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Elemente der Mathematik

The Cramer-Castillon problem and Urquhart's 'most elementary' theorem

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1 The problem of Cramer-Castillon

> "Dans ma jeunesse ... un vieux Géomètre, pour essayer mes forces en ce genre, me proposa le Problème que je vous proposai, tentez de le résoudre et vous verrez, combien il est difficile."
> (G. Cramer in 1742; quoted in Euler's *Opera*, vol. 26, p. xxv)

> "Le lendemain du jour dans lequel je lus à l'Académie ma solution du Problème concernant le cercle et le triangle à inscrire dans ce cercle, en sorte que chaque côté passe par un de trois points donnés, M. de la Grange m'en envoya la solution algébrique suivante."
> (Castillon 1776; see *Oeuvres de Lagrange*, vol. 4, p. 335)

> "Ce problème passe pour difficile, et il a fixé l'attention de plusieurs grands géomètres."
> (L. Carnot, *Géométrie de Position*, 1803, p. 383)

Problem. Given a circle and n points $A_1, A_2, \ldots, A_n$ not on this circle (Fig. 1.1, left), find an n-polygon $B_1, B_2, \ldots, B_n$ inscribed in the circle whose sides $(B_i B_{i+1})$ pass through A_i for $i = 1, \ldots, n$ (where $B_{n+1} = B_1$, see Fig. 1.1, right).

This problem has a long history; a special case for $n = 3$ goes back to Pappus (A.D. 290–350, see [6]). An unknown "vieux Géomètre" proposed the general case for $n = 3$ to Cramer, who in 1742 forwarded it to the young Castillon ("you'll see how difficult it is",

Ein sehr alter Zweig der Mathematik, die ebene Elementargeometrie, ist immer noch voller Rätsel. So kennt man zum Beispiel für den „most" elementaren Satz von Urquhart: „Wenn APQ, ARS, PBS und QBR jeweils in einer Linie liegen und $AP + PB = AR + RB$, dann ist auch $AQ + QB = AS + SB$", keinen eleganten elementaren Beweis. Der hier gegebene Beweis benützt die Möbius-Transformation, welche ehemals für ein anderes berühmtes Problem, das „Problem von Cramer-Castillon", erfunden wurde. Der vorliegende Artikel entspringt einer Einführungsvorlesung über Geometrie.

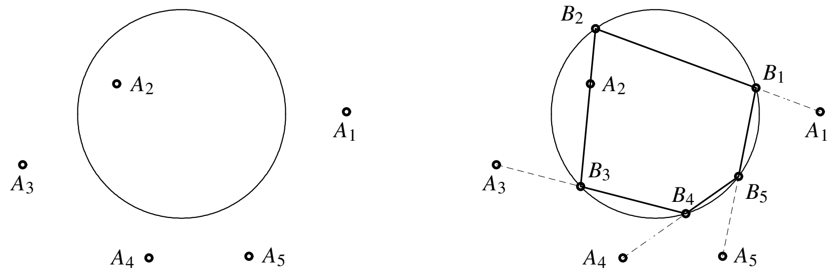

Fig. 1.1 The problem of Cramer-Castillon

see citation). Castillon arrived a third of a century later (1776) at a geometric solution. Other geometric solutions were found by Euler (1783, see [6]) and by Ottaiano (at the age of 16; see [4], p. 141). Throughout the 18th century this problem had the reputation of being very difficult.

One night after Castillon's presentation at the Academy of Berlin, Lagrange found an *analytic* solution (see citations). This solution of Lagrange was simplified by Carnot (1803, see [2]) and generalized to arbitrary n-polygons.

The Möbius transform.

> Wenn man den schlichten, stillen Mann [Möbius] vor Augen hat, muss es einen einigermassen in Erstaunen setzen, dass sein Vater … den Beruf eines Tanzlehrers ausübte. Um die Verschiedenheit der Generationen vollends vor Augen zu führen, erwähne ich, dass ein Sohn des Mathematikers der bekannte Neurologe ist, der Verfasser des vielbesprochenen Buches "Vom physiologischen Schwachsinn des Weibes". (F. Klein, *Entw. der Math. im 19. Jahrh.* (1926), p. 117)

The main tool used in our paper is the so-called *Möbius transform* $u \mapsto v$ where

$$v = \frac{pu+q}{ru+s} \qquad \text{or} \qquad \begin{pmatrix} v \\ 1 \end{pmatrix} = \text{const} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} \tag{1.1}$$

with p, q, r, s known quantities. The matrix is only significant up to a constant factor.

Carnot, in [2], discovered that the *composition* of two such transforms

$$u_2 = \frac{p_1u_1+q_1}{r_1u_1+s_1}\,, \quad u_3 = \frac{p_2u_2+q_2}{r_2u_2+s_2} = \frac{pu_1+q}{ru_1+s}$$
$$\text{where} \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \tag{1.2}$$

is again a Möbius transform with the new coefficient matrix being the *product* of the two coefficient matrices. An analogous result is true for the *inverse* operations, and the transformations with $ps - qr \neq 0$ form a *group*.

The map (1.1) is an *involution*, i.e., it's own inverse, iff $s = -p$.

Analytic solution of the Cramer-Castillon problem. The crucial discovery of Carnot was the fact that the calculations become particularly simple, if the tangents of certain half-angles are used as coordinates. This leads us to the "Pythagorean coordinates" on the circle (which we suppose is of radius 1, see Fig. 1.2, left; see also [8], p. 124)

$$u = \tan\frac{\alpha}{2}\,, \qquad x = \frac{1-u^2}{1+u^2}\,, \qquad y = \frac{2u}{1+u^2}\,. \tag{1.3}$$

The point (x, y) moves through the circle in a counter clockwise sense for $-\infty < u < \infty$ (and the values are connected to the famous Pythagorean triples $(1-u^2, 2u, 1+u^2)$; from there the name).

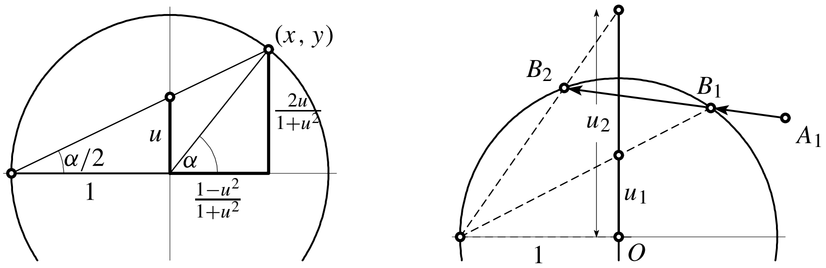

Fig. 1.2 The Pythagorean coordinates on a circle (left) and the involution of the circle with respect to a given point A_1 (right)

The idea is now the following: we start from an arbitrary point B_1 with coordinate u_1 and compute its projection B_2 onto the circle from the point A_1 with given coordinates (a_1, b_1) (see Fig. 1.2, right). We then compute $B_3, B_4, \ldots$ in a similar way and must finally satisfy the condition $B_{n+1} = B_1$.

Computations: the collinearity of B_1, B_2 and A_1 is characterized by

$$\det\begin{pmatrix} \frac{1-u_1^2}{1+u_1^2} & \frac{2u_1}{1+u_1^2} & 1 \\ \frac{1-u_2^2}{1+u_2^2} & \frac{2u_2}{1+u_2^2} & 1 \\ a_1 & b_1 & 1 \end{pmatrix} = 0\,, \qquad \text{or} \qquad \det\begin{pmatrix} 1-u_1^2 & 2u_1 & 1+u_1^2 \\ 1-u_2^2 & 2u_2 & 1+u_2^2 \\ a_1 & b_1 & 1 \end{pmatrix} = 0\,,$$

which, when multiplied out and divided by the trivial factor $u_2 - u_1$, gives

$$u_2 = \frac{-b_1u_1 + 1 - a_1}{-(a_1+1)u_1 + b_1} \quad \text{or} \quad \begin{pmatrix} u_2 \\ 1 \end{pmatrix} = \text{const}\begin{pmatrix} -b_1 & 1-a_1 \\ -a_1-1 & b_1 \end{pmatrix}\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, \tag{1.4}$$

a Möbius transform. Repeating this around the n-polygon of Fig. 1.1 and applying (1.2), we see that we have to multiply all these matrices, and we arrive at the condition

$$\begin{aligned} &u_{n+1} = u_1 = \frac{au_1 + b}{cu_1 + d}, \\ &\text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b_n & 1-a_n \\ -a_n-1 & b_n \end{pmatrix} \cdots \begin{pmatrix} -b_1 & 1-a_1 \\ -a_1-1 & b_1 \end{pmatrix}. \end{aligned} \tag{1.5}$$

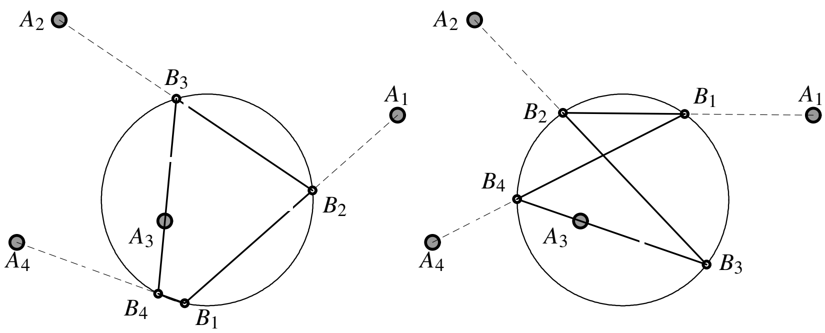

Fig. 1.3 Two solutions of a problem of Cramer-Castillon

This represents a quadratic equation for u_1 with usually two solutions (see example in Fig. 1.3[1]).

The creation of projective geometry and the Möbius transform.

> Die durch den Ponceletschen Traité eingeleitete Bewegung pflanzte sich nach Deutschland fort und ward einerseits von den Analytikern *Moebius* (1790–1868) und *Plücker* (1801–1868) und andererseits von den Synthetikern *Steiner* (1796–1863) und *von Staudt* (1798–1867) weitergeführt.
> (F. Klein, *Vorl. nicht-euklidische Geometrie* (1927), p. 11)

It is now fantastic to see, how the above problem and its solution, which had haunted the greatest minds for centuries, became absolutely natural with the invention of projective geometry. This subject originated from the epoch-making book of Poncelet [12]. Möbius then (in [9] and more explicitly in [10]) showed that the adequate *analytical* tool for describing a one-dimensional "Collineations-Verwandtschaft" were precisely formulas of the type (1.1), which with the operation (1.2) constitute the *Möbius group*. Finally, Steiner (in 1832, see [3], p. 75f) extended projective coordinates to conics. Then the projection $B_1 \mapsto B_2$, as well as $u_1 \mapsto u_2$, must be projective maps, even involutions. At the end, the problem consists in finding the *fixed points* of the involution $u_1 \mapsto u_{n+1}$. For this task, Steiner (in 1833) has found a construction using the ruler alone (see [4], §59 and §33; see also M. Berger [1], vol. 2, p. 280).

2 Billiard in an ellipse

Suppose we have a billiard table in elliptical form with focuses A and B (see Fig. 2.1, left). A fundamental property (already known to Apollonius) of ellipses is that a ball leaving a focus is reflected into the other focus. To see this, we use the fact that $\ell_1 + \ell_2 = \text{const}$: an infinitesimal movement of P by a quantity ds (see Fig. 2.1, right) leads to $d\ell_1 = -d\ell_2$ and the two angles noted α are the same.

Problem. Given the angle φ_1 under which the ball leaves A (or B), find the angle φ_2 under which the ball arrives in B (or A). What happens to φ_3, φ_4, etc.?

Solution. Put

$$c_i = \cos \varphi_i \ . \tag{2.1}$$

[1] The author is grateful to his colleague F. Sigrist (Neuchâtel) for suggesting such an example.

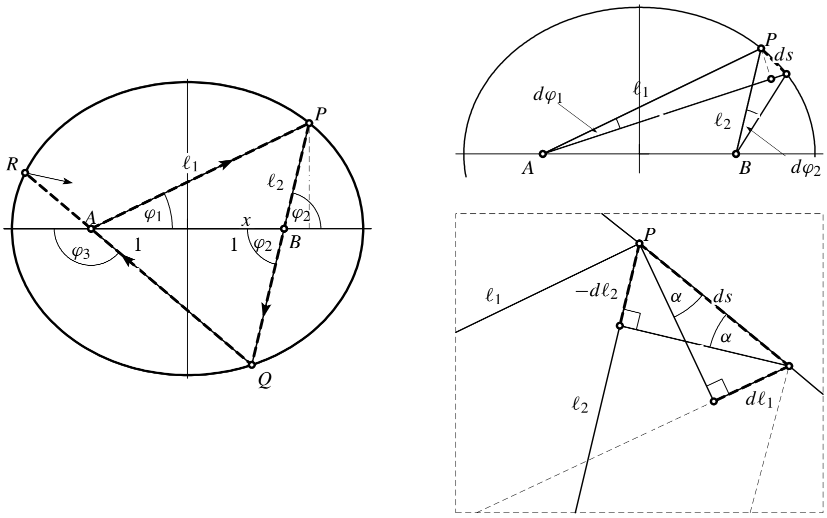

Fig. 2.1 Billiard in an ellipse (left); variation of φ_1, φ_2 and ℓ_1, ℓ_2 (right).

We suppose that the focuses are located at -1 and 1 and that e is the eccentricity of the ellipse. Hence the major semi-axis is $a = \frac{1}{e}$. If x is the abscissa of P, then [2]

$$\ell_{1,2} = a \pm ex = \frac{1}{e} \pm ex \; .$$

Now, by definition of the cosine,

$$c_1 = \frac{x+1}{ex+\frac{1}{e}} \, , \qquad c_2 = \frac{x-1}{-ex+\frac{1}{e}} \, ,$$

which represent Möbius transformations. We invert the first one and insert into the second:

$$\begin{pmatrix} 1 & -1 \\ -e & \frac{1}{e} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ e & \frac{1}{e} \end{pmatrix}^{-1} = \text{const} \cdot \begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \qquad \text{where} \qquad \theta = \frac{2e}{e^2+1} \, .$$

Hence the solution is given by the Möbius transform

$$c_2 = \frac{c_1 - \theta}{-\theta \; c_1 + 1} \qquad \text{with the matrix} \qquad A = \begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \, . \tag{2.2}$$

The subsequent angles φ_3, φ_4, etc. are determined by the *powers* of the matrix A. This matrix has eigenvectors $\binom{1}{1}$ and $\binom{-1}{1}$ with eigenvalues $1 \mp \theta$. In non trivial situations (i.e., the ellipse is not a circle and $\varphi_1 \neq 0$) the cosines c_i will converge to the eigenvector with maximal eigenvalue, i.e., to -1 (see e.g., [11], §4), and the angles φ_i converge to π.

Remark. The above results, without using the relations to Möbius transforms and matrices, were proved in [7].

[2] Remember the fact that $\ell_{1,2}$ are proportional by a factor e to the distances of P to two fixed lines (directrix). According to Zeuthen, this was a discovery of Euclid (see [3], p. 69).

3 Urquhart's 'most elementary' theorem of Euclidean geometry

"Urquhart considered this to be the 'most elementary' theorem, since it involves only the concepts of straight line and distance. The proof of this theorem by purely geometrical methods is not elementary. Urquhart discovered this result when considering some of the fundamental concepts of the theory of special relativity." (D. Elliot, *J. Australian Math. Soc.* (1968), p. 129)

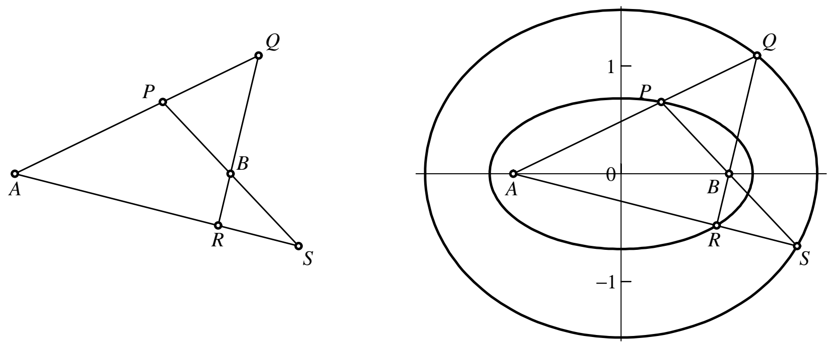

Fig. 3.1 Urquhart's theorem

M.L. Urquhart (1902–1966) was a highly appreciated lecturer of mathematics and physics at several Australian universities; he communicated his mathematical discoveries only to some of his friends. The following theorem became known by his obituary notice [5] and gained wider popularity through the book [13].

Theorem. *Let the points A, B, P, Q, R, S lie on straight lines as sketched in Fig.* 3.1 (*left*)*, then*

$$AP + PB = BR + RA \qquad \textit{implies} \qquad AQ + QB = BS + SA\,. \tag{3.1}$$

Proof. The conditions in (3.1) mean that the points P, R, as well as Q, S, lie on two confocal ellipses with focuses A and B (Fig. 3.1, right). The "billiards" of these ellipses are determined by formula (2.2), the eccentricity (i.e., the θ) being different. Hence, the trajectories

$$A \mapsto P \mapsto B \mapsto S \mapsto A \qquad \text{and} \qquad A \mapsto Q \mapsto B \mapsto R \mapsto A$$

return under the *same angle* φ_3 to A, because the matrices

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -\psi \\ -\psi & 1 \end{pmatrix}$$

commute. □

Corollary. *Under the hypotheses of Urquhart's theorem, we have also* [3]

$$\frac{AP}{PB}\cdot\frac{BS}{SA} = \frac{AQ}{QB}\cdot\frac{BR}{RA}\,. \tag{3.2}$$

[3] This corollary requires to understand not only addition, but also multiplication and division. We may therefore call it "the second most elementary theorem".

Proof. We see from the pictures in Fig. 2.1 (right) that

$$d\varphi_1 \cdot \ell_1 = ds \cdot \cos\alpha = d\varphi_2 \cdot \ell_2 \qquad \Rightarrow \qquad \frac{d\varphi_2}{d\varphi_1} = \frac{\ell_1}{\ell_2} = \frac{AP}{PB}\,.$$

If we now move all four points P, Q, R, S in Fig. 3.1 simultaneously, then the derivative $\frac{d\varphi_3}{d\varphi_1}$ resulting from the two different trajectories must give identical results. This proves (3.2). □

Acknowledgement. The author is grateful to his colleague Pierre de la Harpe for having drawn his attention to Urquhart's theorem and to Tabachnikov's beautiful book. Mr. Stanislaw Bik from the mathematical library in Geneva required only $5\frac{1}{2}$ seconds to find Dörrie's book in its original version, despite the fact that some negligent references wrote him as "Dorrie".

References

[1] Berger, M.: *Géométrie* 1 *et* 2. 3rd ed. Nathan, 1990 (1st ed. 1977).

[2] Carnot, L.: *Géométrie de position*. 1803.

[3] Coxeter, H.S.M.: *The Real Projective Plane*. McGraw-Hill, 1949.

[4] Dörrie, H.: *Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur*. Breslau 1933; engl. transl.: *100 Great Problems of Elementary Mathematics*, Dover 1965.

[5] Elliott, D.: M.L. Urquhart. *J. Austral. Math. Soc.* 8 (1968), 129–133.

[6] Euler, L.: *Problematis cuiusdam Pappi Alexandrini constructio*. 1783, Opera 26, 237–242 (see also the same volume, 243–248 and A. Speiser's remarks xxiv-xxviii).

[7] Frantz, M.: A Focussing Property of the Ellipse. *Amer. Math. Monthly* 101 (1994), 250–258.

[8] Hairer, E.; Wanner, G.: *Analysis by Its History*. Springer, New York 1995, 1997.

[9] Möbius, A.F.: *Der barycentrische Calcul*. 1827.

[10] Möbius, A.F.: Von den metrischen Relationen im Gebiete der Lineal-Geometrie. *J. Reine Angew. Math.* 4 (1829), 101–130.

[11] Perron, O.: Zur Theorie der Matrizen. *Math. Ann.* 64 (1908), 248–263.

[12] Poncelet, J.V.: *Traité des propriétés projectives des figures*. 1822.

[13] Tabachnikov, S.: Billiards. *Panor. Synthèses* 1 (1995).

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