Elem. Math. 61 (2006) 32 – 35
0013-6018/06/010032-4

Elemente der Mathematik

A short note on the Erdös-Debrunner inequality

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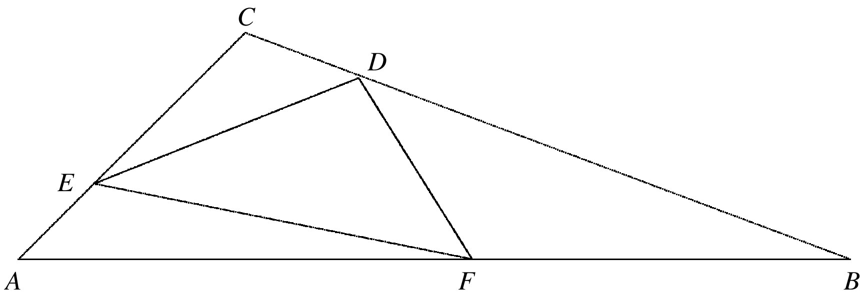
Introduction

Let ABC be an arbitrary triangle and D, E, F arbitrary points on sides BC, CA, AB, resp., all three being different from the vertices of ABC.

Then, triangle ABC is divided into four smaller triangles, a central one DEF, and three corner ones AEF, BDF, CED.

* This research is partially supported by grant 04-39-3265-2/03 (11.XII.2003) of the Federal Ministry of Education and Sciences, Bosnia and Herzegowina.

Ungleichungen, die verschiedene Elemente eines oder mehrerer Dreiecke in Beziehung zueinander setzen, haben in der mathematischen Literatur eine lange Tradition. Wegen ihrer Schönheit und den oft überraschenden Ideen dahinter treten sie häufig in den Aufgabenteilen vieler Zeitschriften und mitunter auch bei mathematischen Wettbewerben auf. Ein Beispiel eines derartigen Ergebnisses ist die in diesem Beitrag betrachtete auf P. Erdös und H. Debrunner zurückgehende Ungleichung, in der die Flächeninhalte von vier Teildreiecken eines beliebigen Ausgangsdreiecks verglichen werden. Im vorliegenden Beitrag wird der beste Exponent einer analogen Ungleichung, in der das Potenzmittel von drei der vier Flächeninhalte durch den vierten abgeschätzt wird, auf ein kleines Intervall eingegrenzt. Für den genauen Wert dieses Exponenten wird eine Vermutung aufgestellt.



Let F_1, F_2, F_3 be the areas of the three corner triangles and F_0 be the one of the central triangle. Then the *Erdös-Debrunner inequality* says

$$F_0 \geq \min(F_1, F_2, F_3), \tag{1}$$

where equality occurs if and only if D, E, F are the midpoints of the respective sides. See [1, p. 81] for an extensive list of references concerning this inequality; furthermore, the appropriate chapters in [3] and [4] report a host of results (and some of their proofs) related to two triangles, one inscribed in the other.

Speaking in the language of power-means, inequality (1) reads

$$F_0 \geq M_{-\infty}(F_1, F_2, F_3),$$

where the p-th power-mean of three positive real numbers x, y, z is defined by

$$M_p(x, y, z) = \begin{cases} \left(\dfrac{x^p + y^p + z^p}{3}\right)^{(1/p)} & p \neq 0, \\ \sqrt[3]{xyz} & p = 0. \end{cases}$$

Then, $M_p(x, y, z)$ is (weakly) increasing as p increases, and

$$M_{-\infty}(x, y, z) = \lim_{p \to -\infty} M_p(x, y, z) = \min(x, y, z).$$

Therefore, it is natural to ask whether or not there do exist inequalities of the type

$$F_0 \geq M_p(F_1, F_2, F_3), \tag{2}$$

where $p > -\infty$.

Subsequently, we will show that this is indeed so and we will give a bound for the maximum value $p_{\max}$ of p. Thereby, we will also falsify a result stated and "proven" in [2]. At the end of this note, we shall state two conjectures for further research.

Bounds for $p_{\max}$

Before stating the announced result, we are going to introduce the method of proof frequently applied in situations as the present one.

Let BC, CA, AB be divided by D, E, F in ratios $t : (1-t)$, $u : (1-u)$, $v : (1-v)$, resp., where $0 < t, u, v < 1$. Then, we have

$$F_1 = (1-u) \cdot v \cdot F_\triangle, \quad F_2 = (1-v) \cdot t \cdot F_\triangle, \quad F_3 = (1-t) \cdot u \cdot F_\triangle,$$

where $F_\triangle$ denotes the area of triangle ABC. For this, note for instance for F_1: $AF = v \cdot AB$, and $AE = (1-u) \cdot AC$. Therefore, $F_0 = F_\triangle - F_1 - F_2 - F_3$ becomes

$$F_0 = (t \cdot u \cdot v + (1-t) \cdot (1-u) \cdot (1-v)) \cdot F_\triangle.$$

Furthermore,

$$\frac{F_0}{F_1} = \frac{1-t-u-v+tu+tv+uv}{(1-u)v} = \frac{1-t}{v} + \frac{t}{1-u} - 1.$$

Since we get similar expressions for F_0/F_2 and F_0/F_3, we introduce the notation

$$x = \frac{t}{1-u}, \quad y = \frac{u}{1-v}, \quad z = \frac{v}{1-t},$$

yielding

$$\frac{F_0}{F_1} = \frac{1}{z} + x - 1, \quad \frac{F_0}{F_2} = \frac{1}{x} + y - 1, \quad \frac{F_0}{F_3} = \frac{1}{y} + z - 1.$$

We now show that p has to be negative for inequality (2) to hold in general. Indeed, let $p = 0$. Then, for (2) the inequality $F_0/F_1 \cdot F_0/F_2 \cdot F_0/F_3 \geq 1$ had to be valid. But $t = 1/2$, $u = 1/3$ and $v = 2/3$ lead to the contradiction $8/9 \geq 1$.

Therefore, we let $p = -q$, where $q > 0$, and thus, obtain for (2) the equivalent inequality

$$F_1^{-q} + F_2^{-q} + F_3^{-q} \geq 3 \cdot F_0^{-q},$$

i.e.,

$$\left(\frac{F_0}{F_1}\right)^q + \left(\frac{F_0}{F_2}\right)^q + \left(\frac{F_0}{F_3}\right)^q \geq 3,$$

hence,

$$\left(\frac{1}{z} + x - 1\right)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + z - 1\right)^q \geq 3, \tag{3}$$

where of course $x, y, z > 0$ have to satisfy

$$\frac{1}{z} + x - 1 \geq 0, \quad \frac{1}{x} + y - 1 \geq 0, \quad \frac{1}{y} + z - 1 \geq 0.$$

We are now in the position to state and prove the following

Theorem. *The quantity* $p_{\max}$ *in inequality* (2) *satisfies*

$$-1 \leq p_{\max} \leq -\frac{\ln(3/2)}{\ln(2)}.$$

Proof. In order to prove this assertion, we have to show that the minimal value $q_{\min}$ such that inequality (3) holds true in general, fulfils $\ln(3/2)/\ln(2) \le q_{\min} \le 1$.

i) Case $q_{\min} \le 1$: Indeed, inequality (3) becomes for $q = 1$

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} \ge 6.$$

But this inequality follows from $t + 1/t \ge 2$, whenever $t > 0$.

ii) Case $q_{\min} \ge \ln(3/2)/\ln(2)$: We let $t = 1/2$, and $v = 1 - u$ $(0 < u < 1)$. Then, we find

$$\frac{F_0}{F_1} = \frac{u}{1-u}, \quad \frac{F_0}{F_2} = \frac{F_0}{F_3} = 2(1-u),$$

whence inequality (3) reads

$$\left(\frac{u}{1-u}\right)^q + 2 \cdot (2(1-u))^q \ge 3$$

with $0 < u < 1$. Since the expression on the left-hand side of this inequality is continuous as $u \to 0$, we arrive at $2 \cdot 2^q \ge 3$, which completes the proof of the theorem. □

Remark. In [2] it is "shown" by an erroneous argument that $p_{\max}$ equals $-1/3$ contradicting the inequality $p_{\max} \le -\ln(3/2)/\ln(2) = -0.58\ldots$

Two conjectures

At the end of this note we state two conjectures. (The second of them is very likely to be settled by non elementary means only.)

Conjecture 1. *Let x, y and z be positive real numbers such that $1/z + x - 1 \ge 0$, $1/x + y - 1 \ge 0$ and $1/y + z - 1 \ge 0$. Then, for any $q > 0$ the minimum of the left-hand expression in* (3) *is attained at x, y and z satisfying $x \cdot y \cdot z = 1$.*

Conjecture 2. *In the above theorem, the equality $p_{\max} = -\ln(3/2)/\ln(2)$ holds true.*

References

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