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Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem when conics are circles not one inside of the other

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1 Introduction

A polygon which is both chordal and tangential will be called a bicentric polygon. The first who was concerned with bicentric polygons was the German mathematician Nicolaus Fuss (1755–1826), a friend of Leonhard Euler (see [5]). He posed the following problem (known as Fuss' problem of the bicentric quadrilateral):

Find the relation between the radii and the line segment joining the centres of the circles of circumscription and inscription of a bicentric quadrilateral.

He found that

$$2\rho^2(r^2 + z^2) = (r^2 - z^2)^2, \tag{1.1}$$

where r and ρ are radii and z is the distance between the centers of the circles of circumscription and inscription.

Die allgemeine Fassung des Schliessungssatzes von Poncelet besagt folgendes: Formen $C, C_1, \ldots, C_n$ ein Kegelschnittbüschel, ist $P \in C$ ein Punkt, konstruiert man $P_1, \ldots, P_n \in C$ derart, dass die Gerade durch PP_1 die Kurve C_1, die Gerade durch P_1P_2 die Kurve $C_2, \ldots$, die Gerade durch $P_{n-1}P_n$ die Kurve C_n berührt und entsteht bei dieser Konstruktion die Gleichheit $P = P_n$, so gilt diese Koinzidenz unabhängig von der Wahl von P. In der vorliegenden Arbeit werden die Spezialfälle $n = 3$ und $n = 4$ betrachtet, wobei zusätzlich vorausgesetzt wird, dass die Kegelschnitte (nicht notwendigerweise verschiedene) Kreise sind. In den genannten Spezialfällen, in denen zudem die Kreise nicht ineinander enthalten sind, wird ein elementarer Beweis des Satzes von Poncelet gegeben.

This problem is listed and considered in [4, p. 188] as one of the 100 great problems of elementary mathematics.

Fuss also found corresponding formulas for bicentric pentagons, hexagons, heptagons and octagons (Nova Acta Petropol., XII, 1798).

The corresponding formula for triangles is

$$r^2 - z^2 = 2r\rho \tag{1.2}$$

and had already been given by Euler.

The very remarkable theorem concerning bicentric polygons is given by the French mathematician Poncelet (1788–1867). In the formulation of this theorem the so-called Poncelet traverse will be used. This in short is:

Let C_1 and C_2 be two circles in a plane. If from any point on C_2 we draw a tangent to C_1, extend the tangent line so that it intersects C_2, and draw from the point of intersection a new tangent to C_1, extend this tangent similarly to intersect C_2, and continue in this way, we obtain the so-called Poncelet traverse which, when it consists of n chords of the circle C_2 (circle of circumscription), is called n-sided.

The Poncelet theorem for circles can be expressed as follows:

> *If on the circle of circumscription there is one point of origin for which the n-sided Poncelet traverse is closed, then the n-sided traverse will also be closed for any other point of origin on the circle.*

Poncelet proved that the analogue holds for conic sections so that the general theorem reads:

Poncelet's closure theorem. *If an n-sided Poncelet traverse constructed for two given conic sections is closed for one position of the point of origin, it is closed for any position of the point of origin.*

Although this problem dates back to the nineteenth century, many mathematicians have been working on a number of problems in connection with it. Many contributions have been made. Very interesting and useful information about this we found in the references concerning Poncelet's closure theorem, particulary in [2], [6] and [8].

In this article we shall restrict ourselves to triangles and bicentric quadrilaterals when the conics are circles not one inside of the other and where instead of incircles there are excircles under consideration. In this case for triangles instead of relation (1.2) Euler's relation holds:

$$z^2 - r^2 = 2r\rho. \tag{1.3}$$

But Fuss' relation (1.1) holds in both of these cases. (More about this will be given in Section 3.)

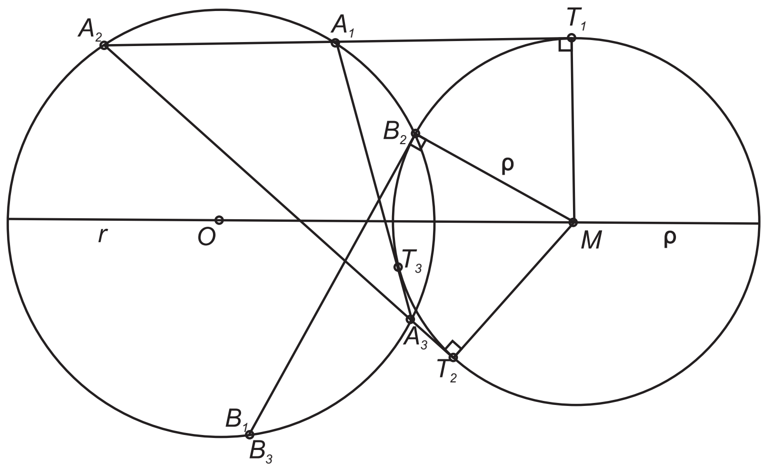


Fig. 1

2 Some relations concerning triangles which have the same excircle and same circumcircle

Notation used in this section:

Let r, z and ρ be any given lengths (positive numbers) such that Euler's relation (1.3) holds, and let M and O be points and C_1 and C_2 be circles such that

$$|MO| = z, \quad C_1 = M(\rho), \quad C_2 = O(r). \tag{2.1}$$

Then, by Poncelet's closure theorem, for every point A_1 on C_2 there is a triangle $A_1A_2A_3$ whose excircle is C_1 and circumcircle C_2. (See Fig. 1, where $r = 3$, $z = 5$, $\rho = \frac{8}{3}$.)

A triangle will be degenerate if one of its vertices belongs to the set $\{P_1, P_2, Q_1, Q_2\}$, where the points P_1, P_2, Q_1, Q_2 are shown in Fig. 2. So, for example, triangle $B_1B_2B_3$ shown in Fig. 1 is a degenerate one.

Now, let us consider Fig. 3. It is easy to see that

$$(t_1 - t_2 - t_3)\rho = \text{area of triangle } \; A_1A_2A_3, \tag{2.2}$$

where $t_i = |A_iT_i|$, $i = 1, 2, 3$. Thus, in this case, instead of t_2 and t_3 we must take $-t_2$ and $-t_3$. It is because in this case we must use oriented angles. Namely, if the angle MA_iT_i is negatively oriented, then instead of t_i we must take $-t_i$.

It can be easily seen that for every triangle $A_1A_2A_3$ whose excircle is C_1, one of the angles MA_iT_i, $i = 1, 2, 3$, is negatively oriented and the other two positively, or conversely, one is negatively oriented and the other two positively.

Also, it is easy to see that

$$|\underline{t}_i + \underline{t}_{i+1}| = |A_iA_{i+1}|, \;\; i = 1, 2, 3,$$

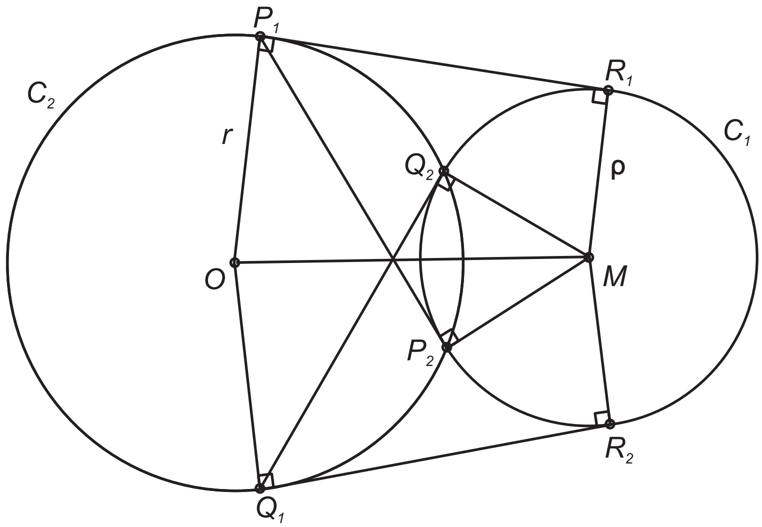


Fig. 2

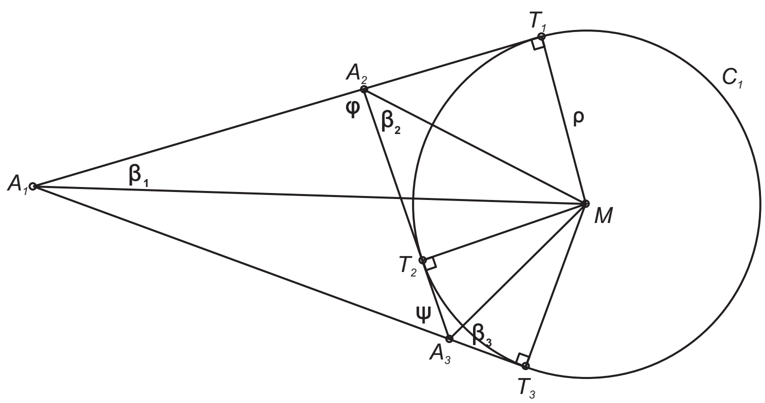


Fig. 3

where

$$\underline{t}_i = t_i \text{ if } \angle MA_iT_i \text{ is positively oriented,}$$
$$\underline{t}_i = -t_i \text{ if } \angle MA_iT_i \text{ is negatively oriented.}$$

Using vertices A_1, A_2, A_3 instead of T_1, T_2, T_3 this can be expressed as follows:

$$\underline{t}_i = t_i \text{ if } \angle MA_iA_{i+1} \text{ is positively oriented,}$$
$$\underline{t}_i = -t_i \text{ if } \angle MA_iA_{i+1} \text{ is negatively oriented.}$$

Of course, if $\angle MA_iA_{i+1}$ is “obtuse” then its supplement is taken.

Remark 1 For simplicity in some of the formulations in this section we shall assume that the vertices of every triangle $A_1A_2A_3$ whose excircle is C_1 and circumcircle C_2 are denoted such that

$$|A_1M| = \max\{|A_1M|, |A_2M|, |A_3M|\}.$$

So, for example, triangle $A_1A_2A_3$ in Fig. 3 is such. Triangle $A_1A_2A_3$ in Fig. 1 becomes such if A_1 and A_2 are mutually interchanged.

Using Fig. 2 it can be said that $A_1 \in \widehat{P_1Q_1}$, where $\widehat{P_1Q_1} \cap \overline{OM} = \emptyset$. As will be seen, doing so, nothing essentially will be changed. First, it can be easily proved that

$$(t_1 - t_2 - t_3)\rho^2 = t_1t_2t_3. \tag{2.3}$$

Namely, from Fig. 3 we see that

$$2\beta_2 = 2\beta_1 + \psi, \qquad 2\beta_3 = 2\beta_1 + \varphi,$$

from which we get

$$-\beta_1 + \beta_2 + \beta_3 = 90°. \tag{2.4}$$

Thus, we can write

$$\begin{gathered}\cot(\beta_2 + \beta_3) = -\tan\beta_1,\\ \cot\beta_1 - \cot\beta_2 - \cot\beta_3 = \cot\beta_1\cot\beta_2\cot\beta_3,\\ \frac{t_1}{\rho} - \frac{t_2}{\rho} - \frac{t_3}{\rho} = \frac{t_1t_2t_3}{\rho^3},\end{gathered}$$

which can be written as (2.3). Now, we can prove the following theorem.

Theorem 2.1 *For every triangle $A_1A_2A_3$ which is such as described in Remark* 1*, the following holds:*

$$|-t_1t_2 + t_2t_3 - t_3t_1| = 4r\rho - \rho^2. \tag{2.5}$$

Proof. From (2.3) we have

$$t_3 = \frac{\rho^2(t_1 - t_2)}{t_1t_2 + \rho^2}. \tag{2.6}$$

Using the above expression for t_3 we get

$$|-t_1t_2 + t_2t_3 - t_3t_1| = \frac{\rho^2(t_1^2 + t_2^2) + t_1^2t_2^2 - \rho^2t_1t_2}{t_1t_2 + \rho^2}. \tag{2.7}$$

Now, we can use the relations

$$J = (t_1 - t_2 - t_3)\rho, \qquad J = \frac{abc}{4r}, \tag{2.8}$$

where

$$J = \text{ area of } ABC, \quad a = t_1 - t_2, \quad b = t_2 + t_3, \quad c = t_1 - t_3.$$

From

$$(t_1 - t_2 - t_3)\rho = \frac{(t_1 - t_2)(t_2 + t_3)(t_1 - t_3)}{4r}$$

and from (2.6) we get

$$4r\rho = \frac{(\rho^2 + t_1^2)(\rho^2 + t_2^2)}{t_1 t_2 + \rho^2} \tag{2.9}$$

or, subtracting ρ^2 from both sides,

$$4r\rho - \rho^2 = \frac{\rho^2(t_1^2 + t_2^2) + t_1^2 t_2^2 - \rho^2 t_1 t_2}{t_1 t_2 + \rho^2}. \tag{2.10}$$

So, equation (2.7) can be written as (2.5). Theorem 2.1 is proved. □

Corollary 2.1.1 *For every triangle $A_1A_2A_3$ whose excircle is C_1 and circumcircle is C_2*

$$|\underline{t}_1\underline{t}_2 + \underline{t}_2\underline{t}_3 + \underline{t}_3\underline{t}_1| = 4r\rho - \rho^2 \tag{2.11}$$

holds, where

$$\underline{t}_i = t_i \ \text{ if } \ \angle MA_iT_i \ \text{ is positively oriented,}$$
$$\underline{t}_i = -t_i \ \text{ if } \ \angle MA_iT_i \ \text{ is negatively oriented.}$$

Proof. The value $|\underline{t}_1\underline{t}_2 + \underline{t}_2\underline{t}_3 + \underline{t}_3\underline{t}_1|$ does not depend upon numeration of vertices of a triangle whose excircle is C_1 and circumcircle is C_2. □

Corollary 2.1.2 *Let $A_1A_2A_3$ and $B_1B_2B_3$ be any two triangles whose excircles have equal radii. Then the circumcircles of these triangles have also equal radii iff*

$$|\underline{t}_1\underline{t}_2 + \underline{t}_2\underline{t}_3 + \underline{t}_3\underline{t}_1| = |\underline{u}_1\underline{u}_2 + \underline{u}_2\underline{u}_3 + \underline{u}_3\underline{u}_1|, \tag{2.12}$$

where

$$|\underline{t}_i + \underline{t}_{i+1}| = |A_iA_{i+1}|, \quad i = 1,2,3,$$
$$|\underline{u}_i + \underline{u}_{i+1}| = |B_iB_{i+1}|, \quad i = 1,2,3.$$

Proof. Iff (2.11) holds, then from

$$|\underline{t}_1\underline{t}_2 + \underline{t}_2\underline{t}_3 + \underline{t}_3\underline{t}_1| = 4r\rho - \rho^2,$$
$$|\underline{u}_1\underline{u}_2 + \underline{u}_2\underline{u}_3 + \underline{u}_3\underline{u}_1| = 4r_1\rho - \rho^2$$

it follows that $r = r_1$. □

Corollary 2.1.3 *Let $B_1B_2B_3$ be the degenerate triangle shown in Fig.* 1. *Then*

$$t_1 = \sqrt{z^2 - (r - \rho)^2}.$$

Proof. From (2.5), since $t_2 = 0$, we get

$$t_1^2 = 4r\rho - \rho^2. \tag{2.13}$$

Now, using Euler's relation (1.3), we can write

$$t_1^2 = 2r\rho + 2r\rho - \rho^2 = z^2 - r^2 + 2r\rho - \rho^2 = z^2 - (r-\rho)^2. \qquad \square$$

For the following use, the length $\sqrt{z^2 - (r-\rho)^2}$ will be denoted by t_0, that is

$$t_0 = \sqrt{z^2 - (r-\rho)^2}. \tag{2.14}$$

See Fig. 2. Let us remark that $t_0 = |P_1P_2| = |Q_1Q_2| = |P_1R_1| = |Q_1R_2|$ since $|P_1R_1| = \sqrt{z^2 - (r-\rho)^2}$.

Corollary 2.1.4 *For degenerate triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ shown in Fig.* 2 *we have*

$$|P_1P_2|^2 = |Q_1Q_2|^2 = |P_1R_1|^2 = |Q_1R_2|^2 = 4r\rho - \rho^2. \tag{2.15}$$

Proof. Note that $t_0^2 = 4r\rho - \rho^2$ holds. $\square$

In the following theorem we shall use the length t_M given by

$$t_M = \sqrt{(r+z)^2 - \rho^2}. \tag{2.16}$$

Let us remark that $t_M \geq t$ for every tangent drawn from C_2 to C_1 (see Fig. 4); $t_M = |PQ|$, and $|PQ| = \sqrt{(r+z)^2 - \rho^2}$.

Also, let us remark that $t_0 \leq t_1 \leq t_M$, where $t_1 = |A_1T_1|$ and $A_1A_2A_3$ is a triangle as noted in Remark 1.

Theorem 2.2 *Let t_1 be such that*

$$t_0 \leq t_1 \leq t_M. \tag{2.17}$$

Then the lengths of the other two tangents are given by

$$t_2 = \frac{2r\rho t_1 + \sqrt{D}}{\rho^2 + t_1^2}, \qquad t_3 = \frac{2r\rho t_1 - \sqrt{D}}{\rho^2 + t_1^2}, \tag{2.18}$$

where

$$D = 4r^2\rho^2 t_1^2 - (\rho^2 + t_1^2)(\rho^2 t_1^2 - 4r\rho^3 + \rho^4). \tag{2.19}$$

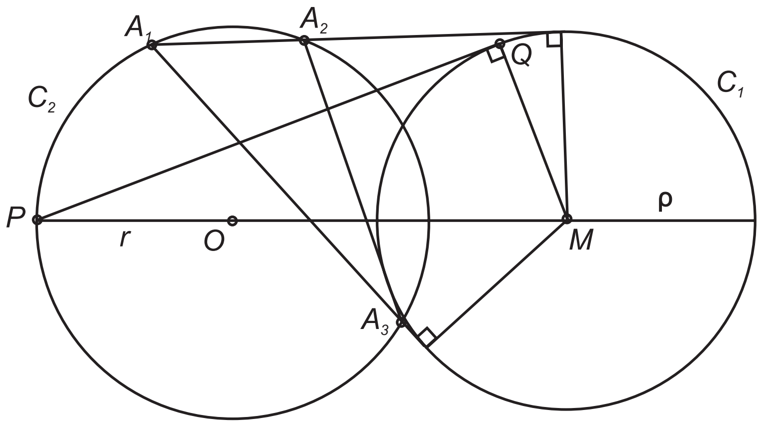


Fig. 4

Proof. The relation (2.9) can be written as

$$(\rho^2+t_1^2)t_2^2-4r\rho t_1t_2+\rho^2t_1^2-4r\rho^3+\rho^4=0,$$

from which, solving for t_2, we get

$$(t_2)_{1,2}=\frac{2r\rho t_1\pm\sqrt{D}}{\rho^2+t_1^2}.$$

Of course, $(t_2)_2=t_3$ since

$$|\underline{t}_1+\underline{t}_2|=|A_1T_1|,\qquad |\underline{t}_1+\underline{t}_3|=|A_1T_3|.$$

Thus, it remains to prove that $D\geq 0$ for every t_1 such that (2.17) holds. For this purpose it is enough to prove that $D=0$ for $t_1=t_M$ and $t_1=-t_M$, that is for $t_1^2=t_M^2$. The proof is as follows: Putting t_M^2 instead of t_1^2 in D/ρ^2 and using Euler's relation (1.3) we can write

$$\begin{aligned}D/\rho^2&=4r^2(r+z)^2-4r^2\rho^2-(r+z)^4+4r\rho(r+z)^2\\&=4r^2(r+z)^2-(z^2-r^2)^2-(r+z)^4+4r\rho(r+z)^2\\&=(r+z)^2\big(4r^2-(z-r)^2-(z+r)^2+2(z^2-r^2)\big)=(r+z)^2\cdot 0=0.\end{aligned}$$

Theorem 2.2 is proved. □

Although t_1 is not given explicitly as are t_2 and t_3, but by condition $t_0\leq t_1\leq t_M$, it is easy to check that for t_1, t_2, t_3 given by (2.17) and (2.18) in the end we get

$$|-t_1t_2+t_2t_3-t_3t_1|=\frac{(4r\rho-\rho^2)(\rho^2+t_1^2)}{\rho^2+t_1^2}=4r\rho-\rho^2.$$

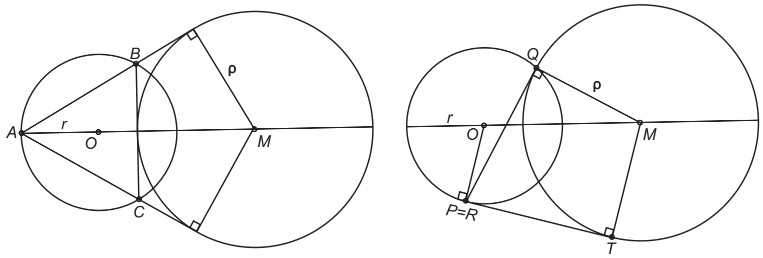
Example 1 Let $r=3$, $z=5$, $\rho=\frac{8}{3}$. Then

$$t_M\approx 7.542472333,\qquad t_0\approx 4.988876516.$$

If we take $t_1=6$, then by (2.18) we get

$$t_2\approx 3.994824489,\qquad t_3\approx 0.458783759.$$

The corresponding triangle $A_1A_2A_3$ is shown in Fig. 4.



a) b)

Fig. 5

In connection with this example let us remark that for $t_1 = t_M$ by (2.18), since $D = 0$, we have

$$t_2 = t_3 = \frac{2r\rho t_M}{\rho^2 + t_M^2} \approx 1.885618083.$$

If we take $t_1 = t_0$, then by (2.18) we have

$$t_2 = t_0, \quad t_3 = 0.$$

In this case, we have $D = 4r^2\rho^2 t_0^2$ since $(\rho^2 + t_0^2)(t_0^2 - 4r\rho + \rho^2) = 0$. Using this example in connection with relation (2.11) we can write

$$|-t_1t_2 + t_2t_3 - t_3t_1| \approx 24.88888889, \; 4r\rho - \rho^2 \approx 24.88888889.$$

Remark 2 It is easy to see that proving Theorem 2.2 we in fact give another proof of Poncelet's closure theorem for triangles where circles are intersecting, using very simple and elementary facts. Therefore, this theorem may be interesting in itself.

Relation (2.11) which has the key role in the proof of Theorem 2.2 has also an important role in the following theorem.

Theorem 2.3 *From* (2.11) *follows Euler's relation given by* (1.3).

Proof. Let ABC be an axially symmetric triangle as shown in Fig. 5a and let PQR be a degenerate triangle as shown in Fig. 5b. Then

$$t_1^2 = (r+z)^2 - \rho^2, \quad t_2^2 = t_3^2 = r^2 - (z-\rho)^2,$$
$$u_1^2 = z^2 - (r-\rho)^2, \quad u_2 = 0, \quad u_3 = -u_1.$$

In connection with u_1 let us remark that $u_1 = |PQ|$ and $|PQ| = |PT|$. Theorem 2.3 immediately follows from

$$|\underline{u}_1\underline{u}_2 + \underline{u}_2\underline{u}_3 + \underline{u}_3\underline{u}_1| = 4r\rho - \rho^2 \quad \text{or} \quad u_1^2 = 4r\rho - \rho^2$$

since

$$z^2 - (r-\rho)^2 = 4r\rho - \rho^2 \Longleftrightarrow z^2 - r^2 = 2r\rho. \qquad \square$$

The following may also be interesting, namely, we can write

$$-t_1t_2 + t_2t_3 - t_3t_1 = -2t_1t_2 + t_2^2, \quad -u_1u_2 + u_2u_3 - u_3u_1 = -u_1^2,$$

and by (2.11) it holds

$$-2t_1t_2 + t_2^2 = -u_1^2$$

or

$$4t_1^2t_2^2 = (t_2^2 + u_1^2)^2,$$

which can be written as

$$(r^2 + 2r\rho - z^2)(r + z - \rho)^2 = 0.$$

Let us remark that from $z^2 - r^2 = 2r\rho$, putting $r + z = \rho$, we get $z = 3r$ and that for $z = 3r$, $\rho = 4r$ it holds $z^2 - r^2 = 2r\rho$. In this limit case we have $4r\rho - \rho^2 = 0$. Thus in this case, $t_1 = t_2 = t_3 = 0$ (the triangle becomes tangential point of C_1 and C_2).

3 Some relations concerning bicentric quadrilaterals when excircles instead of incircles are under consideration

Notation used:

Let r, ρ and z be any given lengths (positive numbers) such that

$$z^2 = r^2 + \rho^2 + \sqrt{4r^2\rho^2 + \rho^4}. \tag{3.1}$$

Let M and O be points and C_1 and C_2 be circles such that

$$|MO| = z, \qquad C_1 = M(\rho), \qquad C_2 = O(r). \tag{3.2}$$

The circles C_1 and C_2 are not intersecting since from (3.1) it follows that

$$z^2 > r^2 + \rho^2 + 2r\rho \ \text{ or } \ z > r + \rho.$$

Let us remark that (3.1) follows from Fuss' relation (1.1), namely, from

$$(r^2 - z^2)^2 = 2\rho^2(r^2 + z^2)$$

it follows that

$$z^2 = r^2 + \rho^2 \pm \sqrt{4r^2\rho^2 + \rho^4}.$$

The condition for a bicentric quadrilateral where C_1 is inside of C_2 is given by

$$z^2 = r^2 + \rho^2 - \sqrt{4r^2\rho^2 + \rho^4}, \tag{3.3}$$

from which it follows that $z < r - \rho$.

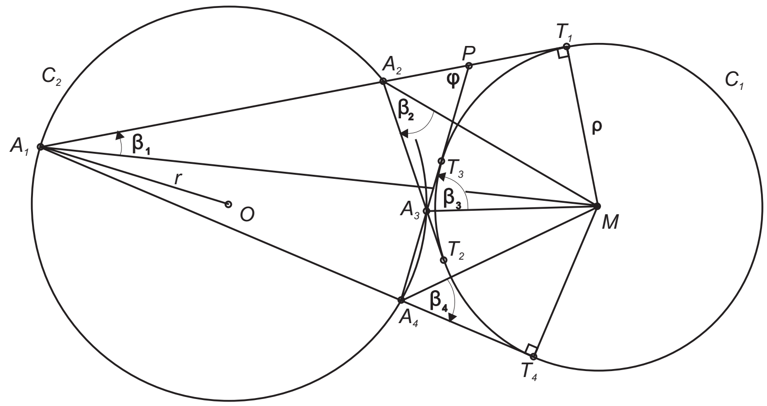


Fig. 6

Now, for example, let $r = 4$, $\rho = 3$, $z = 7.115617418$ (see Fig. 6). It is easy to see that

$$(t_1 - t_2 + t_3 - t_4)\rho = \text{ area of quadrilateral } A_1A_2A_3A_4, \tag{3.4}$$

where

$$|A_1A_2| = t_1 - t_2, \ \ |A_2A_3| = t_2 - t_3, \ \ |A_3A_4| = t_4 - t_3, \ \ |A_4A_1| = t_1 - t_4.$$

Thus, in this case, we must instead of t_2 and t_4 take $-t_2$ and $-t_4$. It is because we must use oriented angles. Namely, if the angle MA_iT_i, $i = 1, 2, 3, 4$, is negatively oriented, then instead of t_i we must take $-t_i$.

It is easy to see that for every quadrilateral $A_1A_2A_3A_4$ whose excircle is C_1 and circumcircle is C_2 either

$$t_1, \ \ -t_2, \ \ t_3, \ \ -t_4 \tag{3.5}$$

or

$$-t_1, \ \ t_2, \ \ -t_3, \ \ t_4 \tag{3.6}$$

holds. Namely, the angles MA_1T_1 and MA_3T_3 are positively oriented and the angles MA_2T_2 and MA_4T_4 are negatively oriented or it is conversely.

Also, it can be easily seen that

$$|\underline{t}_i + \underline{t}_{i+1}| = |A_iA_{i+1}|, \ \ i = 1, 2, 3, 4,$$

where

$$\underline{t}_i = t_i \ \text{ if } \ \angle MA_iT_i \ \text{ is positively oriented,}$$
$$\underline{t}_i = -t_i \ \text{ if } \ \angle MA_iT_i \ \text{ is negatively oriented.}$$

Using vertices A_1, A_2, A_3, A_4 instead of T_1, T_2, T_3, T_4 this can be expressed as follows:

$$\underline{t}_i = t_i \ \text{ if } \ \angle MA_iA_{i+1} \ \text{ is positively oriented,}$$
$$\underline{t}_i = -t_i \ \text{ if } \ \angle MA_iA_{i+1} \ \text{ is negatively oriented.}$$

Of course, if $\angle MA_iA_{i+1}$ is ”obtuse” then its supplement is taken. Now, using Fig. 6, we shall prove that

$$\beta_1 - \beta_2 + \beta_3 - \beta_4 = 0^\circ, \tag{3.7}$$

where

$$\beta_i = \text{ measure of } \angle MA_iT_i, \quad i = 1, 2, 3, 4.$$

First from triangle PA_1A_4, since the measure of $\angle A_3A_4T_4 = 2\beta_4$, we have

$$2\beta_4 = 2\beta_1 + \varphi. \tag{3.8}$$

Now, from triangle PA_2A_3 we see that

$$\varphi + 2\beta_2 + (180 - 2\beta_3) = 180^\circ. \tag{3.9}$$

From (3.8) and (3.9) follows (3.7).

Before we state the following theorem we shall prove that

$$(t_1 - t_2 + t_3 - t_4)\rho^2 = -t_1t_2t_3 + t_2t_3t_4 - t_3t_4t_1 + t_4t_1t_2. \tag{3.10}$$

Starting from (3.7) we can write

$$\tan(\beta_1 + \beta_3) = \tan(\beta_2 + \beta_4),$$

from which, using the relation

$$\frac{\rho}{t_i} = \tan\beta_i, \quad i = 1, 2, 3, 4, \tag{3.11}$$

we readily get (3.10).

Theorem 3.1 *Let $A_1A_2A_3A_4$ be a bicentric quadrilateral whose excircle is C_1 and circumcircle is C_2, where C_1 and C_2 are given by* (3.2)*. Then*

$$t_1t_3 = t_2t_4 = \rho^2, \tag{3.12}$$

where

$$t_i = |A_iT_i|, \quad i = 1, 2, 3, 4.$$

Proof. Since either (3.5) or (3.6) is possible we may assume without loss of generality that (3.5) is valid, namely, that the situation is like that in Fig. 6, where

$$|A_1A_2| = t_1 - t_2, \quad |A_2A_3| = t_2 - t_3, \quad |A_3A_4| = t_4 - t_3, \quad |A_4A_1| = t_1 - t_4.$$

Since (3.4) holds we have the equality

$$(t_1 - t_2 + t_3 - t_4)\rho = \sqrt{(t_1 - t_2)(t_2 - t_3)(t_4 - t_3)(t_1 - t_4)}$$

or

$$(t_1 - t_2 + t_3 - t_4)^2\rho^2 = (t_1 - t_2)(t_2 - t_3)(t_4 - t_3)(t_1 - t_4). \tag{3.13}$$

The above equality, using equality (3.10), can be written as

$$(t_1 - t_2 + t_3 - t_4)(-t_1t_2t_3 + t_2t_3t_4 - t_3t_4t_1 + t_4t_1t_2) = (t_1 - t_2)(t_2 - t_3)(t_4 - t_3)(t_1 - t_4)$$

or

$$t_1^2t_3^2 - 2t_1t_2t_3t_4 + t_2^2t_4^2 = 0,$$

from which it follows that $(t_1t_3 - t_2t_4)^2 = 0$ or

$$t_1t_3 = t_2t_4. \tag{3.14}$$

Now, from (3.10), putting $t_4 = \frac{t_1t_3}{t_2}$, we get

$$\rho^2 = \frac{t_1t_3(t_1 + t_2)(t_2 + t_3)}{(t_1 + t_2)(t_2 + t_3)} = t_1t_3.$$

Also it is valid $\rho^2 = t_2t_4$ since (3.14) is valid. Theorem 3.1 is proved. □

Corollary 3.1.1 *Let $A_1A_2A_3A_4$ be any given tangential quadrilateral whose excircle is C_1. Then this quadrilateral will be a bicentric one whose circumcircle is C_2 iff* (3.12) *holds.*

Proof. From (3.10) and (3.12) follows (3.13). □

Theorem 3.2 *Let $ABCD$ and $PRQS$ be two bicentric quadrilaterals such that their excircles are congruent. Then their circumcircles are also congruent iff*

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = u_1u_2 + u_2u_3 + u_3u_4 + u_4u_1, \tag{3.15}$$

where t_i and u_i, $i = 1, 2, 3, 4$, are the lengths of the consecutive tangents relating to $ABCD$ and $PQRS$, respectively.

Proof. First, let us remark, that from (3.5) and also from (3.6) it follows that

$$\underline{t}_1\underline{t}_2 + \underline{t}_2\underline{t}_3 + \underline{t}_3\underline{t}_4 + \underline{t}_4\underline{t}_1 = -t_1t_2 - t_2t_3 - t_3t_4 - t_4t_1,$$

where $\underline{t}_i = t_i$ or $\underline{t}_i = -t_i$ depending on how the angle MA_iT_i is oriented. Using the expression $-(t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1)$ and the equalities $t_1t_3 = \rho^2$ and $t_2t_4 = \rho^2$ given by (3.12), we find that

$$-(t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1) = \frac{t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^4}{-t_1t_2}. \tag{3.16}$$

Let r be the radius of the circumcircle of $ABCD$. We have to prove that r is also the radius of the circumcircle of $PQRS$ iff (3.15) holds. In the proof we shall use the well-known relations concerning chordal quadrilaterals. These relations are

$$r^2 = \frac{(ad + cd)(ac + bd)(ad + bc)}{16J^2}, \quad J^2 = abcd, \tag{3.17}$$

where

$$a = t_1 - t_2, \quad b = t_2 - t_3, \quad c = t_4 - t_3, \quad d = t_1 - t_4, \quad J = \text{area of } ABCD.$$

From (3.17) it follows that

$$16r^2 = a^2 + b^2 + c^2 + d^2 + \frac{abc}{d} + \frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c},$$

which, using (3.12), can be written as

$$16r^2\rho^2 + 4\rho^4 = \left[\frac{t_1^2t_2^2 + \rho^2(t_1^2 + t_2^2) + \rho^4}{-t_1t_2} + 2\rho^2\right]^2. \tag{3.18}$$

Analogously, for the bicentric quadrilateral $PQRS$ we have

$$16r_1^2\rho^2 + 4\rho^4 = \left[\frac{u_1^2u_2^2 + \rho^2(u_1^2 + u_2^2) + \rho^4}{-u_1u_2} + 2\rho^2\right]^2,$$

where r_1 is the radius of the circumcircle of $PQRS$. Thus, iff (3.15) is valid, then $r_1 = r$. Theorem 3.2 is proved. □

Now, we shall prove that the left-hand side of (3.18) can be written as $4(r^2 + \rho^2 - z^2)^2$, namely, that it holds

$$16r^2\rho^2 + 4\rho^4 = 4(r^2 + \rho^2 - z^2)^2.$$

For this purpose, we shall add $\rho^4 + 2r^2\rho^2 - 2\rho^2z^2$ on both sides of Fuss' relation for a bicentric quadrilateral

$$2\rho^2(r^2 + z^2) = (r^2 - z^2)^2.$$

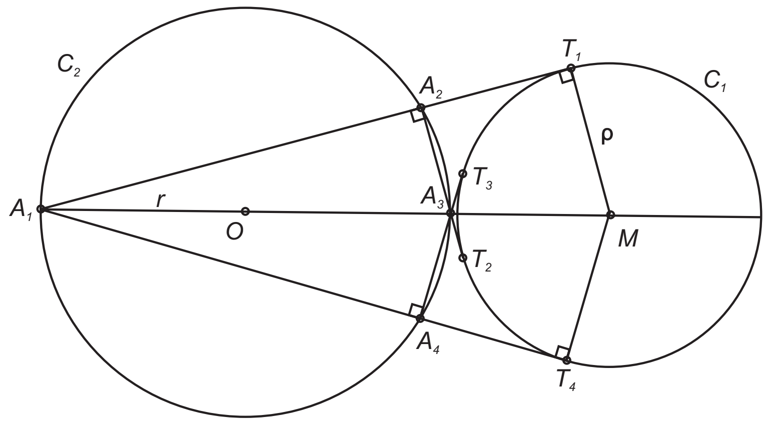


Fig. 7

So, we can write

$$2\rho^2(r^2+z^2)+(\rho^4+2r^2\rho^2-2\rho^2z^2)=(r^2-z^2)^2+(\rho^4+2r^2\rho^2-2\rho^2z^2)$$

or

$$4r^2\rho^2+\rho^4=(r^2+\rho^2-z^2)^2.$$

Thus, the equality (3.18) can be written as

$$\frac{t_1^2t_2^2+\rho^2(t_1^2+t_2^2)+\rho^4}{-t_1t_2}=2(r^2-z^2)$$

or

$$\frac{t_1^2t_2^2+\rho^2(t_1^2+t_2^2)+\rho^4}{t_1t_2}=2(z^2-r^2). \tag{3.19}$$

Since (3.16) holds, we have the following relation

$$t_1t_2+t_2t_3+t_3t_4+t_4t_1=2(z^2-r^2). \tag{3.20}$$

In some of the following theorems we shall use the relations

$$t_m=\sqrt{(z-r)^2-\rho^2}, \qquad t_M=\sqrt{(z+r)^2-\rho^2}. \tag{3.21}$$

See Fig. 7. As can be seen, $t_m=|A_3T_3|$ is the length of the shortest tangent that can be drawn from C_2 to C_1, and $t_M=|A_1T_1|$ is the length of the largest tangent that can be drawn from C_2 to C_1.

By (3.12) it holds

$$t_mt_M=\rho^2. \tag{3.22}$$

Theorem 3.3 *From* (3.22) *follows Fuss' relation given by* (1.1).

Proof. It holds

$$t_m^2 t_M^2 = (r^2 - z^2)^2 - 2\rho^2(r^2 + z^2) + \rho^4,$$

and from (3.22) it follows $t_m^2 t_M^2 - \rho^4 = 0$, that is

$$(r^2 - z^2)^2 - 2\rho^2(r^2 + z^2) = 0.$$

Theorem 3.3 is proved. □

Thus, in this way we can deduce Fuss' relation for bicentric quadrilaterals.

Fuss' relation for bicentric quadrilaterals is closely connected with the relations (3.12) and (3.20). So, for example, using Fig. 7, it is easy to show that (3.20) holds for

$$t_1 = t_M, \qquad t_2 = \rho, \qquad t_3 = t_m, \qquad t_4 = \rho.$$

First, let us remark that from $t_2 t_4 = \rho^2$, since $t_2 = t_4$ and (3.12) holds, it follows that $t_2 = \rho$. So, in this case, we have

$$t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1 = 2\rho(t_m + t_M),$$

and it is easy to show that

$$2\rho(t_m + t_M) = 2(z^2 - r^2). \tag{3.23}$$

Namely, since $2t_m t_M = 2\rho^2$, we can write

$$\rho^2(t_m + t_M)^2 = \rho^2[(z - r)^2 + (z + r)^2 - 2\rho^2] + 2\rho^4 = 2\rho^2(r^2 + z^2).$$

Thus,

$$[2\rho(t_m + t_M)]^2 = [2(z^2 - r^2)]^2,$$

since $2\rho^2(r^2 + z^2) = (z^2 - r^2)^2$ by Fuss' relation (1.1).

Also, using Fuss' relation, it can be easily shown that the following theorem holds.

Theorem 3.4 *It holds*

$$(z + r)^2 t_m = (z - r)^2 t_M, \tag{3.24}$$

$$t_m = \frac{z - r}{z + r}\rho, \qquad t_M = \frac{z + r}{z - r}\rho, \tag{3.25}$$

$$t_m = \frac{z^2 - r^2 - \sqrt{D}}{2\rho}, \qquad t_M = \frac{z^2 - r^2 + \sqrt{D}}{2\rho}, \tag{3.26}$$

where

$$D = (z^2 - r^2)^2 - 4\rho^4. \tag{3.27}$$

Proof. The proof that (3.24) holds:

$$(z+r)^4 t_m^2 - (z-r)^4 t_M^2 = 4rz[(z^2-r^2)^2 - 2\rho^2(z^2+r^2)] = 4rz\cdot 0 = 0.$$

Concerning (3.25), it is easy to show that

$$(r^2-z^2)^2 = 2\rho^2(r^2+z^2) \Longleftrightarrow \sqrt{(r-z)^2-\rho^2} = \frac{r-z}{r+z}\rho,$$
$$(r^2-z^2)^2 = 2\rho^2(r^2+z^2) \Longleftrightarrow \sqrt{(r+z)^2-\rho^2} = \frac{r+z}{r-z}\rho.$$

So, from

$$(r-z)^2-\rho^2 = \left(\frac{r-z}{r+z}\right)^2\rho^2$$

it follows

$$(r^2-z^2)^2 = \rho^2\Big((r-z)^2+(r+z)^2\Big)$$

or

$$(r^2-z^2)^2 = 2\rho^2(r^2+z^2).$$

Obviously, the converse is also valid. Concerning (3.26), using (3.22) and (3.23), we can write

$$t_m t_M = \rho^2, \qquad t_m + t_M = \frac{z^2-r^2}{\rho},$$

from which (3.26) follows. □

Corollary 3.4.1 *The following is true:*

$$z^2 > r^2 + 2\rho^2.$$

Proof. It follows from (3.27). Of course, it also follows from (3.1) since $\sqrt{4r^2\rho^2+\rho^4} > \rho^2$. □

Theorem 3.5 *It holds*

$$A(t_1,\ -t_2,\ t_3,\ -t_4)\cdot H(t_1,\ -t_2,\ t_3,\ -t_4) = \rho^2, \tag{3.28}$$

where $A(t_1,-t_2,t_3,-t_4)$ *and* $H(t_1,-t_2,t_3,-t_4)$ *are the arithmetic and harmonic means of* t_1, $-t_2$, t_3, $-t_4$.

Proof. (3.12), $t_1t_3 = t_2t_4 = \rho^2$, implies $t_1t_2t_3t_4 = \rho^4$. If we divide equation (3.10) by $t_1t_2t_3t_4$, we can write

$$\frac{(t_1-t_2+t_3-t_4)\rho^2}{\rho^4} = \frac{-t_1t_2t_3+t_2t_3t_4-t_3t_4t_1+t_4t_1t_2}{t_1t_2t_3t_4}$$

or

$$\frac{t_1-t_2+t_3-t_4}{4}\cdot\frac{4}{\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}-\frac{1}{t_4}} = \rho^2.$$

Theorem 3.5 is proved. □

Theorem 3.6 *Let $ABCD$ be any given bicentric quadrilateral whose excircle is C_1 and circumcircle is C_2, where C_1 and C_2 are given by* (3.2)*. Then*

$$ef = 2(z^2 - r^2 - 2\rho^2), \tag{3.29}$$

where $e = |AC|$, $f = |BD|$. In other words, for every bicentric quadrilateral whose excircle is C_1 and circumcircle is C_2, the product of the lengths of its diagonals is the constant $2(z^2 - r^2 - 2\rho^2)$.

Proof. Let $a = t_1 - t_2$, $b = t_2 - t_3$, $c = t_4 - t_3$, $d = t_1 - t_4$ be the lengths of the sides of $ABCD$. Then, by Ptolomy's theorem,

$$ef = ac + bd,$$

and we can write

$$\begin{aligned} ac + bd &= (t_1 - t_2)(t_4 - t_3) + (t_2 - t_3)(t_1 - t_4) \\ &= (t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1) - 2(t_1t_3 + t_2t_4) \\ &= 2(z^2 - r^2) - 2(\rho^2 + \rho^2) = 2(z^2 - r^2 - 2\rho^2). \end{aligned}$$

It is easy to see that we have the same result if instead of the possibility (3.5) we take the possibility (3.6). Theorem 3.6 is proved. □

Theorem 3.7 *Let r, ρ and z be any given positive numbers such that* (1.1) *is satisfied, and let t_m and t_M be given by* (3.21)*. Then every positive solution $(t_1, t_2, t_3, t_4) \in \mathbb{R}^4_+$ of the equations*

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(z^2 - r^2), \quad t_1t_3 = \rho^2, \quad t_2t_4 = \rho^2$$

is given by

$$t_1 \text{ is a positive number such that } t_m \le t_1 \le t_M, \tag{3.30}$$

$$t_2 = \frac{(z^2 - r^2)t_1 + \sqrt{D}}{\rho^2 + t_1^2}, \tag{3.31}$$

$$t_3 = \frac{\rho^2}{t_1}, \tag{3.32}$$

$$t_4 = \frac{\rho^2}{t_2}, \tag{3.33}$$

where

$$D = (z^2 - r^2)^2 t_1^2 - \rho^2(\rho^2 + t_1^2)^2. \tag{3.34}$$

Proof. The equation $t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(z^2 - r^2)$, using equations $t_1t_3 = \rho^2$ and $t_2t_4 = \rho^2$, can be written as

$$(\rho^2 + t_1^2)t_2^2 - 2(z^2 - r^2)t_1t_2 + \rho^2(t_1^2 + \rho^2) = 0, \tag{3.35}$$

from which it follows that

$$(t_2)_{1,2} = \frac{(z^2 - r^2)t_1 \pm \sqrt{D}}{\rho^2 + t_1^2}.$$

It is unessential which of $(t_2)_1$ and $(t_2)_2$ will be taken for t_2 since

$$\frac{\rho^2}{(t_2)_1} = \frac{\rho^2(\rho^2 + t_1^2)}{(z^2 - r^2)t_1 + \sqrt{D}} = \frac{(z^2 - r^2)t_1 - \sqrt{D}}{\rho^2 + t_1^2} = (t_2)_2.$$

If we take $t_2 = (t_2)_1$, then $\frac{\rho^2}{t_2} = (t_2)_2$, that is, by (3.33), $(t_2)_2 = t_4$. But if we take $t_2 = (t_2)_2$, then $\frac{\rho^2}{t_2} = (t_2)_1$. Thus, in this case $(t_2)_1 = t_4$.

Now, since in the expression of t_2 in (3.31) appears the term $\sqrt{D}$, we have to prove that $D \geq 0$ for every t_1 such that $t_m \leq t_1 \leq t_M$. Of course, for this purpose it suffices to prove that $D = 0$ for $t_1 = t_m$ and $t_1 = t_M$.

It is easy to show that

$$(z^2 - r^2)^2 t_m^2 - \rho^2(\rho^2 + t_m^2)^2 = 0 \Longleftrightarrow (1.1),$$
$$(z^2 - r^2)^2 t_M^2 - \rho^2(\rho^2 + t_M^2)^2 = 0 \Longleftrightarrow (1.1),$$

where (1.1) stands instead of Fuss' relation given by (1.1). So, for $t_1 = t_m$, we can write

$$(z^2 - r^2)^2 t_m^2 - \rho^2(\rho^2 + t_m^2)^2 = (z - r)^2[(z^2 - r^2)^2 - 2\rho^2(z^2 + r^2)] = (z - r)^2 \cdot 0 = 0.$$

This completes the proof of Theorem 3.7. □

Although t_1 is not given explicitly but by condition $t_m \leq t_1 \leq t_M$, it is easy to check that for $t_1,\ t_2,\ t_3,\ t_4$ given by (3.30)–(3.33) in the end we get

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = \frac{(z^2 - r^2)t_1 + \sqrt{D}}{t_1} + \frac{(z^2 - r^2)t_1 - \sqrt{D}}{t_1} = 2(z^2 - r^2).$$

Corollary 3.7.1 *Let C_1 and C_2 be circles such that* (3.1) *and* (3.2) *holds. Let A_1 be any given point on C_2 and let t_1 be the length of the tangent A_1T_1 drawn from C_2 to C_1. Then the lengths t_2, t_3, t_4 of the other three tangents drawn from C_2 to C_1 are given by* (3.31), (3.32) *and* (3.33)*.*

Here is an example:

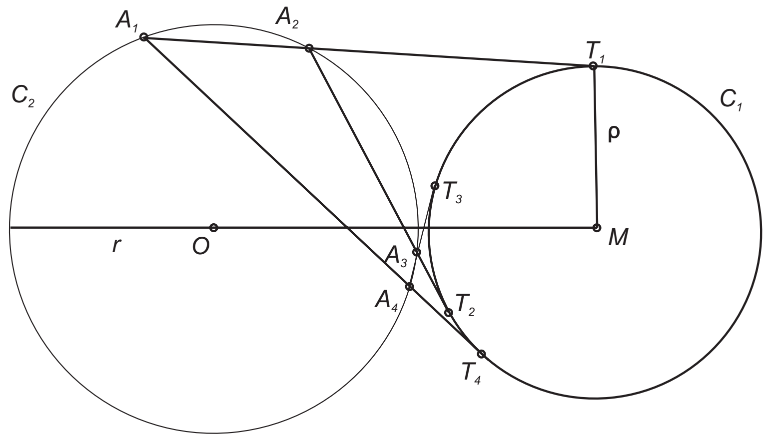


Fig. 8

Example 2 Let $r = 4$, $\rho = 3$, $z = 7.115617418$. Then

$$t_m \approx 0.840875671, \quad t_M \approx 10.70312807, \quad D \approx 28799.07696.$$

If we take $t_1 = 8$, then

$$t_2 \approx 6.119986271, \quad t_3 = 1.125, \quad t_4 \approx 1.470591534.$$

The corresponding quadrilateral $A_1A_2A_3A_4$ is shown in Fig. 8.
It can be checked that

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 \approx 69.26402247 = 2(z^2 - r^2).$$

Also, it can be checked that

$$\beta_1 \approx 20.55604522^\circ, \quad \beta_2 \approx 26.11396343^\circ,$$
$$\beta_3 \approx 69.44395478^\circ, \quad \beta_4 \approx 63.88603657^\circ,$$

$$\beta_1 - \beta_2 + \beta_3 - \beta_4 = 0^\circ,$$

where $\beta_i = \arctan \frac{\rho}{t_i}$, $i = 1, 2, 3, 4$.

If in this figure we write A_2 where is A_4 and A_4 where is A_2, then the angles MA_1T_1 and MA_3T_3 will be negatively oriented and in this case will be

$$-\beta_1 + \beta_2 - \beta_3 + \beta_4 = 0^\circ.$$

Remark 3 As can be seen, by proving Theorem 3.7, we in fact give another proof of Poncelet's closure theorem for bicentric quadrilaterals, when the excircle instead of the incircle is under consideration. In this proof, we use very simple and elementary mathematical facts. Therefore, this proof may be interesting in itself.

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