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Elemente der Mathematik

Note on the diophantine equation $1 + 2p + (2p)^2 + \cdots + (2p)^n = y^p$

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In a note published in 1987, A. Rotkiewicz [2] showed that repunits, i.e., numbers of the form $111\ldots11 = 1 + 10 + \cdots + 10^n$, are never squares or cubes for $n \geq 1$. He did this using a more general result on the diophantine equations $1 + x + \cdots + x^n = y^2$ and $1 + x + \cdots + x^n = y^3$ proved by W. Ljunggren in 1943 [1].

The following result implies that repunits are never fifth powers of integer numbers as well.

Theorem. *Let p be an odd prime number or a Carmichael number and let $n \in \mathbb{N}$. Then, the diophantine equation $1 + 2p + (2p)^2 + \cdots + (2p)^n = y^p$ has no solution y.*

Proof. For $n = 1$ the diophantine equation gives

$$1 + 2p = y^p. \tag{1}$$

Because of $y^p \equiv 1 \bmod p$, y cannot be a multiple of p. Moreover, Fermats little theorem gives $y^p \equiv y \bmod p$. Therefore, y has to be of the form $y = pk + 1$ with a $k \in \mathbb{N}_0$. Hence, the Bernoulli inequality implies $1 + 2p = y^p \geq 1 + p^2k$, i.e., $2 \geq pk$. This is possible only if $k = 0$ (and $y = 1$) contradicting (1).

„Repunits" sind Zahlen mit lauter Einsen: 1, 11, 111, 1111, ... Es ist bekannt, dass sie niemals Quadrate oder Kubikzahlen sein können. In dieser Arbeit wird gezeigt, dass sie auch keine fünfte Potenzen sein können. Dies ist der Spezialfall $p = 5$ des hier durch Kongruenzschlüsse bewiesenen Satzes, dass die im Titel genannte Gleichung keine Lösungen n, y in positiven ganzen Zahlen hat, wenn p eine ungerade Primzahl, oder allgemeiner eine ungerade Zahl ist, für die der kleine Fermatsche Satz in der Form $x^p \equiv x \pmod p$ für alle x gilt.

Suppose now that there is an $n > 1$ leading to the solution $y \in \mathbb{Z}$. Then the two congruences

$$y^p \equiv 1 \bmod 2p \tag{2}$$

and

$$y^p \equiv 2p + 1 \bmod (2p)^2 \tag{3}$$

have to be fulfilled simultaneously. To get equation (2), y must be of the form $(2p)k + 1$ with a $k \in \mathbb{Z}$. This can be seen writing $y = 2pk + r$ with $k \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, 2p - 1\}$. Then

$$(2pk + r)^p \equiv r^p \bmod 2p$$

and again with Fermat

$$(2pk + r)^p \equiv r \bmod p,$$

leading to $y^p = 2pl + r^p$ and $y^p = pm + r$ with $l, m \in \mathbb{Z}$. As r and r^p are both either odd or even, pm is a multiple of 2 and therefore, m is an even number. Hence, $y^p \equiv 1 \equiv r \bmod 2p$, and so $r = 1$.

Using the binomial formula we obtain $y^p = (2pk + 1)^p \equiv 2p^2k + 1 \bmod (2p)^2$. Then an even k leads to $y^p \equiv 1 \bmod (2p)^2$, while we get $y^p \equiv 2p^2 + 1 \bmod (2p)^2$ with k odd. Considering the inequality $1 < 2p + 1 < 2p^2 + 1 < (2p)^2$ this contradicts the congruence (3). Therefore, the assumed n does not exist. □

Choosing $p = 5$, the diophantine equation $1 + 10 + \cdots + 10^n = y^5$ has no solution, and so the numbers $11, 111, 1111, \ldots$ are never fifth powers of integer numbers.

References

[1] Ljunggren, W.: Noen setninger om ubestemte likninger av formen $\frac{x^n-1}{x-1} = y^q$. *Norsk Mat. Tidsskr.* 25 (1943), 17–20.

[2] Rotkiewicz, A.: Note on the diophantine equation $1 + x + x^2 + \cdots + x^n = y^m$. *Elem. Math.* 42 (1987), 76.

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