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Elemente der Mathematik

Stirling numbers of the second kind and Bonferroni's inequalities

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Horst Wegner studierte Mathematik an der Universität Hamburg. Nach dem Diplom 1966 und kurzer Tätigkeit in der Industrie promovierte er 1970 über ein Problem zu Stirlingschen Zahlen zweiter Art an der Universität Köln. Seit 1973 ist er als Akademischer Oberrat an der Universität Duisburg tätig, zunächst in der Lehrerausbildung und seit 1982 im Fachgebiet Stochastik.

1 Introduction

The number of ways of partitioning a set of n elements into k nonempty subsets is usually denoted $S(n,k)$. The numbers $S(n,k)$ are called Stirling numbers of the second kind. It is well-known that

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n$$

for $n, k \in \mathbb{N}$.

By an easy application of Bonferroni's inequalities, it can be shown that the partial sums

$$\frac{1}{k!}\sum_{i=0}^{l}(-1)^i\binom{k}{i}(k-i)^n, \quad l = 0, 1, \ldots, k-1,$$

successively overcount and undercount the number $S(n,k)$ (see Theorem 3).

Auf wie viele Weisen lässt sich eine Menge von n Elementen in k nicht-leere Teilmengen zerlegen? Die Antwort hierauf liefern die Stirlingschen Zahlen $S(n,k)$ zweiter Art. Der Zusatz „zweiter Art" hat historische Gründe. Die Stirlingschen Zahlen erster Art können z.B. als Koeffizienten von x^k in dem Polynom $x(x-1)\cdot\ldots\cdot(x-n+1)$ und die Stirlingschen Zahlen zweiter Art umgekehrt als Koeffizienten in der Darstellung von x^n als Linearkombination der Ausdrücke $(x)_k := x(x-1)\cdot\ldots\cdot(x-k+1)$ eingeführt werden. In der nachfolgenden Arbeit wird mit elementaren kombinatorischen Überlegungen, insbesondere den Ungleichungen von Bonferroni, gezeigt, dass die Partialsummen der bekannten geschlossenen Darstellung für die $S(n,k)$ abwechselnd obere und untere Schranken für die $S(n,k)$ sind.

There are only a few textbooks on combinatorics, which present Bonferroni's inequalities (e.g. [1], [2]), and the representation is sometimes not very satisfactory to the reader (cf. [1]). Therefore, we start with a deduction of Bonferroni's inequalities in a concise and elementary way (see Theorem 1).

2 Bonferroni's inequalities

Bonferroni's inequalities are closely related to the principle of inclusion and exclusion.

Theorem 1 *Let Ω be a nonempty set, and let $A_1, \ldots, A_n$ be finite nonempty subsets of Ω. Putting*

$$m := \max\left\{ |T| : T \subset \{1, 2, \ldots, n\}, \bigcap_{i \in T} A_i \neq \emptyset \right\},$$

then $m \geq 1$ and for $l = 1, 2, \ldots, m$

$$\sum_{j=1}^{l} (-1)^{j-1} \sum_{\substack{T \subset \{1,2,\ldots,n\} \\ |T|=j}} \left| \bigcap_{i \in T} A_i \right| \begin{cases} = \left| \bigcup_{i=1}^{n} A_i \right| & \text{if} \quad l = m, \\ > \left| \bigcup_{i=1}^{n} A_i \right| & \text{if} \quad l < m, \quad l \text{ odd}, \\ < \left| \bigcup_{i=1}^{n} A_i \right| & \text{if} \quad l < m, \quad l \text{ even}. \end{cases}$$

The proof of Theorem 1 becomes short and clear, if we apply the following lemma. For this, a useful device is the so-called indicator function of a set A

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Lemma *Let Ω be a nonempty set, and let $A_1, \ldots, A_n \subset \Omega$ with $\bigcup_{i=1}^{n} A_i \neq \emptyset$. For each $\omega \in \bigcup_{i=1}^{n} A_i$ put $m(\omega) := |\{i : \omega \in A_i\}|$. Then, $m(\omega) \geq 1$ and*

$$\sum_{j=1}^{l} (-1)^{j-1} \sum_{\substack{T \subset \{1,2,\ldots,n\} \\ |T|=j}} \mathbf{1}_{\bigcap_{i \in T} A_i}(\omega) = 1 - (-1)^l \binom{m(\omega) - 1}{l}$$

for $l = 1, 2, \ldots, n$.

Proof. Let $\omega \in \bigcup_{i=1}^{n} A_i$. For abbreviation we put $I(\omega) := \{i : \omega \in A_i\}$. Hence $I(\omega) \neq \emptyset$ and $m(\omega) = |I(\omega)| \geq 1$. Then, for $j = 1, 2, \ldots, n$ we have

$$\sum_{\substack{T \subset \{1,2,\ldots,n\} \\ |T|=j}} \mathbf{1}_{\bigcap_{i \in T} A_i}(\omega) = \sum_{\substack{T \subset I(\omega) \\ |T|=j}} \mathbf{1}_{\bigcap_{i \in T} A_i}(\omega) = \binom{m(\omega)}{j}.$$

Hence, for $1 \leq l \leq n$

$$\sum_{j=1}^{l}(-1)^{j-1}\sum_{\substack{T\subset\{1,2,\dots,n\}\\|T|=j}}\mathbf{1}_{\bigcap_{i\in T}A_i}(\omega)=\sum_{j=1}^{l}(-1)^{j-1}\binom{m(\omega)}{j}$$

$$=\sum_{j=1}^{l}(-1)^{j-1}\left(\binom{m(\omega)-1}{j-1}+\binom{m(\omega)-1}{j}\right)$$

$$=1+(-1)^{l-1}\binom{m(\omega)-1}{l}.$$

□

Proof of Theorem 1. Obviously, $m \geq 1$. For abbreviation, we put $A := \bigcup_{i=1}^{n} A_i$. Then, using the notation and the result of the preceding lemma, we obtain

$$\sum_{j=1}^{l}(-1)^{j-1}\sum_{\substack{T\subset\{1,2,\dots,n\}\\|T|=j}}\left|\bigcap_{i\in T}A_i\right|=\sum_{j=1}^{l}(-1)^{j-1}\sum_{\substack{T\subset\{1,2,\dots,n\}\\|T|=j}}\sum_{\omega\in A}\mathbf{1}_{\bigcap_{i\in T}A_i}(\omega)$$

$$=\sum_{\omega\in A}\left(1-(-1)^{l}\binom{m(\omega)-1}{l}\right)$$

$$=|A|-(-1)^{l}\sum_{\omega\in A}\binom{m(\omega)-1}{l}.$$

Obviously, $m = \max_{\omega\in A} m(\omega)$. Therefore:

$$l=m \quad\Rightarrow\quad \binom{m(\omega)-1}{l}=0 \quad \text{for all } \omega\in A,$$

$$l<m \quad\Rightarrow\quad \binom{m(\omega)-1}{l}>0 \quad \text{for at least one } \omega\in A.$$

Thus, the result of Theorem 1 follows immediately. □

It is obvious that Theorem 1 could be obtained as a corollary to the following Theorem 2, which should also be mentioned here. Nevertheless, we have presented an individual proof of Theorem 1 to show its simplicity.

Theorem 2 *Let* $(\Omega, \mathfrak{A}, \mu)$ *be a measure space, and let* $A_1, \dots, A_n \in \mathfrak{A}$ *with* $0 < \mu\big(\bigcup_{i=1}^{n} A_i\big) < \infty$. *Putting* $m := \max\big\{|T| : T \subset \{1, 2, \dots, n\}, \mu\big(\bigcap_{i\in T} A_i\big) > 0\big\}$, *then* $m \geq 1$ *and for* $l = 1, 2, \dots, m$

$$\sum_{j=1}^{l}(-1)^{j-1}\sum_{\substack{T\subset\{1,2,\dots,n\}\\|T|=j}}\mu\Big(\bigcap_{i\in T}A_i\Big)\begin{cases} = \mu\big(\bigcup_{i=1}^{n}A_i\big) & \text{if } l=m,\\ > \mu\big(\bigcup_{i=1}^{n}A_i\big) & \text{if } l<m,\quad l \text{ odd},\\ < \mu\big(\bigcup_{i=1}^{n}A_i\big) & \text{if } l<m,\quad l \text{ even}.\end{cases}$$

Proof. Since there is at least one A_i with $\mu(A_i) > 0$, it is clear that $m \geq 1$.

Moreover, $\bigcup_{i=1}^{n} A_i \neq \emptyset$. Thus we obtain by the lemma for all $l = 1, 2, \ldots, n$ and all $\omega \in \Omega$

$$(*) \qquad \sum_{j=1}^{l}(-1)^{j-1} \sum_{\substack{T\subset\{1,2,\ldots,n\}\\|T|=j}} \mathbf{1}_{\bigcap_{i\in T} A_i}(\omega) \;=\; \mathbf{1}_{\bigcup_{i=1}^{n} A_i}(\omega) - (-1)^l R_l(\omega)$$

with

$$R_l(\omega) = \begin{cases} \binom{m(\omega)-1}{l} & \text{if } \omega \in \bigcup_{i=1}^{n} A_i, \\ 0 & \text{if } \omega \notin \bigcup_{i=1}^{n} A_i. \end{cases}$$

Since $\mu\big(\bigcup_{i=1}^{n} A_i\big) < \infty$, the functions $\mathbf{1}_{\bigcup_{i=1}^{n} A_i}$, $\mathbf{1}_{\bigcap_{i\in T} A_i}$, R_l are μ-integrable.

First we consider $(*)$ for $l = n$. Since $R_n = 0$, it follows from $(*)$ by integration with respect to μ that

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{j=1}^{n}(-1)^{j-1} \sum_{\substack{T\subset\{1,2,\ldots,n\}\\|T|=j}} \mu\Big(\bigcap_{i\in T} A_i\Big).$$

According to the definition of m, this means

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{j=1}^{m}(-1)^{j-1} \sum_{\substack{T\subset\{1,2,\ldots,n\}\\|T|=j}} \mu\Big(\bigcap_{i\in T} A_i\Big),$$

which is the asserted equality for $l = m$.

Now let $l < m$. From the definition of m there is a subset $T_0 \subset \{1, 2, \ldots, n\}$ with $|T_0| = m$ such that $\mu\big(\bigcap_{i\in T_0} A_i\big) > 0$. Hence $\bigcap_{i\in T_0} A_i \neq \emptyset$ and $m(\omega) = m$ for all $\omega \in \bigcap_{i\in T_0} A_i$. This implies

$$R_l(\omega) = \binom{m-1}{l} \geq 1 \quad \text{for all } \omega \in \bigcap_{i\in T_0} A_i.$$

Hence

$$\int R_l \,\mathrm{d}\mu \geq \int \mathbf{1}_{\bigcap_{i\in T_0} A_i} \mathrm{d}\mu = \mu\Big(\bigcap_{i\in T_0} A_i\Big) > 0.$$

Integrating $(*)$ with respect to μ, the last result completes the proof of the asserted inequalities for $l < m$. □

3 Inequalities for the numbers $S(n, k)$

Theorem 3 *Let $n, k \in \mathbb{N}$ with $k \leq n$. Then*

$$\frac{1}{k!}\sum_{j=0}^{l}(-1)^j\binom{k}{j}(k-j)^n \begin{cases} = S(n,k) & \text{if } l = k-1, \\ > S(n,k) & \text{if } 0 \leq l < k-1, \quad l \text{ even}, \\ < S(n,k) & \text{if } 1 \leq l < k-1, \quad l \text{ odd}. \end{cases}$$

Proof. Because of $S(n, 1) = 1$, the statement is valid for $k = 1$.

Now suppose $2 \leq k \leq n$. Then let X, Y be sets with $|X| = n$, $|Y| = k$. According to the definition of $S(n, k)$, it is evident that the number of surjective functions from X to Y is $k!S(n, k)$. For abbreviation we define the nonempty sets

$$A_y := \{f \in Y^X : y \notin f(X)\}, \quad y \in Y.$$

Then we obtain for the number of non-surjective functions

(i) $$k^n - k!S(n,k) = \Big|\bigcup_{y\in Y} A_y\Big|.$$

This suggests to apply Theorem 1.

Suppose $y_0 \in Y$ and define $g(x) := y_0$ for all $x \in X$. Then $g \in \bigcap_{y\in Y\setminus\{y_0\}} A_y$. Otherwise, it is obvious that $\bigcap_{y\in Y} A_y = \emptyset$. Thus, we have

(ii) $$m := \max\Big\{|T| : T \subset Y, \bigcap_{y\in T} A_y \neq \emptyset\Big\} = k - 1.$$

Furthermore, $\bigcap_{y\in T} A_y = \{f \in Y^X : f(X) \subset Y \setminus T\}$, and hence

$$\Big|\bigcap_{y\in T} A_y\Big| = \Big|(Y \setminus T)^X\Big| = (k - |T|)^n.$$

Thus, for $j = 1, 2, \ldots, k - 1$

(iii) $$\sum_{\substack{T\subset Y\\|T|=j}} \Big|\bigcap_{y\in T} A_y\Big| = \sum_{\substack{T\subset Y\\|T|=j}} (k - |T|)^n = \binom{k}{j}(k-j)^n.$$

By (i), (ii), (iii) and Theorem 1 we obtain thereupon for $l = 1, 2, \ldots, k - 1$

$$\sum_{j=1}^{l}(-1)^{j-1}\binom{k}{j}(k-j)^n \begin{cases} = k^n - k!S(n,k) & \text{if } l = k-1, \\ > k^n - k!S(n,k) & \text{if } l < k-1, \quad l \text{ odd}, \\ < k^n - k!S(n,k) & \text{if } l < k-1, \quad l \text{ even}. \end{cases}$$

Then a simple rearrangement gives us the desired result for $l \geq 1$. Since (i) immediately implies $k^n - k!S(n, k) > 0$, the result of Theorem 3 is also valid for $l = 0$. □

References

[1] Comtet, L.: *Advanced Combinatorics*. Reidel, Dordrecht 1974.

[2] Stanley, R.P.: *Enumerative Combinatorics*. Vol. 1, Cambridge University Press 1997.

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