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The story of Landen, the hyperbola and the ellipse

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1 Introduction

The problem of rectification of conics was a central question of analysis in the 18th century. The goal of this note is to describe Landen's work on rectifying the arc of a hyperbola in terms of an ellipse and a circle. Naturally, Landen's language is that of his time, in terms of *fluents* and *fluxions*, and his arguments are not rigorous in the modern sense.

The main result presented here is a special relation between the length of an ellipse, the length of a hyperbolic segment, and the length of a circle. The proof is based on a generalization of Euler's formula for the lemniscatic curve as described in [4].

Der nachfolgende Beitrag knüpft an einen Artikel der drei selben Autoren gemeinsam mit P.A. Neill über eine gewisse Eigenschaft von Eulers elastischer Kurve (Elem. Math. 55 (2000), 156–162) an. In der nun vorliegenden Arbeit geht es um die Bestimmung der Länge von Hyperbelbögen. Zur Lösung der Aufgabe wird eine Idee von J. Landen aus dem 18. Jahrhundert herangezogen. Damit lässt sich die gesuchte Hyperbelbogenlänge schliesslich mit Hilfe einer Ellipsenbogenlänge ausdrücken, welche in der vorhergehenden Arbeit untersucht wurde.

2 The hyperbola

The arc length of the equilateral hyperbola

$$h(t) = \sqrt{t^2 - 1}, \quad t \geq 1 \tag{2.1}$$

starting at $t = 1$ is given by

$$L_h(x) = \int_1^x \sqrt{\frac{2t^2 - 1}{t^2 - 1}}\, dt \tag{2.2}$$

as a function of the terminal point $t = x$. The tangent line to the hyperbola at $t = x$ is

$$T_h(t) = \sqrt{x^2 - 1} + \frac{x}{\sqrt{x^2 - 1}}(t - x)\,, \tag{2.3}$$

whose intersection with the t-axis is $t = 1/x \in (0, 1)$. The line

$$N_h(t) = -\frac{\sqrt{x^2 - 1}}{x}\, t \tag{2.4}$$

is the perpendicular to T_h passing through the origin. The lines T_h and N_h intersect at the point

$$P_h = \left(\frac{x}{2x^2 - 1}, -\frac{\sqrt{x^2 - 1}}{2x^2 - 1} \right). \tag{2.5}$$

The distance from $(x, h(x))$ to the common point P_h is

$$g_h(x) = 2x\sqrt{\frac{x^2 - 1}{2x^2 - 1}}\,. \tag{2.6}$$

It was observed by Maclaurin, D'Alembert, and Landen that

$$f_h(x) := g_h(x) - L_h(x) = 2x\sqrt{\frac{x^2 - 1}{2x^2 - 1}} - \int_1^x \sqrt{\frac{2t^2 - 1}{t^2 - 1}}\, dt \tag{2.7}$$

is easier to analyze than the arc length $L_h(x)$.

Proposition 2.1 *Let*

$$F_h(z) = \frac{1}{2} \int_z^1 \sqrt{\frac{t}{1 - t^2}}\, dt\,. \tag{2.8}$$

Then

$$F_h(z) = f_h(x)\,, \tag{2.9}$$

where

$$z = \frac{1}{2x^2 - 1}\,. \tag{2.10}$$

Proof. Make the change of variable (2.10) in (2.7). Then $f_h(x)$ becomes

$$F_h(z) = \sqrt{\frac{1-z^2}{z}} + \frac{1}{2}\int_1^z \frac{\mathrm{d}s}{s^{3/2}\sqrt{1-s^2}} \tag{2.11}$$

in terms of the new variable $z = 1/(2x^2-1)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\frac{1-s^2}{s}} = \frac{-1-s^2}{2s^{3/2}\sqrt{1-s^2}}\,,$$

integrating from 1 to z reduces (2.11) to (2.8). □

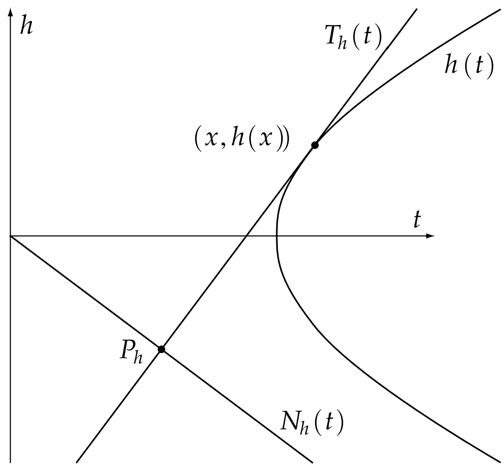

Fig. 1 The hyperbola

3 The ellipse

The equation of the ellipse can be written as

$$e(t) = \sqrt{2(1-t^2)}\,, \quad |t| \le 1\,. \tag{3.1}$$

In this case the tangent line at $t = r$ is

$$T_e(t) = \sqrt{2(1-r^2)} - \sqrt{\frac{2r^2}{1-r^2}}\,(t-r)\,,$$

and the line

$$N_e(t) = \sqrt{\frac{1-r^2}{2r^2}}\,t$$

is the perpendicular to T_e through the origin. These two lines intersect at the point

$$P_e = \left(\frac{2r}{1+r^2}, \frac{\sqrt{r(1-r^2)}}{1+r^2} \right), \tag{3.2}$$

and the distance from $(r, e(r))$ to the common point P_e is

$$g_e(r) = r\sqrt{\frac{1-r^2}{1+r^2}}. \tag{3.3}$$

We express the function g_e in terms of the new variable $z = r^2$ as

$$g_e(z) = \sqrt{\frac{z(1-z)}{1+z}}. \tag{3.4}$$

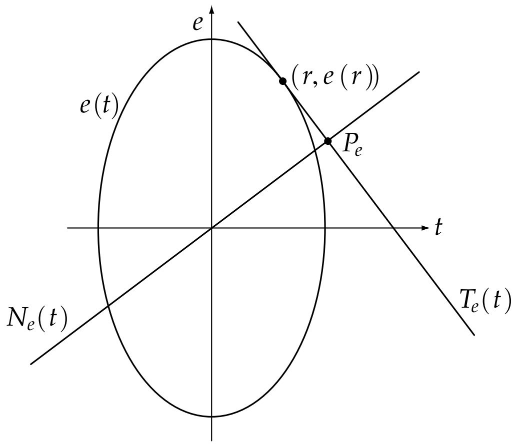

Fig. 2 The ellipse

4 The connection

We now evaluate the function $F_h(z)$ in (2.8) at two points y, $z \in (0,1)$ related via the bilinear transformation $z = (1-y)/(1+y)$. We have

$$F_h(z) + F_h(y) = \frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s.$$

The change of variable $\sigma = (1-s)/(1+s)$ in the second integral yields

$$F_h(z) + F_h(y) = \frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s + \frac{1}{2}\int_0^y \frac{\sqrt{1-\sigma}}{(1+\sigma)^{3/2}\sqrt{\sigma}}\,\mathrm{d}\sigma.$$

Now recall the function $g_e(z)$ in (3.4) and its differential

$$\frac{\mathrm{d}\,g_e}{\mathrm{d}\,z} = \frac{1}{2}\frac{\sqrt{1-z}}{\sqrt{z}(1+z)^{3/2}} - \frac{1}{2}\frac{\sqrt{z}}{\sqrt{1-z^2}}\,.$$

Therefore

$$F_h(z) + F_h(y) = g_e(z) - g_e(1) + \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\,\mathrm{d}t\,.$$

Now observe that $g_e(1) = 0$ and introduce the absolute constant

$$L := \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\,\mathrm{d}t \tag{4.1}$$

so that

$$F_h(z) + F_h(y) = g_e(z) + L\,. \tag{4.2}$$

Thus we have established the following integral relation.

Theorem 4.1 *Let $y \in (0,1)$ and $z = (1-y)/(1+y)$. Then*

$$\frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s = \sqrt{\frac{z(1-z)}{1+z}} + L \tag{4.3}$$

with the absolute constant L in (4.1).

Proof. Let

$$\begin{aligned} G_h(z) &= F_h(z) + F_h(y) \\ &= \frac{1}{2}\int_{(1-z)/(1+z)}^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\,\mathrm{d}s\,, \end{aligned}$$

so that

$$\frac{\mathrm{d}G_h(z)}{\mathrm{d}z} = \frac{1}{2}\frac{\sqrt{1-z}}{\sqrt{z}(1+z)^{3/2}} - \frac{1}{2}\frac{\sqrt{z}}{\sqrt{1-z^2}}\,. \tag{4.4}$$

Integrating (4.4) gives

$$G_h(z) = \sqrt{\frac{z(1-z)}{1+z}} + L\,. \tag{4.5}$$

By letting $z = 0$, the constant L is easily evaluated as

$$\begin{aligned} L &:= \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\,\mathrm{d}\theta = \frac{\pi\sqrt{2\pi}}{\Gamma^2(1/4)} \end{aligned} \tag{4.6}$$

using Wallis' formula. □

We now follow Landen to establish the value of L in terms of elliptic arcs.

The equation (4.2) simplifies if we evaluate it at the fixed point $z^* = \sqrt{2} - 1$ of the transformation $z = (1 - y)/(1 + y)$. In terms of the x variable, the fixed point is

$$x^* = \sqrt{1 + \frac{1}{\sqrt{2}}} = \sqrt{2}\cos(\pi/8)\,. \tag{4.7}$$

Indeed

$$F_h(z^*) = \frac{1}{2}(\sqrt{2} - 1 + L)\,. \tag{4.8}$$

Now introduce the complementary integral

$$M := \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t^2)}} \tag{4.9}$$

and observe that

$$L + M = L_e(1) = \frac{1}{2}\int_0^1 \sqrt{\frac{1+t}{t(1-t)}}\,\mathrm{d}t$$

where $L_e(1)$ is a quarter of the length of the ellipse.

Theorem 4.2 *The integrals L and M satisfy*

$$\begin{aligned} L + M &= L_e(1) \\ L \times M &= \frac{\pi}{4}\,. \end{aligned}$$

Therefore

$$\begin{aligned} L &= \frac{1}{2}\left(L_e(1) - \sqrt{L_e(1)^2 - \pi}\right), \\ M &= \frac{1}{2}\left(L_e(1) + \sqrt{L_e(1)^2 - \pi}\right). \end{aligned}$$

Proof. Observe that for $q \in \mathbb{Q}$ we have

$$\frac{\mathrm{d}(\,t^q\sqrt{1-t^2})}{\mathrm{d}t} = \frac{qt^{q-1} - (q+1)\,t^{q+1}}{\sqrt{1-t^2}} \tag{4.10}$$

and integrating from 0 to 1 we obtain

$$\int_0^1 \frac{t^{q-1}}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{q+1}{q}\int_0^1 \frac{t^{q+1}}{\sqrt{1-t^2}}\,\mathrm{d}t\,. \tag{4.11}$$

The proof now proceeds along the same line as Theorem 3.1 in [4]. □

We now write $\pi/2 = L_c(1)$ as a quarter of the length of the circle in analogy to $L_e(1)$.

Theorem 4.3 *The length of the hyperbolic segment is given by*

$$L_h\left(\frac{1}{\sqrt{2-\sqrt{2}}}\right) = \frac{\sqrt{2}+1}{2} - \frac{1}{4}\sqrt{(L_e(1)^2 - 4L_c(1))} - L_e(1)\,. \tag{4.12}$$

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