Elem. Math. 56 (2001) 62 – 70
0013-6018/01/020062-9 $ 1.50+0.20/0

Elemente der Mathematik

Disjoint empty convex polygons in planar point sets

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1 Introduction

Recently, the study of convex polygons has gained a renewed interest because of their importance in computer graphics, geometric learning theory, and artificial intelligence, for instance. Surprisingly, many simple questions are unanswered in this field. Let us start with a beautiful example. We say that a set of points in the plane is in general position if no three of the points lie on a line. Decades ago, Erdős, Klein, and Szekeres posed the problem of determining the maximum number $f(k)$ of points in general position in the plane so that no k points form the vertex set of a convex polygon. Erdős and Szekeres [3] proved that

$$2^{k-2} \leq f(k) \leq \binom{2k-4}{k-2},$$

Fragen zu speziellen Konfigurationseigenschaften von Punkten in der Ebene sind seit jeher Gegenstand von Untersuchungen in der kombinatorischen Geometrie. Heutzutage kommt diesen Fragestellungen aufgrund der Anwendungsmöglichkeiten beim Design von Computergraphiken eine besondere Bedeutung zu. So stellt sich bei vorgelegter natürlicher Zahl k zum Beispiel die Frage nach der Maximalzahl von Punkten in allgemeiner Lage in der Ebene mit der Eigenschaft, dass in keinem Falle k dieser Punkte die Ecken eines konvexen Polygons bilden. Dieses Problem ist für $k > 5$ bis heute ungelöst! In dem vorliegenden Beitrag von A. Gulyás und L. Szabó wird mit Hilfe eines raffinierten vollständigen Induktionsbeweises eine verwandte Problemstellung gelöst. *jk*

and conjectured that $f(k)$ is equal to the lower bound. Surprisingly, this conjecture has been verified only for $k = 3, 4, 5$. Recently, the upper bound has been slightly improved by many authors, see [2, 6, 7]. The current record, due to Tóth and Valtr [7], is

$$f(k) \leq \binom{2k-5}{k-2} + 1.$$

Later, Erdős also posed the problem of determining the maximum number $g(k)$ of points in general position in the plane so that no k points form the vertex set of an empty convex polygon, i.e., a convex polygon whose interior is disjoint from the point set. It is easy to see that $g(3) = 2$ and $g(4) = 4$. Harborth [4] proved that $g(5) = 9$, and Horton [5] showed that $g(k)$ is infinite for $k \geq 7$. It is a challenging open problem to decide whether $g(6)$ is finite.

Let $g_k(n)$ denote the minimum number of empty convex k-gons induced by the k-tuples of a set of n points in general position in the plane. Bárány and Füredi [1] proved that $g_3(n) \geq n^2 - O(n \log n)$, $g_4(n) \geq \frac{1}{4}n^2 - O(n)$, and $g_5(n) \geq [n/10]$. We note that the last bound can easily be improved to $g_5(n) \geq [(n-4)/6]$. On the other hand, Valtr [8] showed that $g_3(n) \leq 1.8n^2$, $g_4(n) \leq 2.42n^2$, and $g_5(n) \leq 1.46n^2$.

It is obvious that the k-tuples of a set of n points in general position in the plane always induce a family of $[n/(g(k)+1)]$ disjoint empty convex k-gons, and this bound is tight for $k = 3$. In this paper we consider the case $k = 4$ and we prove

Theorem 1 *The quadruples of a set of n points in general position in the plane always induce a family of* $[2n/9]$ *disjoint empty convex quadrangles.*

We also show that the bound $[2n/9]$ cannot be improved for $n \leq 21$.

2 Proof of Theorem 1

First we prove that any set $\mathcal{P}$ of nine points in general position in the plane contains two disjoint empty convex quadrangles. Let $p_1, p_2, \ldots, p_m$ denote the vertices of the convex hull of $\mathcal{P}$ in a counterclockwise order (we will use the convention that $p_i = p_j$ if $i \equiv j \pmod{m}$). Observe that if $\triangle p_{i-1}p_ip_{i+1}$ is an empty triangle of $\mathcal{P}$ for some $1 \leq i \leq m$, then $\mathcal{P}$ contains two disjoint empty quadrangles. Indeed, among $\mathcal{P} \setminus \{p_{i-1}, p_i, p_{i+1}\}$ choose a point r whose distance from the line $p_{i-1}p_{i+1}$ is minimal. Now $p_{i-1}p_ip_{i+1}r$ is an empty convex quadrangle and the remaining five points of $\mathcal{P}$ also contain an empty convex quadrangle which is obviously disjoint from $p_{i-1}p_ip_{i+1}r$. Therefore, in what follows, we will assume that $\triangle p_{i-1}p_ip_{i+1}$ is not empty for $1 \leq i \leq m$. This immediately implies among others that $m \leq 6$.

Case 1. $m = 6$. Let q_1, q_2, q_3 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Without loss of generality we may assume that $q_i \in \triangle p_{2i-2}p_{2i-1}p_{2i} \cap \triangle p_{2i-1}p_{2i}p_{2i+1}$ for $i = 1, 2, 3$ (do not forget that no $\triangle p_{i-1}p_ip_{i+1}$ is empty, $1 \leq i \leq 6$). Then $p_1q_1q_3p_6$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by the line p_2p_5. Now we are done, since the set of the remaining five points necessarily contains an empty convex quadrangle, which is, of course, disjoint from $p_1q_1q_3p_6$.

Case 2. $m = 5$. Let q_1, q_2, q_3, q_4 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. A point of $\{q_1, q_2, q_3, q_4\}$ will be called special if it is contained in $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ for some $1 \le i \le 5$. Obviously, at least one point of $\{q_1, q_2, q_3, q_4\}$ is special.

Case 2.1. Exactly one point of $\{q_1, q_2, q_3, q_4\}$, say q_4, is special. Without loss of generality we may assume that $q_4 \in \triangle p_1p_4p_5 \cap \triangle p_3p_4p_5$ and $q_i \in \triangle p_{i-2}p_ip_{i+2} \cap \triangle p_{i-1}p_ip_{i+1}$ for $i = 1, 2, 3$. Now $p_1q_1q_2p_2$ and $p_3q_3q_4p_4$ are disjoint empty convex quadrangles (they are separated by the line joining p_5 and $\overline{p_1p_3} \cap \overline{p_2p_4}$).

Case 2.2. Exactly two points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, are special. Then $\{q_1, q_2\} \subseteq \triangle p_{i-1}p_ip_{i+1}$ for at most one $1 \le i \le 5$.

Case 2.2.1. For some $1 \le i \le 5$, the set $\{q_1, q_2\} \subseteq \triangle p_{i-1}p_ip_{i+1}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-2}p_{j-1}p_j \cap \triangle p_{j-1}p_jp_{j+1}$ for $j = 1, 2$ and $q_j \in \triangle p_{j-2}p_jp_{j+2} \cap \triangle p_{j-1}p_jp_{j+1}$ for $j = 3, 4$. Now $p_3p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_5.

Case 2.2.2. No $\triangle p_{i-1}p_ip_{i+1}$ contains both q_1 and q_2, $1 \le i \le 5$. Without loss of generality we may assume that $q_1 \in \triangle p_1p_2p_5 \cap \triangle p_1p_2p_3$, $q_2 \in \triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$, and $q_3 \in \triangle p_1p_4p_5 \cap \triangle p_2p_3p_5$. Let $u = p_2q_3 \cap \overline{p_4p_5}$ and $v = p_3q_3 \cap \overline{p_5p_1}$ (see Figure 1).

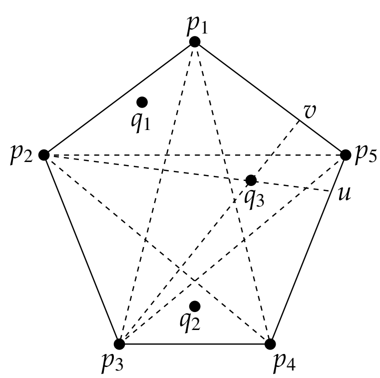

Fig. 1

If q_4 is contained in the quadrangle $p_2p_3p_4u$, then $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3. Similarly, if q_4 is contained in the quadrangle $p_1p_2p_3v$, then $p_4p_5q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_3. Finally, if q_4 is contained in the quadrangle up_5vq_3, then $p_2p_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_1p_4.

Case 2.3. Exactly three points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, q_3, are special.

Case 2.3.1. For some $1 \leq i \leq 5$, $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $\{q_1, q_2\} \subseteq \triangle p_1p_4p_5 \cap \triangle p_1p_2p_5$, $q_3 \in \triangle p_1p_2p_3 \cap \triangle p_2p_3p_4$, and $q_4 \in \triangle p_2p_4p_1 \cap \triangle p_3p_4p_5$. Now $p_3p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_5.

Case 2.3.2. No $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains two points of $\{q_1, q_2, q_3\}$, $1 \leq i \leq 5$. Then, among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \leq i \leq 5$, one or two contain two points of $\{q_1, q_2, q_3\}$.

Case 2.3.2.1. Among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \leq i \leq 5$, exactly one contains two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2$, $q_3 \in \triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$, and q_4 is not separated from q_2 by the line p_2q_3. Now $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3.

Case 2.3.2.2. Among the triangles $\triangle p_{i-1}p_ip_{i+1}$, $1 \leq i \leq 5$, exactly two contain two points of $\{q_1, q_2, q_3\}$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2, 3$, and $q_4 \in \triangle p_2p_3p_5 \cap \triangle p_1p_4p_5$. Now $p_1p_5q_4q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 2.4. All four points of $\{q_1, q_2, q_3, q_4\}$ are special. Then there are three points of $\{q_1, q_2, q_3, q_4\}$, say q_1, q_2, q_3, so that no $\triangle p_{i-2}p_{i-1}p_i \cap \triangle p_{i-1}p_ip_{i+1}$ contains more than one point of $\{q_1, q_2, q_3\}$, $1 \leq i \leq 5$. Without loss of generality we may assume that $q_j \in \triangle p_{j-1}p_jp_{j+1} \cap \triangle p_jp_{j+1}p_{j+2}$ for $j = 1, 2$.

Case 2.4.1. The point q_3 is in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$. Without loss of generality we may assume that q_4 is not separated from q_2 by the line p_2q_3. Now $p_1p_5q_3q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2q_3.

Case 2.4.2. The point q_3 is contained in $\triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$ or $\triangle p_4p_5p_1 \cap \triangle p_5p_1p_2$, say in $\triangle p_2p_3p_4 \cap \triangle p_3p_4p_5$. Then q_4 is contained in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$ or $\triangle p_4p_5p_1 \cap \triangle p_5p_1p_2$, say in $\triangle p_3p_4p_5 \cap \triangle p_4p_5p_1$ (do not forget that $\triangle p_1p_4p_5$ is not empty). Now $p_1p_5q_4q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 3. $m = 4$. Let q_1, q_2, q_3, q_4, q_5 denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Let $u = \overline{p_1p_3} \cap \overline{p_2p_4}$.

Case 3.1. No q_1, q_2, q_3, q_4, q_5 is contained in $\triangle p_iup_{i+1}$ for some $1 \leq i \leq 4$, say in $\triangle p_1up_2$. Without loss of generality we may assume that $\measuredangle q_1p_1p_2 < \measuredangle q_jp_1p_2$ for $2 \leq j \leq 5$, and $\measuredangle p_1q_1q_2 < \measuredangle p_1q_1q_k$ for $3 \leq k \leq 5$. Now $q_1 \in \triangle p_2up_3$ and the line q_1q_2 intersects $\overline{p_1p_4}$ since $\triangle p_1p_2p_3$ and $\triangle p_1p_2p_4$ are not empty. Then $p_1p_2q_1q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2.

Case 3.2. All $\triangle p_iup_{i+1}$ contain at least one point of $\{q_1, q_2, q_3, q_4, q_5\}$, $1 \leq i \leq 4$. Without loss of generality we may assume that $q_i \in \triangle p_iup_{i+1}$ for $1 \leq i \leq 4$ and $q_5 \in \triangle p_4up_1$ (see Figure 2).

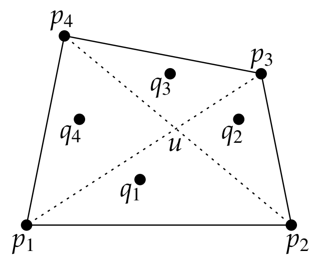

Fig. 2

Case 3.2.1. The line q_4q_5 does not intersect $\overline{p_1p_4}$. If q_4q_5 separates both q_1 and q_3 from p_1 and p_4, then $p_1p_4q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5. If q_4q_5 does not separate q_3 and q_1 from p_1 and p_4, then $p_1p_4q_1q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5. Finally, if exactly one point of $\{q_1, q_3\}$, say q_1, is separated from p_1 and p_4 by q_4q_5, then $p_1p_4q_4q_3$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_5.

Case 3.2.2. The line q_4q_5 intersects $\overline{p_1p_4}$. Without loss of generality we may assume that q_4q_5 is disjoint from $\triangle p_1up_2$. Now $p_1q_1q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_2p_4.

Case 4. $m = 3$. Let $q_1, q_2, q_3, q_4, q_5, q_6$ denote the points of $\mathcal{P}$ lying in the interior of the convex hull of $\mathcal{P}$. Without loss of generality we may assume that $\measuredangle p_2p_1q_1 < \measuredangle p_2p_1q_i$ for $2 \leq i \leq 6$ and $\measuredangle p_1q_1q_2 < \measuredangle p_1q_1q_i < \measuredangle p_1q_1q_6$ for $3 \leq i \leq 5$. Let $u = p_1q_1 \cap \overline{p_2p_3}$ and $v = p_2q_1 \cap \overline{p_1p_3}$ (see Figure 3).

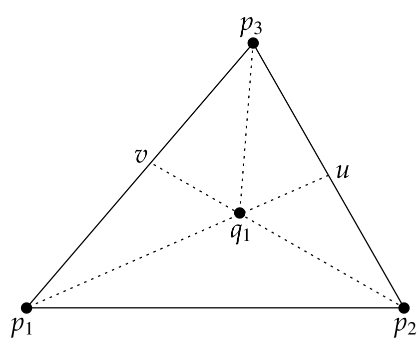

Fig. 3

It is obvious that $\triangle p_1up_2$ is empty. If $\triangle p_1q_1v$ is not empty, then $p_1p_2q_1q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2. Thus, in what follows, we will assume that $\triangle p_1q_1v$ is empty. If $\triangle q_1p_3v$ is empty, then $p_1p_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_2. Thus, in what follows, we will also assume that $\triangle q_1p_3v$ is not empty. Similarly, if $\triangle q_1p_3u$ is empty, then $p_2p_3q_6q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_1q_6. Thus, in what follows, we will also assume that $\triangle q_1p_3u$ is not empty. For technical reasons, in the remaining part of the proof we will disregard the special choice of q_2 and q_6.

Case 4.1. Exactly one point of $\{q_2, q_3, q_4, q_5, q_6\}$ is contained in $\triangle q_1p_3v$ or $\triangle q_1p_3u$. Without loss of generality we may assume that q_2 is contained in $\triangle q_1p_3u$ and q_3, q_4, q_5, q_6 are contained in $\triangle q_1p_3v$.

Case 4.1.1. Not all q_3, q_4, q_5, q_6 are separated from p_3 by p_2q_2. Without loss of generality we may assume that $\measuredangle p_3q_2q_3 < \measuredangle p_3q_2q_i$ for $4 \le i \le 6$. Now $p_2p_3q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_2q_3.

Case 4.1.2. All q_3, q_4, q_5, q_6 are separated from p_3 by p_2q_2. Suppose that p_1p_3 and q_1q_2 are not parallel (the case where p_1p_3 and q_1q_2 are parallel can be settled similarly). Let $w = p_1p_3 \cap q_1q_2$. Without loss of generality we may assume that $\measuredangle q_1wq_3 < \measuredangle q_1wq_i$ for $4 \le i \le 6$ (see Figure 4).

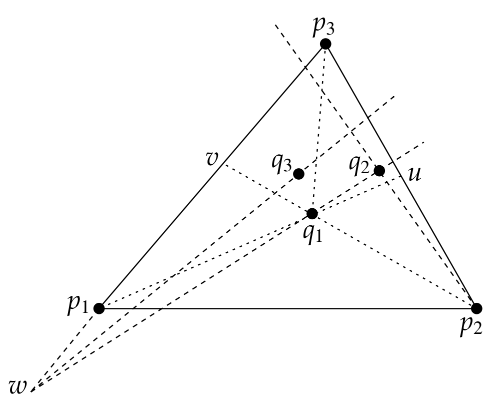

Fig. 4

Now $p_2q_1q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by wq_3.

Case 4.2. Exactly two points of $\{q_2, q_3, q_4, q_5, q_6\}$ are contained in $\triangle q_1p_3v$ or $\triangle q_1p_3u$. Without loss of generality we may assume that q_2, q_3 are contained in $\triangle q_1p_3u$ and q_4, q_5, q_6 are contained in $\triangle q_1p_3v$.

Case 4.2.1. The line q_2q_3 does not intersect $\overline{p_2p_3}$. Now $p_2p_3q_3q_2$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1.

Case 4.2.2. The line q_2q_3 intersects $\overline{p_2p_3}$ and $\overline{q_1p_3}$. Now $p_2q_3q_2q_1$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1.

Case 4.2.3. The line q_2q_3 intersects $\overline{p_2p_3}$ and $\overline{q_1p_2}$. Without loss of generality we may assume that q_3 separates q_2 from $q_2q_3 \cap \overline{p_2p_3}$ (see Figure 5). If $p_1q_4q_5q_6$ is a convex quadrangle then we are done. Indeed, now $p_1q_4q_5q_6$ is empty and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_1. Thus, in what follows, we will assume, without loss of generality, that q_4 is contained in $\triangle p_1q_5q_6$. We will also assume that q_6 separates q_5 from $q_5q_6 \cap \overline{p_1p_3}$ if q_5q_6 intersects $\overline{p_1p_3}$ and that q_6 separates q_5 from $q_5q_6 \cap \overline{p_3q_1}$ if q_5q_6 does not intersect $\overline{p_1p_3}$.

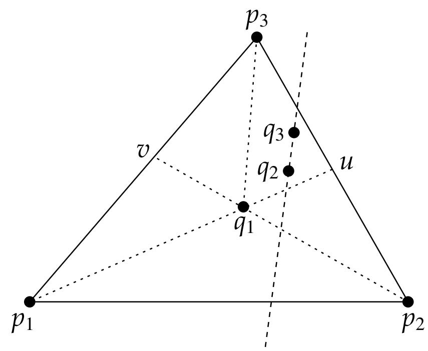

Fig. 5

Case 4.2.3.1. The line q_5q_6 separates p_3 from q_2 and q_3. If q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_1p_3q_6q_4$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_5q_6. On the other hand, if q_5q_6 intersects $\overline{p_1p_3}$, then either $p_1p_3q_6q_4$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_4q_6 or $p_3q_6q_4q_5$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by p_1q_5.

Case 4.2.3.2. The line q_5q_6 separates q_2 and q_3. If q_5q_6 intersects $\overline{p_1p_3}$, then $p_1q_1q_2q_4$ and $p_3q_3q_5q_6$ are disjoint empty convex quadrangles. On the other hand, if q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_3q_4q_6q_3$ and either $p_1q_1q_2q_5$ or $p_2q_1q_5q_2$ are disjoint empty convex quadrangles.

Case 4.2.3.3. The line q_5q_6 does not separate p_3 from q_2 and q_3. If q_5q_6 does not intersect $\overline{p_1p_3}$, then $p_2q_1q_5q_6$ is an empty convex quadrangle and it is separated from the remaining five points of $\mathcal{P}$ by q_5q_6. If q_5q_6 intersects $\overline{p_1p_3}$ and $\overline{p_3q_1}$, then $p_3q_6q_2q_3$ and $p_1q_4q_5q_1$ are disjoint empty convex quadrangles. Thus, we will assume that q_5q_6 intersects $\overline{p_1p_3}$ and $\overline{p_1q_1}$. If $p_1p_3q_6q_4$ is a convex quadrangle, then we are done. Indeed, now $p_1p_3q_6q_4$ is empty and it is separated from the remaining five points of $\mathcal{P}$ by p_3q_6. Thus, we will assume that $p_3q_6q_4q_5$ is an empty convex quadrangle. If p_3 and q_3 are separated by p_1q_5, then $p_3q_6q_4q_5$ is separated from the remaining five points of $\mathcal{P}$ by p_1q_5. On the other hand, if p_3 and q_3 are not separated by p_1q_5, then $p_3q_6q_4q_3$ is an empty convex quadrangle, and either $p_1q_1q_2q_5$ or $p_2q_1q_5q_2$ is an empty convex quadrangle disjoint from $p_3q_6q_4q_3$.

Thus we have proved that any set $\mathcal{P}$ of nine points in general position in the plane contains two disjoint empty convex quadrangles.

The proof of Theorem 1 will be done by induction on n. We know that the assertion is true for $n \leq 9$. Let $n \geq 10$ and consider a set $\mathcal{P}$ of n points in general position. It is obvious that there exists a line which cuts $\mathcal{P}$ into two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of 9 and $n-9$ points, respectively. Then, by the induction hypothesis, $\mathcal{P}_1$ contains two disjoint empty convex quadrangles and $\mathcal{P}_2$ contains $[2(n-9)/9] = [2n/9] - 2$ disjoint empty convex quadrangles. Thus $\mathcal{P}$ contains $[2n/9]$ disjoint empty convex quadrangles.

3 Constructions

It is easy to find a set of eight points in general position in the plane which does not contain two disjoint empty convex quadrangles. Indeed, if p_1, p_2, p_3, p_4 are the vertices of a square in a counterclockwise order and q_i is an interior point of $p_1p_2p_3p_4$ sufficiently close to the midpoint of $\overline{p_ip_{i+1}}$, $1 \leq i \leq 4$, then $\{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$ is just such a point set.

Next we show that, for each $m \geq 3$, there exists a set of $n = 4m+1$ points in general position which does not contain m disjoint empty convex quadrangles.

Let $p_1, p_2, \ldots, p_{2m}$ be the vertices of a regular $2m$-gon C in a counterclockwise order and let q_i be an interior point of C sufficiently close to the midpoint of $\overline{p_ip_{i+1}}$, $1 \leq i \leq 2m$. Furthermore, let r be a point sufficiently close to the centre of C so that $\mathcal{P} = \{p_1, \ldots, p_{2m}, q_1, \ldots, q_{2m}, r\}$ is in general position. Suppose, for contradiction, that $\mathcal{P}$ contains m disjoint empty convex quadrangles $Q_1, Q_2, \ldots, Q_m$.

Let p_{i_1} and p_{i_2} two vertices of C so that they belong to Q_i for some $1 \leq i \leq m$ and the number l of vertices of C on the shorter arc of C bounded by p_{i_1} and p_{i_2} is as small as possible. Now, a very simple counting argument shows that $l \leq 4$. Thus, without loss of generality we may assume that $i_1 = 1$ and $i_2 \in \{2, 3, 4\}$. If $i_2 = 2$, then q_1 cannot be a vertex of a quadrangle. If $i_2 = 3$, then p_2 cannot be a vertex of a quadrangle. Finally, if $i_2 = 4$, then p_2 or p_3 cannot be a vertex of a quadrangle. Next, let p_{j_1} and p_{j_2} be two vertices of the longer arc of C bounded by p_{i_1} and p_{i_2} so that they belong to Q_j for some $1 \leq j \leq m$ and the number l' of vertices of C on the arc of C bounded by p_{j_1} and p_{j_2} is as small as possible. Again, a very simple counting argument shows that $l' \leq 4$ and, similarly as before, we can find a point of $\mathcal{P}$ different from q_1, p_2, p_3 which cannot be a vertex of a quadrangle. Thus $\mathcal{P}$ necessarily contains two points which are not vertices of the quadrangles $Q_1, Q_2, \ldots, Q_{2m}$, a contradiction.

Note that the above constructions show that the bound in Theorem 1 is tight for $n \leq 21$.

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