Elem. Math. 57 (2002) 8 – 18
0013-6018/02/010008-11 $ 1.50+0.20/0

Elemente der Mathematik

Error estimate for the Jacobi method adapted to the weak row sum resp. weak column sum criterion

Ludwig Kohaupt

Ludwig Kohaupt, born in 1945, obtained his doctoral degree in mathematics at the University of Frankfurt am Main in 1973. From 1974 – 1979 he was a teacher at a secondary school, and from 1979 – 1990 he worked as an engineer in the field of technical computations in the automotive industry. Since 1990 he is a professor of mathematics at the Technische Fachhochschule Berlin. His main interests are applied and industrial mathematics with emphasis on numerical methods.

1 Introduction

The Jacobi method with the iteration matrix B converges iff the spectral radius satisfies $\rho(B) < 1$. For the derivation of error estimates, stronger conditions such as $\|B\| < 1$ are needed, as a rule. This is briefly reviewed in Section 2. If the matrix B is, in addition, irreducible and satisfies a weak convergence criterion, then an error estimate in a weighted ∞- resp. 1-norm can be derived containing $\rho(|B|)$ instead of $\|B\|$. The weights are given by the components of the positive eigenvector of $|B|$ resp. of $|B|^T$ pertinent to the positive eigenvalue $\rho(|B|)$. This is described in Section 3. In Section 4, an example illustrates the obtained result. Sections 2, 3, and 4 could be used for class room teaching. Further, Section 5 contains some historical remarks with respect to the presented subject, and Section 6 discusses the role of the Jacobi method and of related methods in today's numerical mathematics.

Sehr große lineare Gleichungssysteme $A\,x = b$ lassen sich oft am besten iterativ lösen. Dazu bringt man $A\,x = b$ beispielsweise auf die Form $x = B\,x + c$ und erhält damit das nach Jacobi benannte Verfahren $x^{(k+1)} = B\,x^{(k)} + c$, $k = 0, 1, 2, \ldots$, wobei $x^{(0)}$ ein Anfangsvektor ist. Die Konvergenz dieses Verfahrens untersucht man mit Normen für Matrizen und Vektoren. Dabei spielen letztlich Überlegungen wie bei der Konvergenz der geometrischen Zahlenreihe $\sum_{j=0}^{\infty} q^j = (1-q)^{-1}$, $|q| < 1$, eine wesentliche Rolle. In diesem Beitrag werden in allgemeinverständlicher Form zwei neue Fehlerabschätzungen für das Jacobi-Verfahren hergeleitet. Hierzu werden historische und didaktische Anmerkungen gemacht, und es werden Verbindungen zur gegenwärtigen numerischen linearen Algebra hergestellt.

2 Error estimate for the Jacobi method in an ordinary norm

Let $\mathbb{F} = \mathbb{R}$ resp. $\mathbb{F} = \mathbb{C}$ be the field of real resp. complex numbers. Further, let $A = (a_{ik})$ be an $n \times n$ matrix, and let x and b be n-vectors with elements in $\mathbb{F}$ such that

$$A x = b . \tag{1}$$

Assume that

$$a_{jj} \neq 0, \quad j = 1, \ldots, n , \tag{2}$$

and define

$$D = \operatorname{diag}(a_{jj}) . \tag{3}$$

Then, (1) is equivalent to

$$x = B x + c \tag{4}$$

with

$$\left.\begin{array}{l} B = -D^{-1}(A - D) \\ c = D^{-1} b \end{array}\right\} . \tag{5}$$

The Jacobi method for the solution of (1) resp. (4) is given by

$$x^{(k+1)} = B x^{(k)} + c, \quad k = 0, 1, \ldots , \tag{6}$$

where $x^{(0)}$ is any initial vector. According to [27, 8.1.2(8), pp. 156–157], this method is convergent for each initial vector $x^{(0)}$ iff

$$B^k \longrightarrow 0 \qquad (k \longrightarrow \infty) , \tag{7}$$

which in turn is equivalent to the condition that the spectral radius $\rho(B)$ satisfies

$$\rho(B) := \max_{j=1,\ldots,n} |\lambda_j(B)| < 1 , \tag{8}$$

where $\lambda_j(B),\ j = 1, \ldots, n$, are the eigenvalues of B, cf. [30, Theorem 1.4 and Definition 1.4, p. 13].

More precisely, the following theorem holds (see [27] as cited above).

Theorem 1 *Let $B = (b_{ij})$ be any quadratic matrix. If* (7) *is fulfilled, then and only then the Jacobi method converges. In this case, the system* (4) *is uniquely soluble and the matrix $E - B$ is nonsingular.*

As a rule, error estimates require a stronger condition than (8). For this, let $\|\cdot\|$ be a vector norm and an associated sup matrix norm. Then, provided that

$$\|B\| < 1 , \tag{9}$$

from [27, p. 158, (21)] it follows that

$$\|x^{(k)} - z\| \leq \frac{\|B\|}{1 - \|B\|} \|x^{(k-1)} - x^{(k)}\| \leq \frac{\|B\|^k}{1 - \|B\|} \|x^{(0)} - x^{(1)}\| , \tag{10}$$

$k = 1, 2, \ldots$, where $x = z$ is the unique solution of (1).

Remark The condition $B^k \to 0$ $(k \to \infty)$ is much more elementary than the condition $\rho(B) < 1$ since the latter requires the knowledge of the eigenvalue theory. So, the first one can be used in an early stage of teaching, whereas the second one might be used for the teaching of advanced students. In this paper, we shall be interested in the condition $\rho(B) < 1$.

Remark The condition $B^k \to 0$ $(k \to \infty)$ is closely related to the convergence of the *Neumann series* $\sum_{j=0}^{\infty} B^j$. In fact, according to [27, p. 170, (7)], it is equivalent to the existence of $(E - B)^{-1}$ and the representation

$$\sum_{j=0}^{\infty} B^j = (E - B)^{-1} . \tag{11}$$

In this context, another form of the spectral radius is important, namely

$$\rho(B) = \lim_{k\to\infty} \|B^k\|^{\frac{1}{k}} , \tag{12}$$

where $\|\cdot\|$ is any matrix norm, cf. [30, p. 95], [29, p. 262], or [15, p. 78]. So, $\lim_{k\to\infty} \|B^k\|^{\frac{1}{k}} < 1$ is the *root test* for the convergence of the series $\sum_{k=0}^{\infty} \|B^k\|$, which in turn is a majorant to $\| \sum_{k=0}^{\infty} B^k \|$ and which has the geometric series $\sum_{k=0}^{\infty} q^k$ with $q = \|B\|$ as a majorant.

Now, there are cases where $\|B\| = 1$, but where $\rho(|B|) < 1$ (cf. Section 4). Here, we have $\rho(B) \le \rho(|B|) < 1$, which follows from (12) for $\|\cdot\| = \|\cdot\|_\infty$. So, the Jacobi method is convergent, but the estimate (10) is nevertheless not applicable.

Under additional conditions, it is possible to evade this problem by introducing a weighted norm.

3 Error estimate for the Jacobi method in a weighted ∞- resp. 1-norm

(i) The main idea

Instead of the usual estimate

$$|(B\,x)_i| = \left| \sum_{j=1}^{n} b_{ij}\, x_j \right| \le \sum_{j=1}^{n} |b_{ij}| \max_{j=1,\ldots,n} |x_j|, \quad i = 1, \ldots, n , \tag{13}$$

one adds the factor $1 = \mu_j/\mu_j$, $j = 1, \ldots, n$, where $\mu_j > 0$, $j = 1, \ldots, n$, and obtains thus

$$|(B\,x)_i| = \left| \sum_{j=1}^{n} b_{ij}\, \mu_j \, \frac{1}{\mu_j}\, x_j \right| \le \left(\sum_{j=1}^{n} |b_{ij}|\, \mu_j \right) \left(\max_{j=1,\ldots,n} \mu_j^{-1}\, |x_j| \right), \; i = 1, \ldots, n ; \tag{14}$$

further, one multiplies this result by μ_i^{-1},

$$\mu_i^{-1}\, |(B\,x)_i| \le \left(\sum_{j=1}^{n} \mu_i^{-1}\, |b_{ij}|\, \mu_j \right) \left(\max_{j=1,\ldots,n} \mu_j^{-1}\, |x_j| \right), \quad i = 1, \ldots, n , \tag{15}$$

in order that both sides of the estimate can be expressed by a weighted norm, see (ii). Then, for μ_j the components κ_j of the eigenvector corresponding to $\rho(|B|)$ are chosen, see (iii).

(ii) General weighted ∞- resp. 1-norm

Let $\mu := [\mu_1, \ldots, \mu_n]^T \in \mathbb{R}^n$ be such that

$$\mu > 0\,. \tag{16}$$

Define

$$\|x\|_{\infty,\mu^{-1}} := \max_{i=1,\ldots,n} |x_i|\,\mu_i^{-1}, \quad x = [x_1, \ldots, x_n]^T \in \mathbb{F}^n, \tag{17}$$

as well as

$$\|x\|_{1,\mu} := \sum_{i=1}^{n} |x_i|\,\mu_i, \quad x \in \mathbb{F}^n\,. \tag{18}$$

This leads to

$$\|B\|_{\infty,\mu^{-1}} := \max_{0\neq x\in\mathbb{F}^n} \frac{\|B\,x\|_{\infty,\mu^{-1}}}{\|x\|_{\infty,\mu^{-1}}} = \max_{i=1,\ldots,n} \sum_{j=1}^{n} \mu_i^{-1}\,|b_{ij}|\,\mu_j \tag{19}$$

and

$$\|B\|_{1,\mu} := \max_{0\neq x\in\mathbb{F}^n} \frac{\|B\,x\|_{1,\mu}}{\|x\|_{1,\mu}} = \max_{j=1,\ldots,n} \sum_{i=1}^{n} \mu_i\,|b_{ij}|\,\mu_j^{-1}\,. \tag{20}$$

We mention that $(\mathbb{F}^n, \|\cdot\|_{\infty,\mu^{-1}})$ and $(\mathbb{F}^n, \|\cdot\|_{1,\mu})$ are dual spaces (see [9], [12], or [29]).

(iii) First additional condition: irreducibility of the matrix B

In the sequel, we need the following

Definition 2 For $n \geq 2$, an $n \times n$ matrix with elements of $\mathbb{F}$ is called *reducible* if there exists an $n \times n$ permutation matrix P such that

$$P\,A\,P^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ O & A_{2,2} \end{bmatrix}, \tag{21}$$

where $A_{1,1}$ is an $r \times r$ submatrix and $A_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is called *irreducible*. If A is a 1×1 matrix, then A is *irreducible* if its single entry is nonzero, and *reducible* otherwise.

This definition can be found in [30, p. 18]. The meaning of a reducible matrix is illustrated in [30, p. 19]. It is mentioned that a permutation matrix is a square matrix which in each row and each column has exactly one entry 1, all others are 0. One can describe an irreducible matrix by the term of *directed graph* $G(A)$ of the matrix A, see [30, pp. 19–20, especially Definition 1.6 and Theorem 1.6], which is often quite useful.

After this preparation, as the first additional condition, suppose that

$$B \quad \text{is irreducible.} \tag{22}$$

Then, the matrix

$$|B| := (|b_{ik}|) \tag{23}$$

is also irreducible (cf. [32, Proposition 1.12]), and $|B| \geq 0$. Now, we make use of the following Perron-Frobenius theorem, cf. [30, Theorem 2.1, p. 30].

Theorem 3 *Let $C \geq 0$ be an irreducible $n \times n$ matrix. Then,*

1. *C has a positive real eigenvalue equal to its spectral radius.*
2. *To $\rho(C)$ there corresponds an eigenvector $w > 0$.*
3. *$\rho(C)$ increases when any entry of C increases.*
4. *$\rho(C)$ is a simple eigenvalue of C.*

We apply this theorem to $C := |B|$. Consequently, the spectral radius $\rho(|B|)$ is a simple (positive) eigenvalue of $|B|$, and the associated eigenvector κ can be chosen such that $\kappa > 0$, that is

$$|B|\,\kappa \;=\; \rho(|B|)\,\kappa, \;\;\kappa \;>\; 0\,. \tag{24}$$

From (24) and (19), because of $\|B\|_{\infty,\mu^{-1}} = \|\,|B|\,\|_{\infty,\mu^{-1}}$ one infers

$$\|B\|_{\infty,\kappa^{-1}} \;=\; \rho(|B|)\,. \tag{25}$$

Further, from (22) it follows that B^T is irreducible (cf. [32, Proposition 1.12 or 1.13]). Thus, similarly as before, the spectral radius $\rho(|B|^T)$ is a simple (positive) eigenvalue of $|B|^T$, and the associated eigenvector χ can be chosen such that $\chi > 0$, that is,

$$|B|^T\,\chi \;=\; \rho(|B|^T)\,\chi, \;\;\chi \;>\; 0\,. \tag{26}$$

From (26) and (19), we obtain

$$\|B^T\|_{\infty,\chi^{-1}} \;=\; \rho(|B|^T)\,; \tag{27}$$

taking into account $\|B^T\|_{\infty,\chi^{-1}} = \|B\|_{1,\chi}$ and $\rho(|B|^T) = \rho(|B|)$, we get

$$\|B\|_{1,\chi} \;=\; \rho(|B|)\,. \tag{28}$$

(iv) Second additional condition: weak convergence criteria

As our second additional condition, suppose that the matrix B satisfies the *weak row sum criterion*

$$\left.\begin{aligned} &\sum_{j=1}^{n} |b_{ij}| \;\leq\; 1, \quad i = 1,\ldots,n\,, \\ &\sum_{j=1}^{n} |b_{i_0 j}| \;<\; 1, \quad \text{for at least one } i_0 \in \{1,\ldots,n\}\,, \end{aligned}\right\} \tag{29}$$

or the *weak column sum criterion*

$$\left.\begin{aligned} &\sum_{i=1}^{n} |b_{ij}| \;\leq\; 1, \quad j = 1,\ldots,n\,, \\ &\sum_{i=1}^{n} |b_{i j_0}| \;<\; 1, \quad \text{for at least one } j_0 \in \{1,\ldots,n\}\,. \end{aligned}\right\} \tag{30}$$

Remark The *strong* row sum criterion is given by

$$\sum_{j=1}^{n} |b_{ij}| < 1, \quad i = 1, \ldots, n \ ,$$

or equivalently by

$$\|B\|_\infty < 1 \, ,$$

and the *strong* column sum criterion is given by

$$\sum_{i=1}^{n} |b_{ij}| < 1, \quad j = 1, \ldots, n \ ,$$

or equivalently by

$$\|B\|_1 < 1 \, .$$

Further, the following somewhat more general criterion due to Sassenfeld [23] is used in [27, p. 161, Formula (14)]:

$$q_1 = \sum_{j=1}^{n} |b_{1j}|, \quad q_i = \sum_{j=1}^{i-1} |b_{ij}|\, q_j + \sum_{j=i}^{n} |b_{ij}|, \quad j = 2, \ldots, n; \quad q = \max_{i=1,\ldots,n} q_i < 1 \, .$$

(v) Estimate in a weighted ∞- resp. 1-norm

Under the conditions (22), (29) or (22), (30), according to [30, p. 75] or [27, pp. 161–163] along with Formula (12), we obtain

$$\rho(B) \ \le \ \rho(|B|) \ < \ 1 \, . \tag{31}$$

Therefore, we have the following result:

Theorem 4 *Let B be irreducible. Further, let the weak row sum resp. the column sum criterion for B be satisfied. Then,*

$$\|x^{(k)} - z\|_{\infty,\kappa^{-1}} \ \le \ \frac{\rho(|B|)}{1-\rho(|B|)} \, \|x^{(k-1)} - x^{(k)}\|_{\infty,\kappa^{-1}} \ \le \ \frac{\rho(|B|)^k}{1-\rho(|B|)} \, \|x^{(0)} - x^{(1)}\|_{\infty,\kappa^{-1}}, \tag{32}$$

$k = 1, 2, \ldots$, *resp.*

$$\|x^{(k)} - z\|_{1,\chi} \ \le \ \frac{\rho(|B|)}{1-\rho(|B|)} \, \|x^{(k-1)} - x^{(k)}\|_{1,\chi} \ \le \ \frac{\rho(|B|)^k}{1-\rho(|B|)} \, \|x^{(0)} - x^{(1)}\|_{1,\chi}, \tag{33}$$

$k = 1, 2, \ldots$

Special Case 1: $B \ge 0$.
For *positive* matrices B, $\rho(|B|)$ is replaced by $\rho(B)$.

Special Case 2: $B \ge 0$ and $B = B^T$.
For *symmetric positive* matrices B, we have additionally $\kappa = \chi$.

Remark Often, one has only an approximation $\sigma > 0$ resp. $\tau > 0$ for $\kappa > 0$ resp. $\chi > 0$. If $\|B\|_{\infty,\sigma^{-1}} < 1$ resp. $\|B\|_{1,\tau} < 1$, then one has at least estimates of the form (10) in the weighted norm $\|\cdot\|_{\infty,\sigma^{-1}}$ resp. $\|\cdot\|_{1,\tau}$.

4 Example

Let the $n \times n$ matrix A be given by

$$A = \text{tridiag}\,[-1\ 2 - 1] = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{34}$$

Then,

$$B = \frac{1}{2}\,\text{tridiag}\,[1\ 0\ 1] = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}. \tag{35}$$

So, here $B \geq 0$ and $B = B^T$. Evidently, for $n \geq 3$ we have $\|B\|_\infty = \|B\|_1 = 1$ so that (10) is not applicable for the norm $\|\cdot\|_\infty$ resp. $\|\cdot\|_1$. Now, according to [34, p. 230ff]

$$\rho(B) = \rho(|B|) = \cos\frac{\pi}{n+1} < 1 \tag{36}$$

and

$$\kappa = \chi = \left[\sin\frac{\pi}{n+1},\ \sin 2\,\frac{\pi}{n+1}, \ldots,\ \sin n\,\frac{\pi}{n+1}\right]^T. \tag{37}$$

Therefore, the estimates (32) and (33) hold with (36) and (37).

The matrix A in (34) often appears in the finite element approximation of simple one-dimensional boundary value problems, cf. [21, p. 110] or in finite difference approximations, cf. [26, p. 55] and [32, p. 117]. Also, similar matrices occur in applications, see [32, pp. 122–123].

The matrix B is symmetric. One can construct nonsymmetric matrices B with $\|B\|_\infty = \|B\|_1 = 1$ and $\rho(|B|) < 1$ by using the results of [26, pp. 154–156], whereby also (36) and (37) can be obtained as special cases. The information of [26, pp. 154–156] is not contained in the earlier editions of the book [26].

Remark In practice, more complicated systems of linear equations occur, see Section 6. To solve these equations is a hard problem. It should be said that the problem to determine the associated quantities $\rho(|B|)$ and $\kappa > 0$ is even harder.

5 Historical remarks

5.1 The Jacobi method

(i) The Jacobi method is introduced in [10, p. 297] in 1845. If one uses the matrix notation, which was not yet known at that time, however, the original version of Jacobi's method is equivalent to

$$\left.\begin{aligned} x^{(0)} &:= D^{-1}\,b =: c \\ \Delta^{(1)} &:= B\,x^{(0)} \\ \Delta^{(2)} &:= B\,\Delta^{(1)} \\ \Delta^{(3)} &:= B\,\Delta^{(2)} \\ &\;\vdots \end{aligned}\right\} \tag{38}$$

and

$$x = x^{(0)} + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)} + \cdots . \tag{39}$$

There was no convergence proof of (39) in [10]. Nowadays, we see at once that, by inserting (38) into (39), one obtains the *Neumann series* and thus convergence if, for example, the strong row sum or column sum criterion is fulfilled since then

$$x = (E + B + B^2 + B^3 + \cdots)\,c = (E - B)^{-1}\,c\,, \tag{40}$$

see [27, p. 170]. We remark that according to [33, p. 1] the *matrix notation* was introduced by Sylvester [28] in 1850 (see [28, p. 369]) and that according to [33, p. 2] the first *matrix calculations* were made by Cayley [2] in 1858.

(ii) In [10], Jacobi applied his method to normal equations (see [27, p. 138]) which result from the least square method for the solution of an overdetermined system of linear equations stemming from a problem of celestial mechanics. Jacobi determines the initial vector $x^{(0)}$ by neglecting the off-diagonal coefficients of the matrix. For strictly diagonally dominant matrices, this is of course a good approximation to the solution. Today, we know that the Jacobi method is convergent even for any $x^{(0)}$ if, e.g., $\|B\| < 1$. This follows directly from Banach's fixed point theorem (see also Section 5.5).

(iii) Before starting his iterative procedure (38), (39), Jacobi makes it sufficiently strictly diagonally dominant. For this, he uses a method which can nowadays be described by a finite number of similarity transformations constructed by so-called elementary rotations and which is used in [27, p. 212ff] to compute the eigenvalues of a symmetric matrix. Jacobi himself applies this method also for the calculation of eigenvalues in [11]. We would like to remark that in modern iterative methods one uses preparatory techniques before starting the iteration itself, too (see Section 6).

(iv) For convergence considerations, originally strong convergence criteria such as $\|B\|_\infty < 1$ and $\|B\|_1 < 1$ were applied, cf. [10] and [27, p. 159]. Weak convergence criteria became necessary in the context of systems of linear equations arising in the approximation of boundary value problems, cf. [3, p. 159ff]. We mention that, in [3], a review of the Jacobi and Gauss-Seidel methods (called there *Einschrittverfahren* and *Gesamtschrittverfahren*) is given, till 1950.

(v) We remark that Jacobi's original work [10] is referenced, e.g., in [6] and [30], and his work [11] in [25].

5.2 The notation of irreducible matrix

The term *irreducible* (*unzerlegbar*) was introduced by Frobenius (1912) in [5], and it is also used by Wielandt (1950) in [31]. Collatz (1950) uses in [3] the term *nichtzerfallend*.

5.3 The Perron-Frobenius theorem

The original work leading to this theorem can be found in Perron [20] (1907) and Frobenius [5] (1912). In [20], also a method for the numerical computation of the spectral radius of a positive matrix is given resembling that of Graeffe. For this, see [4, p. 387]. For a recent paper in this journal on matrices with strictly dominant eigenvalue, see [17].

5.4 Eigenvalue and eigenvector of the example in Section 4

These stem from Lagrange (1759), see [8, pp. 27–29].

5.5 The Banach fixed point theorem (contraction-mapping theorem)

The estimate (10) can be derived by using the Banach fixed point theorem. The original can be found in [1, p. 160, Théorème 6]. However, the author has not found in [1] the error estimate (10) itself. Its derivation relies, by the way, heavily on the formula $\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}$, $|q| < 1$, for the geometric series. In the Anglo-Saxon literature, Banach's fixed point theorem is usually called *contraction-mapping theorem*. For this, see [19, p. 129]. In [19], also Banach's original work is referenced.

5.6 Neumann series

According to [22, p. 146], the convergence of the so-called *Neumann series* was first proved by Carl Neumann in [18] (1877). As far as the author sees, however, the Neumann series (11) itself is not used in [18], but Carl Neumann uses in [18, p. 200] the geometric series as a majorant to prove the convergence of other series. Carl Neumann must not be confused with the famous mathematician J. von Neumann, who is referenced, e.g., in [12, p. 572].

6 The Jacobi method in contemporary numerical mathematics

Iterative methods such as the Jacobi method play an important role in the solution of large systems of linear equations when direct methods such as the Gaussian elimination are no longer used due to, for example, rounding errors or because they take too much time, see [7]. Other standard iterative methods are the *Gauss-Seidel method* as well as *under- and overrelaxation methods*, cf. [30, p. 58ff].

Especially, iterative methods are used to solve large systems of linear equations arising in the finite element method, cf. [24, p. 148ff]. Before the iterative method starts, the *conditioning* of the system is improved by methods such as *scaling* or *preconditioning*, cf. [24, p. 233ff]. A similar idea had already been applied by Jacobi in his pioneering work [10], as mentioned in Section 5.1.

As to the Jacobi method in contemporary mathematics, a version called *damped Jacobi iteration* plays an important role in the *multi-grid methods* for the solution of problems in *fluid dynamics*, cf. [6, p. 19].

To conclude this section, we want to add some remarks.

The result in Theorem 4 can be generalized to Banach spaces with a cone; for this, see [14, Section 1 and Section 4] or look for the notion of u-norm in [15]. Hereby, the corresponding results for integral equations in [13] follow. We leave the details to the interested reader. Further, we note that, according to [16, Chapter I, 1.4], for a given norm $\|\cdot\|$ there exists an equivalent norm $\|\cdot\|_*$ such that $\rho(B) \leq \|B\|_* \leq \rho(B)+\varepsilon$, and a corresponding error estimate can be found (see [16, p. 19]). But, this norm is mainly of theoretical interest. Here, we have constructed a norm by using intrinsic properties of a matrix B, namely the eigenvector of $|B|$ associated with the eigenvalue $\rho(|B|)$. We mention that, in general, the spectral radius of $|B|$ cannot be replaced by the spectral radius of B because, in general, $\rho(B)$ is no eigenvalue of B and the pertinent eigenvector is not positive.

Acknowledgement
The author would like to give thanks to the referee and to J. Kramer, who carefully read a former draft of this paper and made numerous suggestions to improve it.

References

[1] Banach, S.: Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrables. *Fundamentae mathematicae* 3 (1922), 133–181.

[2] Cayley, A.: A memoir of the Theory of Matrices. *Trans. London Philos. Soc.* 148 (1858), 17–37.

[3] Collatz, L.: Über die Konvergenzkriterien bei Iterationsverfahren für lineare Gleichungssysteme. *Math. Z.* 53 (1950), 149–161.

[4] Faddejew, D.K. und Faddejewa, W.N.: *Numerische Methoden der linearen Algebra*. R. Oldenbourg Verlag, München und Wien 1976.

[5] Frobenius, G.: Über Matrizen aus positiven bzw. nicht negativen Elementen. *S.B. preuss. Akad. Wiss.* (1908), 471–476; (1909), 514–518; (1912), 456–477.

[6] Hackbusch, W.: *Multi-Grid Methods and Applications*. Springer-Verlag, Berlin et al. 1985.

[7] Hackbusch, W.: *Iterative Lösung großer schwachbesetzter Gleichungssysteme*. 2.Auflage, B.G. Teubner, Berlin et al. 1993.

[8] Hairer, E., Nørsett, S.P., and Wanner, G.: *Solving Ordinary Differential Equations* I (*Nonstiff Problems*). Second Revised Edition, Springer-Verlag, Berlin et al. 1993.

[9] Halmos, P.R.: *Finite-Dimensional Vector Spaces*. 2nd. Ed., D. van Nostrand, Princeton 1958.

[10] Jacobi, C.G.J.: Über eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden lineären Gleichungen. *Astr. Nachr.* 22, No. 523 (1845), 297–306.

[11] Jacobi, C.G.J.: Über ein leichtes Verfahren, die in der Theorie der Säkularstörungen vorkommenden Gleichungen numerisch aufzulösen. *Crelle's Journal* 30 (1846), 51–94.

[12] Kato, T.: *Perturbation Theory for Linear Operators*. Springer-Verlag, New York 1966.

[13] Kohaupt, L.: The Jacobi Method in Weighted Banach Spaces for Integral Equations, with Emphasis on Green's-Function-Like Kernels. *J. Integral Equations Appl.* 6 (1994), 75–117.

[14] Kohaupt, L.: An Alternative for the Spectral Radius of Positive Integral Operators – A Functional Analytic Approach. *Rocky Mountain Journal of Mathematics* 30 (2) (2000), 609–619.

[15] Krasnosel'skii, M.A., Lifshits, Je.A., and Sobolev, A.V.: *Positive Linear Systems*. Heldermann Verlag, Berlin 1989.

[16] Krasnosel'skii, M.A., Vainikko, G.M, Zabreiko, P.P., Rutitskii, Ya.B., and Stetsenko, V.Ya.: *Approximate solution of operator equations*. Wolters-Noordhoff Publishing, Groningen 1972.

[17] Länger, H.: Matrices with strictly dominant eigenvalue. *Elem. Math.* 56 (2001), 55–61.

[18] Neumann, C.: *Untersuchungen über das logarithmische und Newtonsche Potential*. B.G. Teubner, Leipzig 1877.

[19] Ortega, J.M. and Rheinboldt,W.C.: *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York and London 1970.

[20] Perron, O.: Zur Theorie der Matrizen. *Math. Ann.* 64 (1907), 248–263.

[21] Reddy, J.N.: *An Introduction to the Finite Element Method*. 2nd Ed., McGraw-Hill, Inc., New York et al. 1993.

[22] Riesz, F. and Sz.-Nagy, B.: *Functional Analysis*. Translated from the 2nd French Edition, Dover Publication, Inc., New York 1990.

[23] Sassenfeld, H.: Ein hinreichendes Konvergenzkriterium und eine Fehlerabschätzung für die Iteration in Einzelschritten bei linearen Gleichungen. *ZAMM* 31 (1951), 92–94.

[24] Schwarz, H.R.: *Methode der finiten Elemente*. 2. Auflage, B.G. Teubner, Stuttgart 1984.

[25] Schwarz, H.R.: *Numerische Mathematik*. B.G. Teubner, Stuttgart 1986.

[26] Smith, G.D.: *Numerical Solution of Partial Differential Equations – Finite Difference Methods*. 3rd Ed., Clarendon Press, Oxford 1999.

[27] Stummel, F. und Hainer, K.: *Praktische Mathematik*. B.G. Teubner, Stuttgart 1982.

[28] Sylvester, J.J.: Additions to the articles in the September Number of this Journal, "On a new Class of Theorems," and on Pascal's Theorem. *Philos. Mag.* 37 (1850), 363–370.

[29] Taylor, A.E.: *Introduction to Functional Analysis*. John Wiley & Sons, Inc., London 1958.

[30] Varga, R.S.: *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey 1962.

[31] Wielandt, H.: Unzerlegbare, nicht negative Matrizen. *Math. Z.* 52 (1950), 642–648.

[32] Windisch, G.: *M-matrices in Numerical Analysis*. B.G. Teubner, Leipzig 1989.

[33] Zurmühl, R.: *Matrizen und ihre technischen Anwendungen*. Vierte Auflage, Springer-Verlag, Berlin et al. 1964.

[34] Zurmühl, R. und Falk, S.: *Matrizen und ihre Anwendungen*. Teil 1: Grundlagen. Springer-Verlag, Berlin et al. 1984.

Ludwig Kohaupt
Technische Fachhochschule Berlin
Fachbereich II
Luxemburger Str. 10
D–13353 Berlin, Germany
e-mail: kohaupt@tfh-berlin.de