the length of the curve from $x = 0$ to $x = 1$.*

*Proof.* We have

$$R_n = \int_0^1 \frac{t^n}{\sqrt{1-t^{2n}}}\,dt \quad \text{and} \quad L_n = \int_0^1 \frac{dt}{\sqrt{1-t^{2n}}}.$$

Integrate the relation

$$d\left(t^k\sqrt{1-t^{2n}}\right) = \frac{kt^{k-1}\,dt - (k+n)t^{2n+k-1}\,dt}{\sqrt{1-t^{2n}}}$$

from 0 to 1 to produce the recursive formula

$$\int_0^1 \frac{t^{k-1}}{\sqrt{1-t^{2n}}}\,dt = \frac{k+n}{k}\int_0^1 \frac{t^{2n+k-1}}{\sqrt{1-t^{2n}}}\,dt. \tag{3.3}$$

The value $k = n+1$ in (3.3) yields

$$R_n = \frac{2n+1}{n+1}\int_0^1 \frac{t^{3n}}{\sqrt{1-t^{2n}}}\,dt\,. \tag{3.4}$$

Then the value $k = 3n+1$ produces

$$\int_0^1 \frac{t^{3n}}{\sqrt{1-t^{2n}}}\,dt = \frac{4n+1}{3n+1}\int_0^1 \frac{t^{5n}}{\sqrt{1-t^{2n}}}\,dt\ ,$$

so (3.4) produces

$$R_n = \frac{2n+1}{n+1} \times \frac{4n+1}{3n+1}\int_0^1 \frac{t^{5n}}{\sqrt{1-t^{2n}}}\,dt.$$

Iterating (3.3) we obtain, after $m$ steps,

$$R_n = \prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1} \times \int_0^1 \frac{t^{(2m+1)n}}{\sqrt{1-t^{2n}}}\, dt. \tag{3.5}$$

The next step is to justify the passage to the limit in (3.5) as $m \to \infty$, with $n$ fixed. Observe that the left hand side is *independent* of $m$, so it remains $R_n$ after $m \to \infty$. The difficulty in passing to the limit is that the product in (3.5) diverges. The general term $p_j$ satisfies

$$1 - p_j = \frac{-n}{(2j-1)n+1}$$

and the divergence of the product follows from that of the harmonic series. The divergence is cured by introducing scaling factors both in the integral and the product. The proof is omitted in Eulerian fashion.

**Proposition 3.2** *The functions*

$$\frac{1}{2m+1} \int_0^1 \frac{t^{(2m+1)n}}{\sqrt{1-t^{2n}}}\, dt \quad \text{and} \quad (2m+1) \times \prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1}$$

*have non-zero limits as* $m \to \infty$.

Therefore from (3.5) we obtain

$$R_n = \lim_{m\to\infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n}}{\sqrt{1-t^{2n}}}\, dt$$

where we have employed

$$\prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1} = \prod_{j=1}^{2m} (jn+1)^{(-1)^j}$$

in order to simplify the notation. A similar argument shows that

$$\begin{aligned} L_n &= \prod_{j=1}^{m} \frac{(2j-1)n+1}{2(j-1)n+1} \int_0^1 \frac{t^{2mn}}{\sqrt{1-t^{2n}}}\, dt \\ &= \lim_{m\to\infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^{j+1}} \int_0^1 \frac{t^{2mn}}{\sqrt{1-t^{2n}}}\, dt\, . \end{aligned} \tag{3.6}$$

The final step is to introduce the auxiliary quantities

$$A_n := \int_0^1 \frac{t^{n-1}}{\sqrt{1-t^{2n}}}\, dt \quad \text{and} \quad B_n := \int_0^1 \frac{t^{2n-1}}{\sqrt{1-t^{2n}}}\, dt.$$

We now show that the quotient $L_n/A_n$ can be evaluated explicitly and that the value of $A_n$ is elementary. This produces an expression for $L_n$. A similar statement holds for $R_n/B_n$ and $B_n$.

Observe first that

$$A_n = \int_0^1 \frac{t^{n-1}}{\sqrt{1-t^{2n}}}\,dt = \frac{1}{n}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2n} \tag{3.7}$$

and similarly $B_n = 1/n$. Now consider the recursion (3.3) for odd multiples of $n$ to produce

$$A_n = \lim_{m\to\infty} \prod_{j=1}^{2m} (jn)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n-1}}{\sqrt{1-t^{2n}}}\,dt \tag{3.8}$$

and similarly the even multiples of $n$ yield

$$B_n = \frac{1}{n}\lim_{m\to\infty} \prod_{j=1}^{2m+1} (jn)^{(-1)^{j+1}} \times \int_0^1 \frac{t^{2(m+1)n-1}}{\sqrt{1-t^{2n}}}\,dt,$$

in the exact manner as the derivation of (3.5). Therefore using (3.6) and (3.8), and passing to the limit as $m \to \infty$ so that the integrals disappear, we obtain

$$\frac{L_n}{A_n} = \prod_{j=1}^{\infty} \left[ (jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}} \right],$$

so (3.7) yields

$$L_n = \frac{\pi}{2n} \times \prod_{j=1}^{\infty} \left[ (jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}} \right].$$

Similarly, using $B_n = 1/n$,

$$R_n = \prod_{j=1}^{\infty} \left[ (jn+1)^{(-1)^{j}} \times (jn)^{(-1)^{j}} \right].$$

The formula $R_n \times L_n = \pi/2n$ follows directly from here. □

## 4 Conclusions

In this paper we have established that the main radius $R_n$ of the generalized elastic curve (3.2) and the length $L_n$ of this curve satisfy $R_n \times L_n = \pi/2n$. The case $n = 2$ corresponds to the classical Legendre's formula for elliptic integrals.

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